International Journal of Differential Equations and Applications
Volume 14 No. 3 2015, 252-270
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v14i3.2371

acadpubl.eu

HEAT CONDUCTION EQUATION IN PHYSICALLY INHOMOGENEOUS MOVING COMPOSITE SOLIDS

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Abstract: This paper presents a heat conduction problem inhomogeneous physically moving compound bodies, which consists of two cylinders. Using a sequence of integral transformations(Laplace, Hankel), the Cauchys residue theory, Bessel functions theory and using the results of the roots of one transcendental equation, which results from the conditions of contact between the two cylinders, a solution in the form of the series is obtained. Hence a special case of alternating zeros of transcendental equation is also given.

AMS Subject Classification: 35K05, 35A22, 33C10
Key Words: heat conduction problem, inhomogeneous physically moving solids, sequential integral transforms method, bessel functions theory

1. Introduction

Problems on conduction of heat in composite solids are usually solved by the

Received: April 8, 2015

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sequence of integral transformations method. Thermal conductivity problem of the moving solid bodies which depends on two cylinders has been considered with few researchers among of them Carslaw and Jaeger [2]. A simple form of the thermal conductivity problem has been initiated almost a century ago, where many of simple moving and simple geometrical regionsand. All its earlier detailes have been summaried andgiven in Carslaw and Jaeger [2] and ozisik [3].In recent years, a number of the papers investigated the problem of heat conduction in the circul a hollow cylinder using a Maple11 program see Shahout et al.[4].Analytical solution of the problems thermal conduetivity of the moving bodies with finite size (in continuous and hollow cylinders and in parallelepipeds) is obtained by Gasimov et al. [5] , Lotarev [6] and Kuznetsova [7]. Kholodovskii [8] considers boundary value problems for linear differential equations in piecewise - homogenous cylinders into two half-cylinders by multilayer film,and these problems have a great importance in many engineering fields which intervention in the design of internal combustion engines, material in aviation, and the factories of the production of military. Our motivation in this paper is to investigate the problem of heat propagation in anisotropic composite solid with variable thermic features and with composite region which consists of two cylinders, and moves with velocity in the direction of z-axis. Also we use sequential of integral transformations like Laplace and Hankel transformations. Our idea is to choose appropriate integral transformation with respect to each variable which allows overcoming the main difficulty to find the desired solution with the help of inverse transformations.

2. Formulation of the Problem

Consider the composite solid which consists of two parts; a hollow cylinder($a < r < b, 0 < z < \ell_1$)and entire cylinder($0 < r < b, \ell_1 < z < \ell_1 + \ell_2$),which moves with velocityνin the direction of the axis oz where the heat flow is radial and cylindrically symmetric.At each of the parts the initial temperature is given,the interior wall of the hollow cylinder is maintained at zero temperature.It is also sometimes referred to as 'Newton's Law',and the external surface there happens radiation to the medium with zero temperature.

On the joining surface (for $z = \ell_1$) there is no contact resistance.

The formulation of this problem looks as follows:

$$\frac{\partial T_1}{\partial t} = \varkappa_1\left(\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r} + \frac{\partial^2 T_1}{\partial z^2}\right) - \nu\frac{\partial T_1}{\partial z}, \quad a < r < b,\ 0 < z < \ell_1, \quad t > 0, \ (1)$$

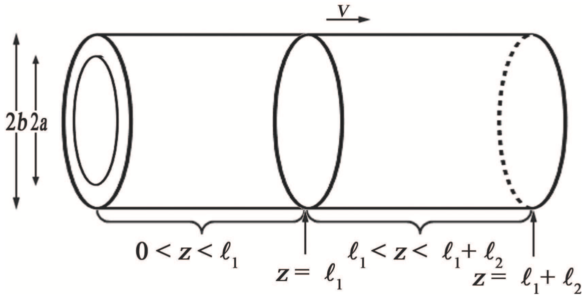

Figure 1: Composite solid which consists of a hollow and entire cylinder.

$$T_1|_{t=0} = f_1(r,z), \tag{2}$$

$$T_1|_{r=a} = 0\ ,\ (\frac{\partial}{\partial r} + h_1)T_1|_{r=b} = 0, \tag{3}$$

$$(\frac{\partial}{\partial z} - g_1)T_1|_{z=0} = 0; \tag{4}$$

$$\frac{\partial T_2}{\partial t} = \varkappa_2(\frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial T_2}{\partial r} + \frac{\partial^2 T_2}{\partial z^2}) - \nu\frac{\partial T_2}{\partial z},\quad 0 < r < b,\ \ell_1 < z < \ell_1 + \ell_2,\ t > 0, \tag{5}$$

$$T_2|_{t=0} = f_2(r,z), \tag{6}$$

$$(\frac{\partial}{\partial r} + h_2)T_2|_{r=b} = 0, \tag{7}$$

$$(\frac{\partial}{\partial z} + g_2)T_2|_{z=\ell_1+\ell_2} = 0, \qquad 0 < r < b, \tag{8}$$

$$T_2|_{z=\ell_1} = 0, \qquad 0 \le r < a. \tag{9}$$

In the first part, if we assume that there is no contact resistance at the surface of separation $z = \ell_1$ the boundary conditions become

$$T_1|_{z=\ell_1} = T_2|_{z=\ell_1}, \qquad K_1\frac{\partial T_1}{\partial z}|_{z=\ell_1} = K_2\frac{\partial T_2}{\partial z}|_{z=\ell_1}, \quad a < r < b, \tag{10}$$

where a and b are radii of the hollow cylinder and entire cylinder, respectively. In the first the region $0 < z < \ell_1, T_1, \rho_1, c_1$ and $\varkappa_1$ are the temperature, density, specific heat and diffusivity, where as T_2, ρ_2, c_2 and $\varkappa_2$ for the corresponding quantities in $\ell_1 < z < \ell_1 + \ell_2$, where K_1, K_2 are the thermal conductivity of the substance,that move with velocity ν is constant, and h_1, h_2, g_1, g_2 are the coefficient of surface heat transfer are constants, and $f_1(r,z)$ and $f_2(r,z)$ are given functions.

3. Solving Method

Using successive transformations, the solution is organized follows.
The first stage: We apply the Laplace transform with respect to t namely:

$$\widetilde{T}_i \equiv \widetilde{T}_i(r,z;p) = \int_0^{\infty} e^{-pt} T_i(r,z,t)dt \ , (Re(p) > 0), i = 1, 2.$$

The subsidiary equation is

$$\varkappa_1 \Big(\frac{\partial^2 \widetilde{T}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{T}_1}{\partial r} + \frac{\partial^2 \widetilde{T}_1}{\partial z^2}\Big) - \nu \frac{\partial \widetilde{T}_1}{\partial z} - p\widetilde{T}_1 = -f_1(r,z), \quad a < r < b, \quad 0 < z < \ell_1, \tag{11}$$

with

$$\widetilde{T}_1|_{r=a} = 0, \qquad \Big(\frac{\partial}{\partial r} + h_1\Big)\widetilde{T}_1|_{r=b} = 0, \tag{12}$$

$$\Big(\frac{\partial}{\partial z} - g_1\Big)\widetilde{T}_1|_{z=0} = 0, \tag{13}$$

$$\varkappa_2 \Big(\frac{\partial^2 \widetilde{T}_2}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{T}_2}{\partial r} + \frac{\partial^2 \widetilde{T}_2}{\partial z^2}\Big) - \nu \frac{\partial \widetilde{T}_2}{\partial z} - p\widetilde{T}_2 = -f_2(r,z),$$
$$0 < r < b, \ \ell_1 < z < \ell_1 + \ell_2, \tag{14}$$

with

$$\Big(\frac{\partial}{\partial r} + h_2\Big)\widetilde{T}_2|_{r=b} = 0, \tag{15}$$

$$\Big(\frac{\partial}{\partial z} + g_2\Big)\widetilde{T}_2|_{z=\ell_1+\ell_2} = 0, \qquad 0 < r < b, \tag{16}$$

$$\widetilde{T}_2|_{z=\ell_1} = 0, \quad 0 \le r < a. \tag{17}$$

At the junction $z = \ell_1$ the boundary conditions there are

$$\widetilde{T}_1|_{z=\ell_1} = \widetilde{T}_2|_{z=\ell_1}, \quad K_1 \frac{\partial \widetilde{T}_1}{\partial z}|_{z=\ell_1} = K_2 \frac{\partial \widetilde{T}_2}{\partial z}|_{z=\ell_1}, \quad a < r < b, \tag{18}$$

The second stage: Applying the Hankel transform to Eq.(11) - Eq.(13) in $a < r < b$.The problem is transformed in to

$$\varkappa_1 \Big(-\lambda^2 \overline{\overline{T}}_1 + \frac{d^2 \overline{\overline{T}}_1}{dz^2}\Big) - \nu \frac{d\overline{\overline{T}}_1}{dz} - p\overline{\overline{T}}_1 = -\overline{f}_1(z), \quad 0 < z < \ell_1, \tag{19}$$

$$\left(\frac{d}{dz} - g_1\right)\overline{\overline{T}}_1|_{z=0} = 0, \tag{20}$$

writing $\overline{\overline{\widetilde{T}}}_1$ for the Hankel transform of $\overline{\overline{\widetilde{T}}}_1$ with respect to r, we have

$$\overline{\overline{\widetilde{T}}}_1 \equiv \overline{\overline{\widetilde{T}}}_1(z;p,\eta) = \int_a^b r\widetilde{T}_1(r,z,p)\overline{K}_\eta^{(1)}(r)dr.$$

$$(\overline{f}_1(z) = \int_a^b rf_1(r,z)\overline{K}_\eta^{(1)}(r)dr).$$

$\overline{K}_\eta^{(1)}(r)$ is consider the solution of the spectral problem of Bessel's equation.

$$\frac{d^2\overline{K}^{(1)}}{dr^2} + \frac{1}{r}\frac{d\overline{K}^{(1)}}{dr} + \lambda^2\overline{K}^{(1)} = 0, \qquad a < r < b \tag{21}$$

$$\overline{K}^{(1)}|_{r=a} = 0, \qquad (\frac{d\overline{K}^{(1)}}{dr} + h_1\overline{K}^{(1)})|_{r=b} = 0. \tag{22}$$

The general solution of Eq.(21)is:

$$\overline{K}^{(1)}(r) = A_0J_0(\lambda r) + B_0Y_0(\lambda r);$$

where J_0, Y_0 are Bessel functions of zero order A_0 and B_0 are arbitrary constants. The unknowns A_0 and B_0 are to be found from Eq.(22), as follows:

$$\begin{aligned} &A_0J_0(\lambda a) + B_0Y_0(\lambda a) = 0, \\ &A_0\left[\lambda J_0'(\lambda b) + h_1J_0(\lambda b)\right] + B_0\left[\lambda Y_0'(\lambda b) + h_1Y_0(\lambda b)\right] = 0. \end{aligned} \tag{23}$$

So, eigenvalues of the spectral problem Eq.(21) and Eq.(22) are obtained from the determinant system Eq.(23):

$$\begin{vmatrix} J_0(\lambda a) & Y_0(\lambda a) \\ \lambda J_0'(\lambda b) + h_1J_0(\lambda b) & \lambda Y_0'(\lambda b) + h_1Y_0(\lambda b) \end{vmatrix} = 0$$

or

$$h_1\left[J_0(\lambda a)Y_0(\lambda b) - J_0(\lambda b)Y_0(\lambda a)\right] + \lambda\left[J_1(\lambda b)Y_0(\lambda a) - J_0(\lambda a)Y_1(\lambda b)\right] = 0. \tag{24}$$

The roots of Eq.(24) are all real and simple; we need to prove the following theorem:

Theorem(Type theorem Dixon's[1]):If $AD = BC$,then the positive zeros of$AY_0(x)+BxY_0'(x)$ are interlaced with those of $CJ_0(x)+DxJ_0'(x)$. To prove this theorem, we enter a function in the following form

$$\varphi(x) = \frac{CJ_0(x) + DxJ_0'(x)}{AY_0(x) + BxY_0'(x)},$$

let us denote $d \equiv [AY_0(x) + BxY_0'(x)]^2$ and we need to prove that this an increasing function or is decreasing function between any two consecutive zeros of the denominator. Then

$$\begin{aligned}\varphi'(x) &= \frac{1}{d}\{\left[(C+D)J_0'(x) + DxJ_0''(x)\right]\left[AY_0(x) + BxY_0'(x)\right] \\ &- \left[CJ_0(x) + DxJ_0'(x)\right]\left[(A+B)Y_0'(x) + BxY_0''(x)\right]\} \\ &= \frac{1}{d}\{\left[(C+D)J_0'(x) - DJ_0'(x) - DxJ_0(x)\right]\left[AY_0(x) + BxY_0'(x)\right] \\ &- \left[CJ_0(x) + DxJ_0'(x)\right]\left[(A+B)Y_0'(x) - BY_0'(x) - BxY_0(x)\right]\} \\ &= \frac{1}{dx}\left\{\frac{2}{\pi}\left[AC + BDx^2\right] + [BC - AD]\left[x^2J_0(x)Y_0(x) + J_0'(x)Y_0'(x)\right]\right\}.\end{aligned}$$

$$BC - AD = 0 \implies \exists S : \frac{A}{C} = \frac{B}{D} \equiv S \implies A = C \cdot S\ , B = D \cdot S \implies$$

If $S > 0$, then $AC = C^2 \cdot S > 0$ and $BD = D^2 \cdot S > 0$,if $S < 0$, then $AC < 0,\ BD < 0$. So form the sign of $\varphi'(x)$ fixed, between any two zeros consecutive for the denominator . Again, we next use this theorem to rewrite Eq.(24) as

$$\frac{J_0(\beta x)}{Y_0(\beta x)} = \frac{h_1 bJ_0(x) + xJ_0'(x)}{h_1 bY_0(x) + xY_0'(x)} \quad ,where\ \ \beta \equiv \frac{a}{b},\quad x \equiv \lambda b. \tag{25}$$

Using this theorem for the right hand side of Eq.(25) at $A = C = h_1 b$, $B = D = 1$, and using the properties of Bessel function known $J_0,\ Y_0$ [1], it produces all roots of equation Eq.(24) are all real, simple and have an infinite numbers. Now in the special case for Eq.(24, we calculated using program Maple and part of this programming and suppose that the thickness of the cylinder $10^{-3}m^2$.To consider that the center material and the cylinder material are silver ($K_1 = 1.00,\ h_1 = 0.07$).Calculating the roots of the equation Eq.(24) and can be rewritten in Maple language,as well as taken the values to $a = 1, b = 2$,we obtain the following [4]:

restart;

$$\begin{aligned}\Phi\ \ :\ &= unapply(\lambda \cdot (BesselJ(0,\lambda) \cdot \frac{\partial}{\partial\lambda}(BesselY(0, 2\cdot\lambda)) \\ &-BesselY(0,\lambda) \cdot \frac{\partial}{\partial\lambda}(BesselJ(0, 2\cdot\lambda))) + 0.07 \cdot (BesselJ(0,\lambda) \\ &\cdot BesselY(0, 2\cdot\lambda) - BesselY(0,\lambda) \cdot BesselJ(0, 2\cdot\lambda))) : \Phi(0);\end{aligned}$$

$$\lambda(-2BesselJ(0,\lambda)BesselY(1,2\lambda)+2BesselY(0,\lambda)BesselJ(1,2\lambda))$$
$$+0.07BesselJ(0,\lambda)BesselY(0,2\lambda)-0.07BesselY(0,\lambda)BesselJ(0,2\lambda)$$

$$lambda[1,1]:=fsolve(\Phi(1),\lambda,1..2)$$

$$1.3886$$

$$for\ \ k\ \ from\ 2\ to\ 10\ do\ lambda[1,k]:=fsolve(\Phi(1),\lambda,\%+\pi..\%+2\cdot\pi)\ od$$

$$4.6534\ ,\ 7.8186\ ,\ 10.970\ ,\ 14.117\ ,\ 17.262$$
$$20.406\ ,\ 23.550\ ,\ 26.693\ ,\ 29.835$$

The two solutions of the equations can be found as:

$$\begin{aligned} A_0 &= \lambda_{1,\eta}Y_0'(\lambda_{1,\eta}b)+h_1Y_0(\lambda_{1,\eta}b)\ , \\ B_0 &= -\lambda_{1,\eta}J_0'(\lambda_{1,\eta}b)-h_1J_0(\lambda_{1,\eta}b). \end{aligned}$$

Therefore one can be consider the following functions as eigenfunctions for problems Eq.(21)-Eq.(22):

$$\begin{aligned} \Phi_{1,\eta}(r) &= [\lambda_{1,\eta}Y_0'(\lambda_{1,\eta}b)+h_1Y_0(\lambda_{1,\eta}b)]J_0(\lambda_{1,\eta}r)-[\lambda_{1,\eta}J_0'(\lambda_{1,\eta}b) \\ &\quad +h_1J_0(\lambda_{1,\eta}b)]Y_0(\lambda_{1,\eta}r), \end{aligned}$$

that is to say the kernel can be found from a relation:

$$\overline{K}_\eta^{(1)}(r)=\frac{1}{N_{1,\eta}}\Phi_{1,\eta}(r),$$

where

$$N_{1,\eta}=\int_a^b r\left[\Phi_{1,\eta}(r)\right]^2dr.$$

Thus, our solution for Eq.(19)-Eq.(20) is in the form

$$\begin{aligned} \overline{\widetilde{T}}_{1,\eta}(z;p,\eta) &= e^{\nu z/(2\varkappa_1)}[\frac{2}{\omega_1-q_{1,\eta}}\left(\omega_1\sinh q_{1,\eta}z-q_{1,\eta}\cosh q_{1,\eta}z\right)A_{1,\eta} \\ &\quad -\frac{1}{\varkappa_1 q_{1,\eta}}\int_0^z e^{-\nu\zeta/(2\varkappa_1)}\sinh q_{1,\eta}(z-\zeta)\cdot\overline{f}_{1,\eta}(\zeta)d\zeta] \end{aligned}$$

where

$$q_{1,\eta}^2\equiv\left[p+(\varkappa_1\lambda_{1,\eta}^2+\frac{\nu^2}{4\varkappa_1})\right]/\varkappa_1,\qquad \omega_1\equiv\frac{\nu}{2\varkappa_1}-g_1,$$

$A_{1,\eta}$ *are arbitrary constants.*

The third stage: Applying the zero-order Hankel transform with respect to r to Eq.(14)-Eq.(16). This gives

$$\varkappa_2(-\lambda^2\overline{\overline{T}}_2 + \frac{d^2\overline{\overline{T}}_2}{dz^2}) - \nu\frac{d\overline{\overline{T}}_2}{dz} - p\overline{\overline{T}}_2 = -\overline{f}_2(z)\ , \qquad \ell_1 < z < \ell_1 + \ell_2, \quad (26)$$

$$\left(\frac{d}{dz} + g_2\right)\overline{\overline{T}}_2|_{z=\ell_1+\ell_2} = 0, \quad (27)$$

where $\overline{\overline{T}}_2$ is the Hankel transform, by kernel for$\overline{K}_\eta^{(2)}(r)$ namely:

$$\overline{\overline{T}}_2 \equiv \overline{\overline{T}}_2(z;p,\eta) = \int_0^b r\widetilde{T}_2(r,z;p)\overline{K}_\eta^{(2)}(r)dr.$$

$$(\overline{f}_2(z) = \int_0^b rf_2(r,z)\overline{K}_\eta^{(2)}(r)dr.$$

$\overline{K}_\eta^{(2)}(r)$ is considered as a solution of the spectral problem:

$$\frac{d^2\overline{K}^{(2)}}{dr^2} + \frac{1}{r}\frac{d\overline{K}^{(2)}}{dr} + \lambda^2\overline{K}^{(2)} = 0, \qquad 0 < r < b, \quad (28)$$

$$\overline{K}^{(2)}|_{r=0} < \infty, \qquad (\frac{d\overline{K}^{(2)}}{dr} + h_2\overline{K}^{(2)})|_{r=b} = 0. \quad (29)$$

The eigenvalues $\lambda_{2,\eta}^2$ for the problems Eq.(28)-Eq.(29),can be obtained from the equation:

$$\lambda J_0'(\lambda b) + h_2 J_0(\lambda b) = 0. \quad (30)$$

It is known that at[1] the roots of Eq.(30) are all real,simple and have an infinite numbers. $\overline{K}_\eta^{(2)}(r)$ the kernels is calculated from the relation

$$\overline{K}_\eta^{(2)}(r) = \frac{1}{N_{2,\eta}}J_0(\lambda_{2,\eta}r),$$

where

$$N_{2,\eta} = \int_0^b rJ_0^2(\lambda_{2,\eta}r)dr \ = \frac{b^2}{2\lambda_{2,\eta}^2}(h_2^2 + \lambda_{2,\eta}^2)J_0^2(\lambda_{2,\eta}b).$$

Thus,the solution of the problem Eq.(26)-Eq.(27) takes the form:

$$\overline{\overline{T}}_{2,\eta}(z,p,\eta) = e^{\nu z/(2\varkappa_2)}\{\frac{2e^{(\ell_1+\ell_2)q_{2,\eta}}}{\omega_2 - q_{2,\eta}}[\omega_2 \sinh q_{2,\eta}(z-(\ell_1+\ell_2))$$

$$- q_{2,\eta} \cosh q_{2,\eta}(z - (\ell_1 + \ell_2))]A_{2,\eta} + \frac{1}{\varkappa_2 q_{2,\eta}}$$
$$\cdot \int_z^{\ell_1+\ell_2} e^{\nu\zeta/(2\varkappa_2)} \sinh q_{2,\eta}(z - \zeta) \cdot \overline{f}_{2,\eta}(\zeta) d\zeta\},$$

where

$$q_{2,\eta}^2 \equiv \left[p + (\varkappa_2 \lambda_{2,\eta}^2 + \frac{\nu^2}{4\varkappa_2})\right] / \varkappa_2, \qquad \omega_2 \equiv \frac{\nu}{2\varkappa_2} + g_2,$$

$A_{2,\eta}$ *are arbitrary constants.*

The fourth stage: Using the inverse Hankel transform,we find two solutions of the two problems Eq.(19)-Eq.(20) and Eq.(26)-Eq.(27) in the form:

$$\widetilde{T}_1(r, z; p) = 2e^{\nu z/(2\varkappa_1)} \sum_{\eta=1}^{\infty} \frac{A_{1,\eta}}{\omega_1 - q_{1,\eta}} (\omega_1 \sinh q_{1,\eta} z - q_{1,\eta} \cosh q_{1,\eta} z) \Phi_{1,\eta}(r)$$
$$- \frac{1}{\varkappa_1} \sum_{\eta=1}^{\infty} \left[\int_0^z e^{\nu(z-\zeta)/(2\varkappa_1)} \sinh q_{1,\eta}(z - \zeta) \cdot \overline{f}_{1,\eta}(\zeta) d\zeta\right] \frac{1}{q_{1,\eta}} \Phi_{1,\eta}(r), \tag{31}$$

$$\widetilde{T}_2(r, z; p) = 2e^{\nu z/(2\varkappa_2)} \sum_{\eta=1}^{\infty} \frac{A_{2,\eta}}{\omega_2 - q_{2,\eta}} e^{(\ell_1+\ell_2) q_{2,\eta}} [\omega_2 \sinh q_{2,\eta}(z - (\ell_1 + \ell_2))$$
$$- q_{2,\eta} \cosh q_{2,\eta}(z - (\ell_1 + \ell_2))]\Phi_{2,\eta}(r) + \frac{1}{\varkappa_2} \sum_{\eta=1}^{\infty} [\int_z^{\ell_1+\ell_2} e^{\nu(z-\zeta)/(2\varkappa_2)}$$
$$\cdot \sinh q_{2,\eta}(z - \zeta) \cdot \overline{f}_{2,\eta}(\zeta) d\zeta] \frac{\Phi_{2,\eta}(r)}{q_{2,\eta}}, \tag{32}$$

where $\Phi_{2,\eta}(r) \equiv J_0(\lambda_{2,\eta} r)$. When $a < r < b$, the constants $A_{1,\eta}$ and $A_{2,\eta}$ are found from Eq.(18) in the following form

$$A_{1,\eta} = (C_{2,2} \cdot G_{1,\eta} - C_{1,2} \cdot G_{2,\eta}) / (\Delta_\eta \cdot \Phi_{1,\eta}), \tag{33}$$

$$A_{2,\eta} = (C_{1,1} \cdot G_{2,\eta} - C_{2,1} \cdot G_{1,\eta}) / (\Delta_\eta \cdot \Phi_{2,\eta}), \tag{34}$$

where

$$C_{1,1} \equiv C_{1,1}(\eta) = \frac{2}{\omega_1 - q_{1,\eta}} (\omega_1 \sinh q_{1,\eta} \ell_1 - q_{1,\eta} \cosh q_{1,\eta} \ell_1) \cdot e^{\nu \ell_1/(2\varkappa_1)},$$
$$C_{1,2} \equiv C_{1,2}(\eta) = \frac{2}{\omega_2 - q_{2,\eta}} (\omega_2 \sinh q_{2,\eta} \ell_2 + q_{2,\eta} \cosh q_{2,\eta} \ell_2)$$
$$\cdot e^{\nu \ell_1/(2\varkappa_2)} \cdot e^{(\ell_1+\ell_2) q_{2,\eta}},$$
$$C_{2,1} \equiv C_{2,1}(\eta) = \frac{1}{\omega_1 - q_{1,\eta}} [(\frac{\nu \omega_1}{\varkappa_1} - 2q_{1,\eta}^2) \sinh q_{1,\eta} \ell_1 + q_{1,\eta}(2\omega_1 - \frac{\nu}{\varkappa_1})$$

$$\cdot \cosh q_{1,\eta}\ell_1]e^{\nu\ell_1/(2\varkappa_1)},$$

$$C_{2,2} \equiv C_{2,2}(\eta) = \frac{k}{\omega_2 - q_{2,\eta}}[(\frac{\nu\omega_2}{\varkappa_2} - 2q_{2,\eta}^2)\sinh q_{2,\eta}\ell_2 - q_{2,\eta}(2\omega_2 - \frac{\nu}{\varkappa_2})$$

$$\cdot \cosh q_{2,\eta}\ell_2]e^{\nu\ell_1/(2\varkappa_2)}\cdot e^{(\ell_1+\ell_2)q_{2,\eta}},$$

$$G_{1,\eta} \equiv G_{1,\eta}(p,r) = \overline{F}_{2,\eta}^{(s)}(p)\cdot\Phi_{2,\eta}(r) + \overline{F}_{1,\eta}^{(s)}(p)\Phi_{1,\eta}(r)$$

$$G_{2,\eta} \equiv G_{2,\eta}(p,r) = k\left[\frac{\nu}{2\varkappa_2}\overline{F}_{2,\eta}^{(s)}(p) + \overline{F}_{2,\eta}^{(c)}(p)\right]\Phi_{2,\eta}(r) + \left[\frac{\nu}{2\varkappa_1}\overline{F}_{1,\eta}^{(s)}(p) + \overline{F}_{1,\eta}^{(c)}(p)\right]\Phi_{1,\eta}(r),$$

$$\Delta_\eta \equiv \Delta_\eta(p) = C_{1,1}\cdot C_{2,2} - C_{1,2}\cdot C_{2,1}, \qquad k \equiv \frac{K_2}{K_1}$$

$$\overline{F}_{1,\eta}^{(s)} = \frac{1}{\varkappa_1 q_{1,\eta}}\int_0^{\ell_1} e^{\nu(\ell_1-\zeta)/(2\varkappa_1)}\sinh q_{1,\eta}(\ell_1-\zeta)\cdot\overline{f}_{1,\eta}(\zeta)d\zeta \quad ,$$

$$\overline{F}_{1,\eta}^{(c)} = \frac{1}{\varkappa_1}\int_0^{\ell_1} e^{\nu(\ell_1-\zeta)/(2\varkappa_1)}\cosh q_{1,\eta}(\ell_1-\zeta)\cdot\overline{f}_{1,\eta}(\zeta)d\zeta,$$

$$\overline{F}_{2,\eta}^{(s)} = \frac{1}{\varkappa_2 q_{2,\eta}}\int_{\ell_1}^{\ell_1+\ell_2} e^{\nu(\ell_1-\zeta)/(2\varkappa_2)}\sinh q_{2,\eta}(\ell_1-\zeta)\cdot\overline{f}_{2,\eta}(\zeta)d\zeta,$$

$$\overline{F}_{2,\eta}^{(c)} = \frac{1}{\varkappa_2}\int_{\ell_1}^{\ell_1+\ell_2} e^{\nu(\ell_1-\zeta)/(2\varkappa_2)}\cosh q_{2,\eta}(\ell_1-\zeta)\cdot\overline{f}_{2,\eta}(\zeta)d\zeta.$$

when $0 < r < a$,the constants $A_{2,\eta}$ are obtained from the relation Eq.(17):

$$A_{2,\eta} = \overline{F}_{2,\eta}^{(s)}(p)/C_{1,2}(\eta). \tag{35}$$

The fifth stage: we look for the zeros of function$\Delta_\eta(p)$.Using these relations $for\ C_{i,j}(i,j=1,2)$ we write the equation $\Delta_\eta(p) = 0$ in the form:

$$\frac{\nu}{2}\left(\frac{k}{\varkappa_2} - \frac{1}{\varkappa_1}\right)(\omega_1\sinh q_{1,\eta}\ell_1 - q_{1,\eta}\cosh q_{1,\eta}\ell_1)(\omega_2\sinh q_{2,\eta}\ell_2 + q_{2,\eta}\cosh q_{2,\eta}\ell_2)$$

$$- q_{1,\eta}(\omega_1\cosh q_{1,\eta}\ell_1 - q_{1,\eta}\sinh q_{1,\eta}\ell_1)(\omega_2\sinh q_{2,\eta}\ell_2 + q_{2,\eta}\cosh q_{2,\eta}\ell_2)$$

$$-kq_{2,\eta}\left(\omega_1\sinh q_{1,\eta}\ell_1-q_{1,\eta}\cosh q_{1,\eta}\ell_1\right)\left(\omega_2\cosh q_{2,\eta}\ell_2+q_{2,\eta}\sinh q_{2,\eta}\ell_2\right)=0. \tag{36}$$

If $\varkappa_1\lambda_{1,\eta}^2+\frac{\nu^2}{4\varkappa_1}>\varkappa_2\lambda_{2,\eta}^2+\frac{\nu^2}{4\varkappa_2}$,then the equation Eq.(36)takes the form

$$\frac{\nu}{2}(\frac{k}{\varkappa_2}-\frac{1}{\varkappa_1})(\omega_1\sin\beta_\eta\ell_1-\beta_\eta\cos\beta_\eta\ell_1)(\omega_2\sin\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\ell_2+\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}$$
$$\cdot\cos\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\ell_2)-\beta_\eta(\omega_1\cos\beta_\eta\ell_1+\beta_\eta\sin\beta_\eta\ell_1)(\omega_2\sin\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\ell_2$$
$$+\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\cos\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\ell_2)-k\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}(\omega_1\sin\beta_\eta\ell_1-\beta_\eta\cos\beta_\eta\ell_1)$$
$$\cdot(\omega_2\cos\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\ell_2-\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\sin\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\ell_2)=0. \tag{37}$$

where

$$\varkappa\equiv\varkappa_1/\varkappa_2\ ,\quad \gamma_\eta^2\equiv\frac{1}{\varkappa_2}\left[\left(\varkappa_1\lambda_{1,\eta}^2+\frac{\nu^2}{4\varkappa_1}\right)-\left(\varkappa_2\lambda_{2,\eta}^2+\frac{\nu^2}{4\varkappa_2}\right)\right]$$

The Eq.(37) is an algebraic equation for β_η , to prove that it has no complex root for this the equation,we introduce the following function:

$$U(z;\beta_\eta)=\begin{cases} a_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta), & \text{if } 0<z<\ell_1,\\ a_{2,\eta}e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta), & \text{if } \ell_1<z<\ell_1+\ell_2,\end{cases}$$

(the conjugate subsidiary function to β_η), where β_η is the root of the Eq.(37),

$$U_1(z;\beta_\eta)\equiv\omega_1\sin\beta_\eta z-\beta_\eta\cos\beta_\eta z,$$
$$U_2(z;\beta_\eta)\equiv\omega_2\sin\beta_{2,\eta}((\ell_1+\ell_2)\ -z)+\beta_{2,\eta}\cos\beta_{2,\eta}((\ell_1+\ell_2)-z),$$
$$\beta_{2,\eta}\equiv\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2},$$

$$a_{1,\eta}\equiv K_2e^{\nu\ell_1/(2\varkappa_2)}\left(\frac{\nu}{2\varkappa_2}U_2(\ell_1;\beta_\eta)+U_2'(\ell_1;\beta_\eta)\right),$$
$$a_{2,\eta}\equiv K_1e^{\nu\ell_1/(2\varkappa_1)}\left(\frac{\nu}{2\varkappa_1}U_1(\ell_1;\beta_\eta)+U_1'(\ell_1;\beta_\eta)\right),$$

It is evident that U satisfies the following relations

$$\left(\frac{d}{dz}-g_1\right)U(z;\beta_\eta)|_{z=0}=0, \tag{38}$$

$$\left(\frac{d}{dz}+g_2\right)U(z;\beta_\eta)|_{z=\ell_1+\ell_2}=0. \tag{39}$$

Since β_η is the root of the equation Eq.(37),we can prove that the function $U(z;\beta_\eta)$ satisfies two the conditions of continuity when $z=\ell_1$ as follows

$$a_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)|_{z=\ell_1}=a_{2,\eta}e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)|_{z=\ell_1}, \tag{40}$$

$$K_1a_{1,\eta}\frac{d}{dz}\left(e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)|_{z=\ell_1}=K_2a_{2,\eta}\frac{d}{dz}\left(e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)|_{z=\ell_1}. \tag{41}$$

On the other hand U satisfies the following relations:

$$\left(\frac{d^2}{dz^2}+\beta_\eta^2\right)U_1(z;\beta_\eta)=0, \qquad 0<z<\ell_1, \tag{42}$$

$$\left(\frac{d^2}{dz^2}+\beta_{2,\eta}^2\right)U_2(z;\beta_\eta)=0, \qquad \ell_1<z<\ell_1+\ell_2, \tag{43}$$

we suppose that $\beta_\eta{=}\beta_\eta^{(1)}+i\beta_\eta^{(2)},\;\; \alpha_\eta\equiv\overline{\beta}_\eta=\beta_\eta^{(1)}-i\beta_\eta^{(2)}$, then the two roots are complex conjugate of the equation Eq.(37), thus, we obtain

$$\left(\frac{d^2}{dz^2}+\alpha_\eta^2\right)U_1(z;\alpha_\eta)=0, \qquad 0<z<\ell_1, \tag{44}$$

$$\left(\frac{d^2}{dz^2}+\alpha_{2,\eta}^2\right)U_2(z;\alpha_\eta)=0 \quad , \qquad \ell_1<z<\ell_1+\ell_2 \quad , \tag{45}$$

$$\alpha_{2,\eta}^2\equiv\varkappa\alpha_\eta^2+\gamma_\eta^2.$$

From Eq.(42)and Eq.(44) we find that:

$$\begin{aligned}&\left(\beta_\eta^2-\alpha_\eta^2\right)\int_0^{\ell_1}\left(a_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)\left(\overline{a}_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)dz\\&+\int_0^{\ell_1}|a_{1,\eta}|^2\,[(e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta))''\left(e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)\\&-\left(e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)''\left(e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)]dz=0,\end{aligned} \tag{46}$$

and from Eq.(43)-Eq.(45)we get:

$$\varkappa\left(\beta_\eta^2-\alpha_\eta^2\right)\int_{\ell_1}^{\ell_1+\ell_2}\left(a_{2,\eta}e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)\left(\overline{a}_{2,\eta}e^{\nu z/(2\varkappa_2)}U_2(z;\alpha_\eta)\right)dz$$

$$+\int_{\ell_1}^{\ell_1+\ell_2}|a_{2,\eta}|^2\,[\left(e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)''\left(e^{\nu z/(2\varkappa_2)}U_2(z;\alpha_\eta)\right)$$

$$-\left(e^{\nu z/(2\varkappa_2)}U_2(z;\alpha_\eta)\right)''\left(e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)]dz=0 \qquad . \qquad (47)$$

From the relations Eq.(46)-Eq.(47), using Eq.(38)-Eq.(41), we obtain:

$$\left(\beta_\eta^2-\alpha_\eta^2\right)[K_1\int_0^{\ell_1}|a_{1,\eta}|^2\left(e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)\left(e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)dz$$

$$+K_2\varkappa\int_{\ell_1}^{\ell_1+\ell_2}|a_{2,\eta}|^2\left(e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)\left(e^{\nu z/(2\varkappa_2)}U_2(z;\alpha_\eta)\right)dz]$$

$$=K_1\int_0^{\ell_1}|a_{1,\eta}|^2\,[\left(e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)\left(e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)''$$

$$-\left(e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)\left(e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)'']dz$$

$$+K_2\int_{\ell_1}^{\ell_1+\ell_2}|a_{2,\eta}|^2\,[\left(e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)\left(e^{\nu z/(2\varkappa_2)}U_2(z;\alpha_\eta)\right)''$$

$$-\left(e^{\nu z/(2\varkappa_2)}U_2(z;\alpha_\eta)\right)\left(e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)'']dz$$

$$=K_1\{\left[\left(a_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)\left(\overline{a}_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)'\right]_0^{\ell_1}$$

$$-\left[\left(\overline{a}_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)\left(a_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)'\right]_0^{\ell_1}$$

$$-\int_0^{\ell_1}\left(a_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)'\left(\overline{a}_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)'dz$$

$$+\int_0^{\ell_1}\left(\overline{a}_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\alpha_\eta)\right)'\left(a_{1,\eta}e^{\nu z/(2\varkappa_1)}U_1(z;\beta_\eta)\right)'dz\}$$

$$+K_2\{\left[\left(a_{2,\eta}e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)\left(\overline{a}_{2,\eta}e^{\nu z/(2\varkappa_2)}U_2(z;\alpha_\eta)\right)'\right]_{\ell_1}^{\ell_1+\ell_2}$$

$$-\left[\left(\overline{a}_{2,\eta}e^{\nu z/(2\varkappa_2)}U_2(z;\alpha_\eta)\right)\left(a_{2,\eta}e^{\nu z/(2\varkappa_2)}U_2(z;\beta_\eta)\right)'\right]_{\ell_1}^{\ell_1+\ell_2}$$

$$- \int_{\ell_1}^{\ell_1+\ell_2} \left(a_{2,\eta} e^{\nu z/(2\varkappa_2)} U_2(z;\beta_\eta)\right)' \left(\overline{a}_{2,\eta} e^{\nu z/(2\varkappa_2)} U_2(z;\alpha_\eta)\right)' dz$$

$$+ \int_{\ell_1}^{\ell_1+\ell_2} \left(\overline{a}_{2,\eta} e^{\nu z/(2\varkappa_2)} U_2(z;\alpha_\eta)\right)' \left(a_{2,\eta} e^{\nu z/(2\varkappa_2)} U_2(z;\beta_\eta)\right)' dz\}$$

$$= a_{1,\eta} e^{\nu z/(2\varkappa_1)} U_1(z;\beta_\eta) \mid_{z=\ell_1} [K_1(\overline{a}_{1,\eta} e^{\nu z/(2\varkappa_1)} U_1(z;\alpha_\eta))' \mid_{z=\ell_1}$$

$$- K_2(\overline{a}_{2,\eta} e^{\nu z/(2\varkappa_2)} U_2(z;\alpha_\eta))' \mid_{z=\ell_1}] - (\overline{a}_{1,\eta} e^{\nu z/(2\varkappa_1)}(z;\alpha_\eta)) \mid_{z=\ell_1}$$

$$\cdot \left[K_1 a_{1,\eta} e^{\nu z/(2\varkappa_1)} U_1(z;\beta_\eta) \mid_{z=\ell_1} - K_2(a_{2,\eta} e^{\nu z/(2\varkappa_2)} U_2(z;\beta_\eta)) \mid_{z=\ell_1}\right]$$

$$- K_1 \left|a_{1,\eta}\right|^2 [-\beta_\eta(\frac{\nu}{2\varkappa_1}(-\alpha_\eta) + \omega_1\alpha_\eta) + \alpha_\eta(\frac{\nu}{2\varkappa_1}(-\beta_\eta) + \omega_1\beta_\eta)]$$

$$+ K_2 \left|a_{2,\eta}\right|^2 [e^{\nu(\ell_1+\ell_2)/(2\kappa_2)} U_2(\ell_1+\ell_2;\beta_\eta)(-g_2) e^{\nu(\ell_1+\ell_2)/(2\varkappa_2)} U_2(\ell_1+\ell_2;\alpha_\eta)$$

$$- e^{\nu(\ell_1+\ell_2)/(2\varkappa_2)} U_2(\ell_1+\ell_2;\alpha_\eta)(-g_2) e^{\nu(\ell_1+\ell_2)/(2\varkappa_2)} U_2(\ell_1+\ell_2;\beta_\eta)] = 0.$$

Since α_η and β_η are two complex conjugate numbers,so, $U_1(z;\beta_\eta)$ and $U_1(z;\alpha_\eta)$, also $U_2(z;\beta_\eta)$ and $U_2(z;\alpha_\eta)$, are two conjugate functions.So from the last relation, we deduce that:
$\left(\beta_\eta^2 - \alpha_\eta^2\right) . [\langle \textit{the positive number} \rangle] = 0.$
Hence, we find that $\beta_\eta^{(1)} \beta_\eta^{(2)} = 0$.
This is contradiction.
Thus we see that Eq.(37) has only real roots.
We can write equation Eq.(37) in the form

$$\frac{\left(\frac{\nu\omega_1}{2\varkappa_1\beta_\eta} + \beta_\eta\right) - \left(\frac{\nu}{2\varkappa_1} - \omega_1\right)\cot\beta_\eta\ell_1}{k\left(\frac{\omega_1}{\beta_\eta} - \cot\beta_\eta\ell_1\right)} = \frac{\left(\frac{\nu\omega_2}{2\varkappa_2\beta_{2,\eta}} + \beta_{2,\eta}\right) + \left(\frac{\nu}{2\varkappa_2} - \omega_2\right)\cot\beta_{2,\eta}\ell_2}{\left(\frac{\omega_1}{\beta_{2,\eta}} + \cot\beta_{2,\eta}\ell_2\right)}, \tag{48}$$

so,(dividing the equation Eq.(37) by $\sin\beta_\eta\ell_1 . \sin\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\ell_2$; if it were the common roots for equations $\sin\beta_\eta\ell_1 = 0$, $\sin\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\ell_2 = 0$; there are m_1, m_2 are two natural numbers, where $\beta_\eta\ell_1 = m_1\pi$, $\beta_{2,\eta}\ell_2 = m_2\pi$,and substituting in the equation Eq.(37),we obtain:

$$(-1)^{m_1+m_2}\frac{m_1\pi}{\ell_1}\cdot\frac{m_2\pi}{\ell_2}\left[\frac{\nu}{2}\left(\frac{k}{\varkappa_2}-\frac{1}{\varkappa_1}\right)+\omega_1-k\omega_2\right]=0$$

$$\Longrightarrow g_1+g_2=0,$$

and this is impossible).

We can prove that the left hand side of the relation Eq.(48)is a decreasing function in β_η in each interval:

$$(\beta.1): \qquad \left(\beta_\eta^{(n)},\frac{n\pi}{\ell_1}\right)\cup\left(\frac{n\pi}{\ell_1},\beta_\eta^{(n+1)}\right), \qquad n=1,2,...$$

where $\beta_\eta^{(n)}$ are the roots of the equation

$$\cot\beta_\eta\ell_1=\frac{\omega_1}{\beta_\eta},$$

and the right hand side of the relation Eq.(48)is an increasing function with respect to $\beta_{2,\eta}$ in each interval:

$$(\beta.2): \qquad \left(\beta_{2,\eta}^{(n)},\frac{n\pi}{\ell_2}\right)\cup\left(\frac{n\pi}{\ell_2},\beta_{2,\eta}^{(n+1)}\right), \qquad n=1,2,...$$

where $\beta_{2,\eta}^{(n)}$ are the roots of the equation

$$\cot\beta_{2,\eta}\ell_2=-\frac{\omega_2}{\beta_{2,\eta}}.$$

We have thus proved that the equation has only real, simple and an infinite numbers; in each interval in$(\beta.1)\,,(\beta.2)$ the positive roots for this equation, and the negative roots are equal in absolute value to the positive ones.

The sixth stage: we apply the inverse Laplace transform with respect to(t) of the two relations Eq.(25),Eq.(34). Substituting the relations Eq.(31),Eq.(34) and Eq.(35) in two the relations Eq.(25),Eq.(34)and using Cauchy's theory of residues, on the basis of the results of the fifth stage, we obtain a solution to the problem under study in the form:

$$T_1(r,z,t)=e^{-\nu(\ell_1-z)/(2\varkappa_1)}\sum_{\eta=1}^{\infty}\sum_{m=1}^{\infty}\varphi_1(z,\beta_{\eta m})\{\varphi_2(\ell_2,\beta_{2,\eta m})[\frac{1}{\varkappa_1}\beta_{2,\eta m}$$

$$\cdot\int_0^{\ell_1}(\frac{\nu}{2}(\frac{k}{\varkappa_2}-\frac{1}{\varkappa_1})R^s(\zeta;\beta_{\eta m})-R^c(\zeta;\beta_{\eta m}))\overline{f}_{1,\eta}(\zeta)d\zeta\Phi_{1,\eta}(r)$$

$$- \frac{k}{\varkappa_2} \beta_{\eta m} \int_{\ell_1}^{\ell_1+\ell_2} R_2^c(\zeta; \beta_{2,\eta m}) \overline{f}_{2,\eta}(\zeta) d\zeta \Phi_{2,\eta}(r)]$$
$$- k\varphi'_{2,z}(\ell_2, \beta_{2,\eta m}) Q(r, \beta_{\eta m})\} S_{\eta m}(t),$$

$$T_2(r,z,t) = e^{-\nu(\ell_1-z)/(2\varkappa_2)} \sum_{\eta=1}^{\infty} \sum_{m=1}^{\infty} \varphi_2(\ell_1+\ell_2-z, \beta_{\eta m}) \{\varphi_1(\ell_1, \beta_{\eta m}) \frac{1}{\varkappa_2}$$
$$\cdot [-\beta_{2,\eta m} \int_0^{\ell_1} R^c(\zeta; \beta_{\eta m}) \overline{f}_{1,\eta}(\zeta) d\zeta \Phi_{1,\eta}(r)$$
$$+ \beta_{\eta m} \int_{\ell_1}^{\ell_1+\ell_2} (\frac{\nu}{2}(\frac{1}{\varkappa_1} - \frac{k}{\varkappa_2}) R_2^s(\zeta; \beta_{2,\eta m}) - R_2^c(\zeta; \beta_{2,\eta m}))$$

$$\cdot \overline{f}_{2,\eta}(\zeta) d\zeta \Phi_{2,\eta}(r)] + \varphi'_{1,z}(\ell_1, \beta_{\eta m}) Q(r, \beta_{\eta m})\} \cdot S_{\eta m}(t),$$

when $a < r < b$,

$$T_2(r,z,t) = -2e^{-\nu(\ell_1-z)/(2\varkappa_2)} \sum_{\eta=1}^{\infty} \sum_{j=1}^{\infty} \frac{\Phi_{2,\eta}(r)\varphi_2(\ell_1+\ell_2-z, \beta_{2,\eta j})}{\varkappa_2 \beta_{2,\eta j} \varphi'_{2,\beta}(\ell_2, \beta_{2,\eta j})}$$
$$\cdot \int_{\ell_1}^{\ell_1+\ell_2} R_2^s(\zeta; \beta_{2,\eta j}) \overline{f}_{2,\eta}(\zeta) d\zeta \cdot e^{-\varkappa_2(\beta_{2,\eta j}^2 + \lambda_{2,\eta}^2 + (\nu/(2\varkappa_2))^2)t},$$

when $0 < r < a$,where

$$\varphi_1(z,\beta) \equiv \omega_1 \sin \beta z - \beta \cos \beta z,$$

$$\varphi_2(z,\beta) \equiv \omega_2 \sin \beta z + \beta \cos \beta z,$$

$$R^s(z;\beta) \equiv e^{\nu(\ell_1-z)/(2\varkappa_1)} \sin \beta(\ell_1 - z),$$

$$R_2^s(z;\beta) \equiv e^{\nu(\ell_1-z)/(2\varkappa_2)} \sin \beta(\ell_1 - z),$$

$$R^c(z;\beta) \equiv e^{\nu(\ell_1-z)/(2\varkappa_1)} \beta \cos \beta(\ell_1 - z),$$

$$R_2^c(z;\beta) \equiv e^{\nu(\ell_1-z)/(2\varkappa_2)} \beta \cos \beta(\ell_1 - z),$$

$$Q(r,\beta_{\eta m}) \equiv \frac{1}{\varkappa_1}\beta_{2,\eta m}\int_0^{\ell_1} R^s(\zeta;\beta_{\eta m})\overline{f}_{1,\eta}(\zeta)d\zeta\Phi_{1,\eta}(r)$$
$$+\frac{1}{\varkappa_2}\beta_{\eta m}\int_{\ell_1}^{\ell_1+\ell_2} R_2^s(\zeta;\beta_{2,\eta m})\overline{f}_{2,\eta}(\zeta)d\zeta\Phi_{2,\eta}(r),$$

if $\left(\varkappa_1\lambda_{1,\eta}^2+\frac{\nu^2}{4\varkappa_1}\right) > \left(\varkappa_2\lambda_{2,\eta}^2+\frac{\nu^2}{4\varkappa_2}\right)$, *then*

$\beta_{\eta m}$ are the positive roots of the equation

$$\delta(\beta_\eta) \equiv \left[\frac{\nu}{2}\left(\frac{k}{\varkappa_2}-\frac{1}{\varkappa_1}\right)\varphi_1(\ell_1,\beta_\eta)-\varphi_{1,z}'(\ell_1,\beta_\eta)\right]\varphi_2\left(\ell_2,\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\right)$$
$$-k\cdot\varphi_1(\ell_1,\beta_\eta)\,\varphi_{2,z}'\left(\ell_2,\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\right)=0,$$

$$\beta_{2,\eta m}=\sqrt{\varkappa\beta_{\eta m}^2+\gamma_\eta^2}\;;\qquad \gamma_\eta^2\equiv\frac{1}{\varkappa_2}\left[\left(\varkappa_1\lambda_{1,\eta}^2+\frac{\nu^2}{4\varkappa_1}\right)-\left(\varkappa_2\lambda_{2,\eta}^2+\frac{\nu^2}{4\varkappa_2}\right)\right]$$

$$S_{\eta m}(t)=\frac{2}{\beta_{\eta m}\beta_{2,\eta m}\delta'(\beta_{\eta m})}\exp\{-\varkappa_1\left(\beta_{\eta m}^2+\lambda_{1,\eta}^2+(\nu/(2\varkappa_1))^2\right)t\},$$

if $\left(\varkappa_2\lambda_{2,\eta}^2+\frac{\nu^2}{4\varkappa_2}\right) > \left(\varkappa_1\lambda_{1,\eta}^2+\frac{\nu^2}{4\varkappa_1}\right)$, *then*

$\beta_{2,\eta m}$ are the positive roots of the equation

$$\delta_2(\beta_{2,\eta}) \equiv \left[\frac{\nu}{2}\left(\frac{k}{\varkappa_2}-\frac{1}{\varkappa_1}\right)\varphi_1\left(\ell_1,\sqrt{\frac{1}{\varkappa}\beta_{2,\eta}^2+\gamma_{2,\eta}^2}\right)-\varphi_{1,z}'\left(\ell_1,\sqrt{\frac{1}{\varkappa}\beta_{2,\eta}^2+\gamma_{2,\eta}^2}\right)\right]$$
$$\cdot\varphi_2(\ell_2,\beta_{2,\eta})-k\cdot\varphi_1\left(\ell_1,\sqrt{\frac{1}{\varkappa}\beta_{2,\eta}^2+\gamma_{2,\eta}^2}\right)\varphi_{2,z}'(\ell_2,\beta_{2,\eta})=0,$$

$$\beta_{\eta m}=\sqrt{\frac{1}{\varkappa}\beta_{2,\eta m}^2+\gamma_{2,\eta}^2}\;;\qquad \gamma_{2,\eta}^2\equiv\frac{1}{\varkappa_1}\left[\left(\varkappa_2\lambda_{2,\eta}^2+\frac{\nu^2}{4\varkappa_2}\right)-\left(\varkappa_1\lambda_{1,\eta}^2+\frac{\nu^2}{4\varkappa_1}\right)\right],$$

$$S_{\eta m}(t)=\frac{2}{\beta_{\eta m}\beta_{2,\eta m}\delta_2'(\beta_{2,\eta m})}\exp\{-\varkappa_2(\beta_{2,\eta m}^2+\lambda_{2,\eta}^2+(\nu/(2\varkappa_2))^2)t\},$$

$\beta_{2,\eta j}$ are the positive roots of the equation

$$\omega_2\sin\beta_{2,\eta}\ell_2+\beta_{2,\eta}\cos\beta_{2,\eta}\ell_2=0.$$

4. Conclusion

In this paper, the spread heat in body which in two cylinders is studied, the solution behavior of the boundary problem is given as an infinite series using two sequences of integral transformation forms, Bessel function theory and the conditions of contact between the two cylinders. Hence we get the transcendental equation and using the theorem of type Dixon, we proved that equation has infinitely, simple and real roots. The behavior of eigenvalues spectral problem is determined. The obtained eigenfunctions are the kernel of integral transformations of kind Bessel function. This problem has numerous engineering applications, such as, the administration motion of bodies which consists of two cylinders or more. Also, machining, welding, grinding, internal combustion engines, as well as in a factory for the production of military, are all others practical examples.

Acknowledgements

The authors are very much grateful to Prof. Ahmed M. K. Tarabia and Dr. R. Tantawy members of mathematics department at Damietta university, Egypt for their valuable comments to bring the paper in the present form.

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