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THE RICCATI DIFFERENTIAL EQUATIONS AND THE ADOMIAN DECOMPOSITION METHOD

G. Nhawu[1 §], P. Mafuta[2], J. Mushanyu[3]
[1,2,3]University of Zimbabwe
P.O. Box MP167, Mount Pleasant, Harare, ZIMBABWE

Abstract: In this paper, the Adomian Decomposition method is applied to the solution of Riccati differential equations. The numerical results obtained by this way have been compared with the exact solution to show that the Adomian method is a powerful method for the solution of non-linear differential equations. The method does not need linearization, weak non-linearity assumptions or perturbation theory.

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1. Introduction

The Adomian decomposition method (ADM), first envisioned by Adomian [1], is a simple and yet powerful method for solving a wide range of non-linear problems. The ADM has successfully been applied to many situations. In this paper, Riccati differential equation is solved by using Adomian decomposition method. The numerical results are compared with the exact solutions. It is shown that the errors are very small.

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§Correspondence author

2. Adomian Decomposition Method

Let the general form of a differential equation be

$$Fy = g \tag{2.1}$$

where F is the non-linear differential operator with linear and non-linear terms. The linear term is decomposed as

$$F = L + R \tag{2.2}$$

where L is easily invertible operator and R is the remainder of the linear operator. For convenience, L is taken as the highest order derivative. Thus the equation may be written as

$$Ly + Ry + Ny = g \tag{2.3}$$

where Ny corresponds to the non-linear terms. Solving from (2.3) we have

$$Ly = g - Ry - Ny. \tag{2.4}$$

Because L is invertible, the equivalent expression is

$$L^{-1}(Ly) = L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny). \tag{2.5}$$

If L is a second order operator, then L^{-1} is a two fold integration operator

$$L^{-1} = \int\int (\cdot)\, dt_1 dt_2$$

and

$$L^{-1}L(y) = y(t) - y(0) - ty'(0).$$

Then equation (2.5) for y yields

$$y(t) = y(0) + ty'(0) + L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny). \tag{2.6}$$

Therefore, y can be represented as a series

$$y(t) = \sum_{n=0}^{\infty} y_n \tag{2.7}$$

with y_0 identified as $y(t_0) + ty'(t_0) + L^{-1}(g)$ and $y_n(n > 0)$ is to determined. The non-linear term $N(y)$ will be decomposed by the infinite series of Adomian polynomials [2, 3]

$$N(y) = \sum_{n=0}^{\infty} A_n \tag{2.8}$$

where A_n's are obtained by writing

$$\mu(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n \quad \text{and} \quad N[\mu(\lambda)] = \sum_{n=0}^{\infty} \lambda^n A_n. \tag{2.9}$$

From (2.9) we deduce

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty} \lambda^n y_n\right)\right]_{\lambda=0}. \tag{2.10}$$

Now, substituting (2.7) and (2.8) into (2.6), we obtain

$$\sum_{n=0}^{\infty} y_n = y_0 - L^{-1}\left[R\left(\sum_{n=0}^{\infty} y_n\right)\right] - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{2.11}$$

Consequently, we can write

$$\begin{aligned} y_0 &= y(t_0) + ty'(t_0) + L^{-1}(g) \\ y_1 &= -L^{-1}R(y_0) - L^{-1}(A_0) \\ y_2 &= -L^{-1}R(y_1) - L^{-1}(A_1) \\ \vdots &= \vdots \;\; \vdots \;\; \vdots \\ y_{n+1} &= -L^{-1}R(y_n) - L^{-1}(A_n). \end{aligned} \tag{2.12}$$

Based on the Adomain decomposition method, we considered the solution $y(t)$ as

$$y = \lim_{n\to\infty} \varphi_n$$

where the $(n+1)$ term approximation of the solution is defined in the following form

$$\varphi_n = \sum_{k=0}^{n} y_k(t), \quad n > 0.$$

3. Analysis of General Riccati Differential Equation

In this paper, we present numerical and analytic solutions for the general Riccati differential equation[4]

$$\frac{dy}{dt} = Q(t)y + R(t)y^2 + P(t), \quad y(0) = G(t) \tag{3.1}$$

where $Q(t)$, $R(t)$, $P(t)$ and $G(t)$ are scalar functions. To solve (3.1) by means of Adomian decomposition method, we construct an Adomian polynomials A_n.

4. Numerical Example

Consider the following example

$$\frac{dy}{dt} = y^2(t) - \frac{y(t)}{t} - \frac{1}{t^2}, \quad t > 0, \quad y(0) = 0. \tag{4.1}$$

Here $Q(t) = -\frac{1}{t}$, $R(t) = 1$, $P(t) = -\frac{1}{t^2}$ and $G(t) = 0$. Applying the Adomian decomposition method to the problem, we have

$$y(t) = y(0) + L^{-1}(y^2) - L^{-1}\left(\frac{y}{t}\right) - L^{-1}\left(\frac{1}{t^2}\right) \tag{4.2}$$

where $L^{-1}(\cdot) = \int_0^t (\cdot)\, dt$. Let $y(t) = \sum_{n=0}^{\infty} y_n$ and $y^2 = \sum_{n=0}^{\infty} A_n$. Then the Adomian polynomials can be derived as follows :

$$A_0 = y_0^2, \quad A_1 = 2y_0y_1, \quad A_2 = 2y_0y_2 + y_1^2, \quad A_3 = 2y_0y_3 + 2y_1y_2$$

and so on. The recurrent scheme of ADM is written as

$$\begin{aligned} y_0 &= y(0) - L^{-1}\left(\frac{1}{t^2}\right) \\ y_{n+1} &= -L^{-1}\left(\frac{y_n}{t}\right) + L^{-1}(A_n), \quad n \geq 0. \end{aligned} \tag{4.3}$$

Then $y_0 = \frac{1}{t}$, $\quad y_1 = 0$, $\quad y_2 = 0$ and so on. Now using $\varphi_n(t)$ as the approximation of $y(t)$, we have

$$y(t) \approx \varphi_2(t) = \frac{1}{t} + 0 + 0 = \frac{1}{t}, \tag{4.4}$$

which is the same as the analytic solution.

5. Conclusion

In this paper, the Adomian decomposition method was applied to find the approximate solution of the general Riccati differential equation. This method gives a very better approximation.

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