

ON THE RESONANT OSCILLATION OF CERTAIN FOURTH
ORDER NONLINEAR DIFFERENTIAL EQUATIONS
WITH DELAY

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Abstract: We prove the existence of periodic solutions for the periodic boundary value problem (1.1) under some resonant conditions on the asymptotic behaviour of $\frac{g(t, y)}{ay}$ for $|y| \rightarrow \infty$.

The uniqueness of periodic solutions is also examined.

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1. Introduction

In this paper, we study the periodic boundary value problem

$$x^{(iv)}(t) + a \ddot{x}(t) + f(\dot{x})\ddot{x}(t) + g(t, \dot{x}(t - \tau)) + h(x) = p(t) \quad (1.1)$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \quad (1.2)$$

with fixed delay $\tau \in [0, 2\pi)$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $p : [0, 2\pi] \rightarrow \mathbb{R}$ and $g : [0, 2\pi] \times \mathbb{R} \rightarrow \mathbb{R}$ are 2π -periodic in t and g satisfies caratheodory conditions. The unknown function $x : [0, 2\pi] \rightarrow \mathbb{R}$ is defined for $0 < t \leq \tau$ by $x(t - \tau) = [2\pi - (t - \tau)]$. We are specifically concerned with the existence of

periodic solutions of (1.1) - (1.2) under some resonant conditions.

In a recent paper [4] we studied the above equation with $f(\dot{x}) = b$ and $h(x) = d$ where b and d are constants with $g(t, y)$ satisfying certain non-resonant conditions. In our present study, we will allow $g(t, y)$ to satisfy certain resonant conditions.

In what follows, we shall use the spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ and $L^k([0, 2\pi])$ of continuous, k times continuously differentiable or measurable real functions whose k th power of the absolute value is Lebesgue integrable. We shall use the Sobolev spaces $W_{2\pi}^{4,2}$ and $H_{2\pi}^1$ respectively defined by $W_{2\pi}^{4,2} = \{x : [0, 2\pi] \rightarrow \mathbb{R} | x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}$ are absolutely continuous on $[0, 2\pi]$, $x^{(i)}(0) = x^{(i)}(2\pi)$, $i = 0, 1, 2, 3\}$ with the norm

$$|x|_{W_{2\pi}^{4,2}}^2 = \sum_{i=0}^4 \frac{1}{2\pi} \int_0^{2\pi} |x^{(i)}(t)|^2 dt$$

and

$$H_{2\pi}^1 = \{x : [0, 2\pi] \rightarrow \mathbb{R} | x$$

is absolutely continuous on $[0, 2\pi]$ and $\dot{x} \in L_{2\pi}^2\}$, with norm

$$|x|_{H_{2\pi}^1}^2 = \left(\frac{1}{2\pi} \int_0^{2\pi} x(t) dt \right)^2 + \frac{1}{2\pi} \int_0^{2\pi} |\dot{x}|^2 dt$$

2. The Linear Problem

Let us consider the equation

$$x^{(iv)}(t) + a \ddot{x}(t) + b\dot{x}(t) + c\dot{x}(t - \tau) + dx = 0 \tag{2.1}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{2.2}$$

where a, b, c and d are constants.

Lemma 2.1. *Let $a \neq 0$ and let*

$$(n - 1)^2 < \frac{c}{a} < n^2 \tag{2.3}$$

where $n \geq 1$ is a positive integer. Then the bvp (2.1) - (2.2) has no non trivial 2π -periodic solution.

Proof. We consider a solution of the form $x(t) = Ae^{\lambda t}$ where $\lambda = in$ with $i^2 = -1$ and $A \neq 0$ is a constant.

Then Lemma 2.1 will follow if

$$\phi(n, \tau) = -an^2 + c \cos n\tau \neq 0 \tag{2.4}$$

for all $n \geq 1$ and $\tau \in [0, 2\pi)$. By (2.3) we get

$$a^{-1}\phi(n, \tau) = -n^2 + \frac{c}{a} \cos n\tau \leq -n^2 + \frac{c}{a} < 0$$

Thus $\phi(n, \tau) \neq 0$ and the result follows.

If $x \in L^1_{2\pi}$ we shall write

$$\bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t)dt, \quad \tilde{x}(t) = x(t) - \bar{x} \tag{2.5}$$

so that

$$\int_0^{2\pi} \tilde{x}(t)dt = 0$$

We shall consider next the delay equation

$$x^{(iv)}(t) + a \ddot{x}(t) + b\dot{x}(t) + c(t)\dot{x}(t - \tau) + dx = 0 \tag{2.6}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{2.7}$$

where a, b, d are constants and $c \in L^2_{2\pi}$.

Theorem 2.1. *Let $a \neq 0$ and $d \neq 0$ and set $\Gamma(t) = a^{-1}c(t)$. Suppose that*

$$0 < \Gamma(t) < 1 \tag{2.8}$$

then the bvp (2.6) - (2.7) has no nontrivial solution for every $\tau \in [0, 2\pi)$.

Proof. Let $x(t)$ be any solution of (2.6) - (2.7). Then we can verify that

$$\frac{1}{2\pi} \int_0^{2\pi} -\dot{x}(t)[a^{-1}(x^{(iv)} + b\ddot{x} + dx)]dt = 0$$

Thus we have

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_0^{2\pi} -\dot{x}(t)[a^{-1}\{x^{(iv)} + b\ddot{x}(t) + dx\} + \ddot{x} + \Gamma(t)\dot{x}(t - \tau)]dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} -\dot{x}(t)[\ddot{x} + \Gamma(t)\dot{x}(t - \tau)]dt \end{aligned}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\ddot{x}^2(t) - \Gamma(t)\dot{x}(t)\dot{x}(t - \tau)) dt$$

using

$$-ab = \frac{(a - b)^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

we have

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_0^{2\pi} (\ddot{x}^2(t) - \frac{\Gamma(t)}{2}\dot{x}^2(t) - \frac{\Gamma(t)}{2}\dot{x}^2(t - \tau))dt \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \frac{\Gamma(t)}{2}(\dot{x}(t) - \dot{x}(t - \tau))^2 dt \end{aligned}$$

From (2.8) we have

$$\begin{aligned} 0 &\geq \frac{1}{2\pi} \int_0^{2\pi} (\ddot{x}^2(t) - \frac{\Gamma(t)}{2}(\dot{x}^2(t) + \dot{x}^2(t - \tau)))dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\ddot{x}^2 - \Gamma(t)\dot{x}^2(t)) dt \geq \delta|\dot{x}|_{H^1_{2\pi}}^2 \end{aligned}$$

By Lemma 1 of [6], where we have used the fact that $\int_0^{2\pi} \dot{x}^2(t)dt = \int_0^{2\pi} \dot{x}^2(t - \tau)dt$. Here, $\delta > 0$. Thus $\dot{x} = 0$ and since $d \neq 0$ it is clear that $x = \text{constant}$ cannot be a solution of (2.6) - (2.7). Therefore $x = 0$.

3. The Nonlinear Case

We shall now consider a preliminary Lemma which will enable us obtain a priori estimates required for our results.

Lemma 3.1. *Let all the conditions of Theorem 2.1 hold and let δ be related to $\Gamma(t)$ by Theorem 2.1. Suppose that for $V \in L^2_{2\pi}$, $0 \leq V(t) \leq \Gamma(t) + \varepsilon$ a.e. $t \in [0, 2\pi]$, $\varepsilon > 0$.*

Then

$$\frac{1}{2\pi} \int_0^{2\pi} -\dot{x}[a^{-1}\{x^{(iv)} + b\ddot{x} + dx\} + \ddot{x} + v(t)\dot{x}(t - \tau)]dt \geq (\delta - \varepsilon)|\ddot{x}|_2^2.$$

Proof. From the proof of Lemma 2.1 we have

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} -\dot{\tilde{x}}[a^{-1}\{x^{(iv)} + b\ddot{x} + dx\} + \ddot{x} + v(t)\dot{x}(t - \tau)]dt \\ & \geq \frac{1}{2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} [\ddot{\tilde{x}}^2 - \Gamma(t)\dot{\tilde{x}}^2] dt \right\} \\ & \quad + \frac{1}{2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} [\ddot{\tilde{x}}^2(t - \tau) - \Gamma(t)\dot{\tilde{x}}^2(t - \tau)] dt \right\} \\ & \quad - \frac{\varepsilon}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} (\dot{\tilde{x}}^2(t - \tau) + \dot{\tilde{x}}^2(t))dt \\ & = \frac{1}{2\pi} \int_0^{2\pi} [\ddot{\tilde{x}}^2(t - \tau) - \Gamma(t)\dot{\tilde{x}}^2(t - \tau)] dt \\ & \quad - \frac{1}{2} \left[\frac{\varepsilon}{2\pi} \int_0^{2\pi} \dot{\tilde{x}}^2(t - \tau)dt + \frac{\varepsilon}{2\pi} \int_0^{2\pi} \dot{\tilde{x}}^2(t - \tau) \right] dt \\ & \geq \delta|\ddot{\tilde{x}}|_2^2 - \varepsilon|\dot{\tilde{x}}|_2^2 \\ & = (\delta - \varepsilon)|\ddot{\tilde{x}}|_2^2 \end{aligned}$$

We shall next consider the nonlinear delay equation

$$x^{(iv)}(t) + a \ddot{x}(t) + f(\dot{x})\ddot{x}(t) + g(t, \dot{x}(t - \tau)) + h(x) = p(t) \tag{3.1}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{3.2}$$

where $f, h : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and $g : [0, 2\pi] \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $g(\cdot, x)$ is measurable on $[0, 2\pi]$ for each $x \in \mathbb{R}$ and $g(t, \cdot)$ is continuous on \mathbb{R} for almost each $t \in [0, 2\pi]$. We assume moreover that for each $r > 0$ there exists $Y_r \in L^2_{2\pi}$ such that $g(t, y) \leq Y_r(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in [-r, r]$ such a g is said to satisfy Caratheodory's conditions.

Theorem 3.1. *Let $a \neq 0$. Suppose that g is a Caratheodory function with respect to the space $L^2_{2\pi}$ such that:*

(i) *There exists $r > 0$ such that*

$$axg(t, x) \geq 0 \text{ for } |x| \geq r$$

(ii) $\limsup_{|x| \rightarrow \infty} \frac{g(t, x)}{ax} \leq \Gamma(t)$
 for $t \in [0, 2\pi]$ with $\Gamma(t)$ satisfying $0 < \Gamma(t) < 1$.

(iii) $\lim_{|x| \rightarrow +\infty} \text{sgn} x h(x) = +\infty$

Suppose further that $p \in L^2_{2\pi}$, then for arbitrary continuous function f , equation (3.1) has at least one 2π -periodic solution.

Proof. Let $\delta > 0$ be related to Γ by Lemma 3.1. Then by (i) and (ii) there exists a constant $R > 0$ such that for a.e $t \in [0, 2\pi]$ and all y with $|y| \geq R$ we have

$$0 \leq \frac{g(t, y)}{ay} \leq \Gamma(t) + \delta/2 \tag{3.3}$$

We define

$$\tilde{Y}(t, x) = \begin{cases} (ax)^{-1}g(t, x), & |x| > R \\ (aR)^{-1}g(t, R), & 0 < x < R \\ -(aR)^{-1}g(t, -R), & -R_1 < x < 0 \\ \Gamma(t), & x = 0 \end{cases}$$

Then

$$0 \leq \tilde{Y}(t, x) \leq \Gamma(t) + \delta/2 \tag{3.5}$$

for a.e $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$.

Clearly the function

$$\tilde{g} = a\tilde{Y}(t, \dot{x}(t - \tau))\dot{x}(t - \tau)$$

is a Caratheorody function. So also is g_0 defined by

$$g_0(t, \dot{x}(t - \tau)) = g(t, \dot{x}(t - \tau)) - \tilde{g}(t, \dot{x}(t - \tau)) \tag{3.6}$$

Thus there exists $\alpha \in L^2_{2\pi}$ such that

$$|g_0(t, \dot{x}(t - \tau))| \leq \alpha(t) \tag{3.7}$$

for a.e $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$. Problem (3.1) is thus equivalent to

$$x^{(iv)}(t) + a \ddot{x}(t) + f(\dot{x})\ddot{x}(t) + \tilde{Y}(t, \dot{x}(t - \tau))\dot{x}(t - \tau) + g_0(t, \dot{x}(t - \tau)) + h(x) = p(t) \tag{3.8}$$

to which we shall apply coincidence degree theory.

Let $X = C^3[0, 2\pi]$, $Z = L^2_{2\pi}$
 $\text{dom}L = \{x \in X : x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \text{ and } \dot{x}, \ddot{x}, \ddot{\ddot{x}} \text{ are absolutely continuous on } [0, 2\pi]\}$.

Define as in [6]

$$L : \text{dom}L \subset X \longrightarrow Z, x \longrightarrow x^{(iv)} + a \ddot{\ddot{x}}$$

$$F : \text{dom}L \subset X \longrightarrow Z, x \longrightarrow f(\dot{x})\ddot{x}$$

$$G : \text{dom}L \subset X \longrightarrow Z, x \longrightarrow \tilde{Y}(t, \dot{x}(t - \tau))\dot{x}(t - \tau)$$

$$H : \text{dom}L \subset X \longrightarrow Z, x \longrightarrow h(x)$$

$$A : \text{dom}L \subset X \longrightarrow Z, x \longrightarrow \Gamma(t)\dot{x}(t - \tau)$$

$$G_0 : \text{dom}L \subset X \longrightarrow Z, x \longrightarrow g_0(t, \dot{x}(t - \tau))$$

$$T : \text{dom}L \subset X \longrightarrow Z, x \longrightarrow -p(t)$$

It is seen that G and G_0 are well defined and L -compact on bounded subsets of X and that L is a linear Fredholm mapping of index zero.

Therefore the proof of the theorem will follow from theorem 4.5 of [7] if we show that the possible solutions of the equation

$$Lx + \lambda Fx + (1 - \lambda)Ax + \lambda Gx + \lambda G_0x + (1 - \lambda)dx + \lambda Hx + \lambda Tx = 0 \quad (3.9)$$

or equivalently the equation

$$a^{-1}[x^{(iv)} + \lambda f(\dot{x})\ddot{x}] + \ddot{\ddot{x}} + (1 - \lambda)\Gamma(t)\dot{x}(t - \tau) + \lambda \tilde{Y}(t, \dot{x}(t - \tau))\dot{x}(t - \tau) + (1 - \lambda)a^{-1}dx + a^{-1}\lambda g_0(t, \dot{x}(t - \tau)) + a^{-1}\lambda h(x) - a^{-1}\lambda p(t) = 0 \quad (3.10)$$

where $d > 0$ are apriori bounded independently of $\lambda \in [0, 1]$.

For $\lambda = 0$ we get the equation

$$x^{(iv)} + a \ddot{\ddot{x}} + \Gamma(t)\dot{x}(t - \tau) + dx = 0$$

which by Theorem 2.1 has only the trivial solution. We observe that

$$0 < (1 - \lambda)\Gamma(t) + \lambda \tilde{Y}(t, \dot{x}(t - \tau)) \leq \Gamma(t) + \delta/2$$

for a.e $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$.

Hence using $V(t) = (1 - \lambda)\Gamma(t) + \lambda\tilde{Y}(t, \dot{x}(t - \tau))$, Lemma 3.1 and Cauchy Schwartz inequality we get

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_0^{2\pi} -\dot{\tilde{x}}(t) \{ a^{-1}[x^{(iv)} + \lambda f(\dot{x})\ddot{x}] + \ddot{x} + [(1 - \lambda)\Gamma(t) \\ &\quad + \lambda\tilde{Y}(t, \dot{x}(t - \tau))]\dot{x}(t - \tau) + (1 - \lambda)a^{-1} dx + \lambda a^{-1}g_0(t, \dot{x}(t - \tau) \\ &\quad + a^{-1}\lambda h(x) - a^{-1}\lambda p(t)) \} dt \\ &\geq \delta/2|\ddot{x}|_2^2 - (|\alpha|_2 + |p|_2)|\dot{x}|_2 \end{aligned}$$

Using Wirtinger’s inequality gives

$$0 \geq \frac{\delta}{2}|\ddot{x}|_2^2 - \beta|\dot{x}|_2^2, \quad \text{for some } \beta > 0$$

Hence

$$|\ddot{x}|_2^2 \leq \frac{2\beta}{\delta} = \beta_1, \quad \beta_1 > 0 \tag{3.11}$$

Thus,

$$|\dot{x}|_2 \leq \beta_2, \quad \beta_2 > 0$$

and inequality (3.11) implies that

$$|\ddot{x}|_\infty \leq \beta_3, \quad \beta_3 > 0$$

By the continuity of f , we derive

$$|f(\dot{x})|_\infty \leq \beta_4, \quad \beta_4 > 0.$$

Taking the average of equation (3.10) on $[0, 2\pi]$ we get by the mean value theorem

$$\begin{aligned} |(1 - \lambda)dx(t^*) + \lambda h(x(t^*))| &= \left| (1 - \lambda)d\frac{1}{2\pi} \int_0^{2\pi} x(t)dt + \lambda\frac{1}{2\pi} \int_0^{2\pi} h(x(t))dt \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |(1 - \lambda)\Gamma(t) + \lambda\tilde{Y}(t, \dot{x}(t - \tau))||\dot{x}(t - \tau)|dt \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \lambda|g_0(t, \dot{x}(t - \tau))|dt + \frac{1}{2\pi}\lambda \int_0^{2\pi} |p(t)| dt \\ &\leq \beta_3(1 + \delta/2) + |\alpha|_1 + |p|_1 = \beta_5 \end{aligned}$$

for some $t^* \in [0, 2\pi]$. By (iii) we have for $k > 0$ there is $q = q_k > 0$ such that

$$|h(x)| = Sgn(x)h(x) > 0 \quad \text{for every } |x| > \max\left\{\frac{k}{d}, q\right\}$$

Hence for any $\lambda \in [0, 1]$ we have

$$|(1 - \lambda)dx + \lambda h(x)| = Sgn(x)(1 - \lambda)dx + \lambda h(x) \geq (1 - \lambda)k + \lambda k = k$$

for every $|x| > \max\{\frac{k}{d}, q\}$. We now choose $k > \beta_5$ and derive that

$$|x(t^*)| \leq \max\left\{\frac{k}{d}, q\right\} = \beta_6$$

Now,

$$x(t) = x(t^*) + \int_{t^*}^{2\pi} \dot{x}(t) dt$$

Hence

$$|x|_\infty \leq \beta_7 + 2\pi\beta_3 = \beta_7$$

multiplying (3.10) by $-\ddot{x}(t)$ and integrating over $[0, 2\pi]$ we obtain

$$\begin{aligned} |\ddot{x}|_2^2 \leq & |f(\dot{x})|_\infty |\ddot{x}|_2^2 + |a^{-1}|\delta/2 + 1||\dot{x}|_2 |\ddot{x}|_2 + |a|^{-1}|\alpha|_2 |\ddot{x}|_2 \\ & + |p|_2 |\ddot{x}|_2 + |h(x)|_\infty |\ddot{x}|_2 + |d||\dot{x}|_2^2. \end{aligned}$$

Thus

$$|\ddot{x}|_2 \leq \beta_8, \quad \beta_8 > 0$$

and

$$|\ddot{x}|_\infty \leq \beta_9, \quad \beta_9 > 0$$

Multiplying (3.10) by $x^{(iv)}(t)$ and integrating over $[0, 2\pi]$ we get

$$\begin{aligned} |x^{(iv)}|_2^2 \leq & |f(\dot{x})|_\infty |\dot{x}|_2 |x^{(iv)}|_2 + |a|^{-1}|1 + \delta/2||\dot{x}|_2 |x^{(iv)}|_2 \\ & + |d||x|_2 |x^{(iv)}|_2 + |\alpha|_2 |x^{(iv)}|_2 |a|^{-1} + |h(x)|_\infty |x^{(iv)}|_2 + |p|_2 |x^{(iv)}|_2 \end{aligned}$$

Therefore

$$|x^{(iv)}|_2 \leq \beta_{10}, \quad \beta_{10} > 0$$

and thus

$$|\ddot{x}|_\infty \leq \beta_{11}, \quad \beta_{11} > 0$$

Therefore

$$\begin{aligned} |x|_{C^3} &= |x|_\infty + |\dot{x}|_\infty + |\ddot{x}|_\infty + |x^{(iv)}|_\infty \\ &\leq \beta_7 + \beta_3 + \beta_9 + \beta_{11} = \beta_{12} \end{aligned}$$

Choosing $\rho > \beta_{12} > 0$ we obtain the required a priori bound in $C^3[0, 2\pi]$ independently of x and λ .

4. Uniqueness Result

If in (1.1), $f(\dot{x}) = b$, $h(x) = d$ where b and d are constants, then we have the following uniqueness result.

Theorem 4.1. *Let a, b, d be constants, with $a \neq 0$, $d > 0$. Suppose g is a Caratheodory function satisfying*

$$0 < \frac{g(t, \dot{x}_1) - g(t, \dot{x}_2)}{a(\dot{x}_1 - \dot{x}_2)} \leq \Gamma(t) \tag{4.1}$$

for all $\dot{x}_1, \dot{x}_2 \in \mathbb{R}$, $\dot{x}_1 \neq \dot{x}_2$, where $\Gamma(t) \in L^2_{2\pi}$ is such that

$$0 < \Gamma(t) < 1$$

then for all arbitrary constant b and every $\tau \in [0, 2\pi)$ the boundary value problem

$$x^{(iv)} + a \ddot{x} + b\ddot{x} + g(t, \dot{x}(t - \tau))t dt = p(t) \tag{4.2}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{4.3}$$

has at most one solution.

Proof. Let x_1, x_2 be any two solutions of (4.2) - (4.3). Set $x = x_1 - x_2$. Then x satisfies the boundary value problem

$$a^{-1}x^{(iv)} + \ddot{x} + a^{-1}b\ddot{x} + \Gamma(t)\dot{x}(t - \tau) + a^{-1}dx = 0$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3$$

where the function $\Gamma(t) \in L^2_{2\pi}$ is defined by

$$\Gamma(t) = \begin{cases} \frac{g(t, \dot{x}_1(t - \tau)) - g(t, \dot{x}_2(t - \tau))}{\dot{x}(t)} & \text{if } \dot{x}(t) \neq 0 \\ \frac{1}{2} & \text{if } \dot{x}(t) = 0 \end{cases}$$

If $\dot{x}(t) = 0$ on every subset of $[0, 2\pi]$ of positive measure, then $x = \text{constant} = 0$ since $d \neq 0$. Hence $x_1 = x_2$. Suppose on the other hand that $\dot{x}(t) \neq 0$ on a certain subset of $[0, 2\pi]$ of positive measure. Then using the argument of theorem 2.1 we obtain that $x = 0$ and hence $x_1 = x_2$ a.e.

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