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MEMORY EFFECT ON LEARNING-BY-DOING EXTERNALITIES PROCESS GENERATES ENDOGENOUS FLUCTUATIONS ON BUSINESS CYCLE

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Abstract: With a generalized version of the endogenous growth model by Romer in [20] it is analysed the dynamical characteristics of the effects of the learning-by-doing (LBD) externalities on the memory-dependent production process. It is regarded as an quasi-homogeneous production function whose factors of substitutability are non-constant. It is assumed that the consumer maximizes the utility of consumption according to a constant relative risk-aversion function. The functional forms for the optimal capital formation trajectory and externalities with two-delayed arguments are solutions obtained by optimality principle. The optimal problem admits a steady state. Taking consumption elasticity as a function of one of the delays, we observe economic fluctuations which can be attenuated by the actions of both the delay effect and damping. But, there is a critical value for consumption elasticity at which economic fluctuations become unstable.

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1. Introduction

The sharing of benefits that are generated by technological knowledge involves

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a variety of nonlinear flows that govern the actions of economic agents and consumers of technological innovation [1]. Among others, the authors in [2], [8] and [24], have emphasized the importance of the efforts that have been made to describe dynamical characteristics of the process of capital formation which include stable, unstable and oscillatory behaviours. In [17], Mulligan and Sala-i-Martin examined the dynamical properties that are generated by a class of nonlinear models of economies with two goods: the former what is produced, i.e, physical capital and the latter named by authors externalities, which varies across a range of variable stocks accumulated through an investment process which includes anothers human capital effort by means of education . Such formulation amounts to linking the externalities to the output per capita, which is the natural assumption in LBD [2].

In [5] the authors make it clear that dynamical characteristics are induced by effects in relationships among innovation, capital investment, and other costly activities that might, over time, either enable or restrict the productive activity of firm [3] [4]. Intuitively, if we begin at the initial low level of experience, private capital returns and investment will be high. This increases the level of experience and decreases the private capital returns. Investments are slowed down. By using intertemporal consumption utility and informational externalities to control undesirable behaviour, the level of experience is reduced and this creates new private capital returns (see (16)).

This phenomenon is complex because it has a mechanism for generating cascade-type behaviour which relies on exogenous information flow, either of strategic complementarities or strategic substitutabilities [5]. Utility function allows us to analyse possible feedback effects of consumption and economic growth generated by externalities, which are characterized in the memory of the

[1] The arguments presented by Arrow in [1] and Romer in [20] show how it is possible that technological knowledge and perpetual growth can be generated by learning by doing and knowledge diffusion.

[2] In [24], the author makes it clear that LBD requires repetition of activities and specialization that is conducive to repetition. This is consistent with Arrow‘s argument [1], who stresses that LBD is an important factor to determine the level of stock knowledge of workers in an economy.

[3] When production level is low, firms wait to produce until they have learned about other firms investments.

[4] As we will see later, delay phenomena also influence the dynamic characteristic of capital-formation process and they will be useful to explain persistent oscillations around a steady state

[5] Since investment involves creation and destruction as separate activities, those oscillations are the result of subsequent replacement

process of formation of both stock of knowledge and physical capital[6]. Memory-dependence is built by Boucekkine and la Croix in [11] from an optmization process of the schooling and pension time and change-rate of aggegate stock of human capital is determined by micro-fundations. Memory-dependence may be not only due to regulation, but also to the inertia of institutional, technical systems including the deployment of research and development results as the economic agents compete for the best place in the decision-making queue [7] (see [5]). We show that, under special conditions, the dynamics of the mechanism of regulation and capital formation admits one steady state [8] and close to it we able to observe fluctuations (that may be damped) caused by lagged LBD effects [9]. The elasticity intertemporal substutition of consumption has lower and higher bound and between them there a critical value. If it is less than the critical value we have stability and it is possible to control undesirable behaviour, but if it assume the critical value the it occurs persistent oscillations.

In [7], the authors have proposed a complex, but realistic model considering that there is a lag period from the beginning of development of human capital to full capacity utilization in the production process. They have proved the existence of an equilibrium and have described the dynamic behaviour close to steady state of the LBD-effects on the production process with inelastic labor and their findings have shown that a slight memory effect is enough to generate business-cycle fluctuations (see also [25]).

Throughout this paper we substantially follow the ideas of the authors in [7] and analyse a benchmark two-delayed model with a standard-preference function. We regarded an almost-homogeneous production function whose factors of substitutability are non-constant[10]. Unfortunately, the analysis is not easy, since the optimality problem must be formulated in an infinite-dimensional Ba-

[6]A theoretic and accurate argumentation on learning with several delay patterns is presented in [23].

[7]The effective formation of the amount of human and physical capital employed in the final output relies on the memory of the production process. The fundamental reasons for delay in adopting new mechanism of acceleration of physycal capital formation are associated to the inertia due to past commitments that generate gestation periods for both, physical capital and human capital and so, a time is required to arrange finance.

[8]By allowing that formation-capital has a smooth damping scheme, we are able to show that, close to the steady state, the persistent oscillations in investment can occur with concave utility.

[9]The effects of the damping oscillations in capital, which are induced by the delays in production, can trigger the same dynamics on consumption.

[10]Our approach admits that flows of externalities can differ from the ones regarded in [7], here, a benchmark model can also be applied to examine externalities like as environment pollution.

nach space (see [13]). The presence of delays parameters greatly complicates the task of an analytical study of the dynamics such models. To describe a local stability such as fluctuations of capital-formation system it is used Poincaré-Andronov-Hopf theorem which predicts a bifurcation to limit cycle (see [10], [15] and [18]).

We are able to show how a slight memory effect with delayed damping on capital formation is enough to generate business-cycle fluctuations that can be either controlled or not [11].

2. Production Function

Let $\alpha, \beta, \nu \in R$, $\nu > -1$, $0 < \alpha, \beta < 1$, $\Omega = \{(k, \varrho) \in R^2, k \geq 0, \text{ and } \varrho \geq 0\}$, $\zeta : \Omega \to R$ a non-negative C^2-function and $\varphi : [0, \infty) \to R$, be positive C^2-function fulfilling

$$\begin{aligned} &(a)\ \lim_{k\to 0} \varphi'(k) \in R,\ \lim_{k\to\infty} \varphi'(k) \geq 0. \\ &(b)\ k\varphi'(k) < \nu \min\{1, 1-\beta, \alpha(1-\alpha)^{-1}\}\varphi(k). \\ &(c)\ \varphi'(k) > 0 \ \text{ and } \varphi''(k) < 0. \end{aligned} \tag{1}$$

We define $s : \Omega \to R$ by

$$s(k, \varrho) =: \frac{\alpha\varphi(k)}{\alpha\varphi(k) + (1-\alpha)\zeta(k, \varrho)} \text{ and } \mathcal{E}_\varrho = \frac{\varrho\partial_\varrho s(k, \varrho)}{s(k, \varrho)}. \tag{2}$$

Since φ and ζ are non-negative functions, $0 < s(k, \varrho) < 1$. Observe that $\mathcal{E}_\varrho$ is negative. It follows from (1b) that if ϱ is fixed, $k \to s(\cdot, k)$ is a decreasing function.

Production employs capital (K), labor (L) and earning power depends on the amount and composition of work and education experience (E) according to the technology production function $Y = F(K, \mathcal{A}L, E)$, the constant $\mathcal{A}$ represents the increase rate of labor productivity through productivity function. For the sake of logical completeness and mathematical requirements, the production function represents technological alternatives in production, showing the maximal output Y obtainable from any given combination of labor L and capital K, and E externalities.

$$F(0, 0, 0) = 0, F \in C^2 - \text{class}. \tag{3}$$

[11] In fact we have prove that instability, non-linearity and delays are significant sources for of cyclic behaviour birth.

where $Y, L, K \geq 0$ are production factors. Each factor may or may not be essential for positive output ($F(L,0,0) = F(0,K,0) = 0$, or $F(L,0,0) > 0, F(0,K,0) > 0$).

Denoting $L \neq 0$, $k = \dfrac{K}{\mathcal{A}L}$ as capital stock per labor unit and $\varrho = \dfrac{E}{\mathcal{A}L}$ as per capta accumulated amount of human capital due to education, we consider a generalized production function in intensive form given by

$$f(k,\varrho) = Ak\big[s(k,\varrho)(\alpha\varphi(k))^{-1}\big]^{\frac{1}{\nu}}, \tag{4}$$

where, $\nu \neq 0$ and $A > 0$ (see [7] and [19]).

Remark 1. *If in (4), ζ is the null function and $\varphi(k) = k$, then $f(k,\varrho) = f(k) = A\alpha^{-\frac{1}{\nu}}k^{(1-\frac{1}{\nu})}$.*

- *We can verify that f is convex and increasing if $-1 < \nu < 0$ and is convex and decreasing if $0 < \nu < 1$.*
- *If $\nu > 1$ f is a familiar concave-increasing production function.*
- *If $\zeta(k,\varrho) = \varrho$ and $\varphi(k) = k$, the production function f becomes familiar from the context of a production technology with two inputs, however, with the form (4) it is quite non-standard.*

The parameter ν is a measure of the elasticity of substitution of labor by shock. It is measure of how much labor we can substitute in case of a bad shock.

Propositon 1. *If ζ satify*

$$k\frac{\partial\zeta(k,\varrho)}{\partial k} = \nu\zeta(k,\varrho) \;\text{ and }\; \varrho\frac{\partial\zeta(k,\varrho)}{\partial\varrho} = -\beta\nu\zeta(k,\varrho), \tag{5}$$

ζ is quasi-homogeneous function with weights $(1,\sigma)$ and degree $\nu(1-\sigma\beta)$. Moreover, if φ is homogeneous of degree $\nu(1-\sigma\beta)$, f is quasi-homogeneous with weights $(1,\sigma)$ and degree $\sigma\beta$.

Proof. Setting $\xi(\tau) = \zeta(\tau k, \tau^\sigma\varrho)$, we have

$$\xi'(\tau) = \tau^{-1}\Big[\tau k\frac{\partial\zeta(\tau k,\tau^\sigma\varrho)}{\partial k} + \sigma\tau^\sigma\varrho\frac{\partial\zeta(\tau k,\tau^\sigma\varrho)}{\partial\varrho}\Big]. \tag{6}$$

Now, if we put $(\tau k, \tau^\sigma\varrho) = (u,v)$, we can see that $\tau k\frac{\partial\zeta(\tau k,\tau^\sigma\varrho)}{\partial k} = u\frac{\partial\zeta(u,v)}{\partial u} = \nu\zeta(u,v)$ and $\tau k\frac{\partial\zeta(\tau k,\tau^\sigma\varrho)}{\partial\varrho} = v\frac{\partial\zeta(u,v)}{\partial v} = -\beta\nu\zeta(u,v)$. Then,

$$\nu(1-\sigma\beta)\tau^{-1}\zeta(\tau k,\tau^\sigma\varrho) = \nu(1-\sigma\beta)\tau^{-1}\xi(\tau).$$

Since $\xi(1) = \zeta(k, \varrho)$, $\xi(\tau) = \tau^{\nu(1-\sigma\beta)}\zeta(k, \varrho)$ if follows of the Theorem 1 pp.260 in [14] that ζ is quasi-homogeneous function with weights $(1, \sigma)$ and degree $\nu(1 - \sigma\beta)$.

The condition (1d) implies that f is quasi-homogeneous with weights $(1, \sigma)$ and degree $\sigma\beta$.

At the optimum of a firm, the interest rate and the wage rate are given by

$$\begin{aligned} \bar{r}(t) &= \partial_k f(k(t), \varrho(t)) - \delta \text{ and} \\ \varpi(t) &= f(k(t), \varrho(t)) - k(t)\partial_k f(k(t), \varrho(t)). \end{aligned} \tag{7}$$

being $\delta_1 \geq 0$ depreciation rate of capital, and $\partial_x f$ indicates a partial derivative. The share of capital in total income is given by

$$\varepsilon_k = \varepsilon_k(k, \varrho) = \frac{k\partial_k f(k, \varrho)}{f(k, \varrho)} = \nu^{-1} s(k, \varrho)\Big(\nu - \frac{k\varphi'(k)}{\varphi(k)}\Big), \text{(see [19])} \tag{8}$$

the following share and elasticity are related to the externalities,

$$\varepsilon_\varrho = \varepsilon_\varrho(k, \varrho) = \frac{\varrho\partial_\varrho f(k, \varrho)}{f(k, \varrho)} = \beta(1 - s(k, \varrho)) \in (0, 1), \text{(see [19])} \tag{9}$$

which is the measure of the size of externalities. It follows from (1b) that the output elasticity of capital ε_k is positive, but non-constant. It is important to point out that only the family of Cobb-Douglas functions exhibits constant elasticities of the marginal productivities.

From (1) and (2), respectively, follow that both marginal products $\partial_k f(k, \varrho)$ and $\partial_\varrho f(k, \varrho)$ are always positive, but since

$$\begin{aligned} &(a)\ \partial_{kk} f(k, \varrho) = k^{-2} f(k, \varrho)[\varepsilon_k^2 - s(k, \varrho)], \\ &(b)\ \partial_{ee} f(k, \varrho) = e^{-2} f(k, \varrho)[\varepsilon_\varrho^2 - \varepsilon_\varrho - (\beta - \varepsilon_\varrho)\mathcal{E}_\varrho] \text{ and} \\ &(c)\ \partial_{k\varrho} f(k, \varrho) = (k\varrho)^{-1} f(k, \varrho)(\nu + 1)\varepsilon_k\varepsilon_\varrho, \end{aligned} \tag{10}$$

$\partial_k f(k, \varrho)$ and $\partial_\varrho f(k, \varrho)$ decrease as a single factor exhibiting diminishing returns depending on, if $\varepsilon_k^2 - s(k, \varrho)$ and $\varepsilon_\varrho^2 - \varepsilon_\varrho - (\beta - \varepsilon_\varrho)\mathcal{E}_\varrho$ being negative, respectively ($\mathcal{E}_\varrho$ is defined in (2)). Since $\nu > -1$, from (10c) we can see that an increase of the quantities of one of the factors k or ϱ will increase the marginal productivity of the other factor which is the interacting productivity property of factors known as complementaries.

If the Hessian determinant

$$D = \partial_{kk} f(k, \varrho)\partial_{\varrho\varrho} f(k, \varrho) - (\partial_{k\varrho} f(k, \varrho))^2 \tag{11}$$

is negative, then f is neither concave nor convex. Hence, it is required $D > 0$, if it is necessary.

The elasticity of substitution can be variable depending upon relationship between factor combination (see [21]). Because of (4), the rate of technical substitution which is provided by marginal transformation ratio between the externalities factors and capital is given by

$$\mathrm{MRS}_{\varrho k} = \mathrm{MRS}_{\varrho k}(k,\varrho) = \frac{\varepsilon_k}{\varepsilon_\varrho} = (\nu\beta)^{-1}\Big[\frac{s(k,\varrho)}{1-s(k,\varrho)}\Big]\Big[\nu - \frac{k\varphi'(k)}{\varphi(k)}\Big]. \tag{12}$$

In general, we assume that $\mathrm{MRS}_{\varrho k}$ obeys the law of diminishing marginal substitution rate, which in turn is commonly considered equivalent to the classic diminishing marginal returns, however, such equivalence does not hold, not even for homogeneous production functions[12]. Keeping with our framework, $\mathrm{MRS}_{\varrho k}$ is the proportional change in the factor ratio ϱ by k resulting from the proportional change in marginal substitution rate of ϱ by k (see [23] and [16]). On the level curves of $s(k,\varrho)$, $\mathrm{MRS}_{\varrho k}$ is independent of the effects externalities. It follows from (1b) that $\partial_k s$ is negative, then if $\varphi(k)\varphi''(k) < (\varphi(k))^2(1-\nu^{-1})$, $\mathrm{MRS}_{\varrho k}$ decreases monotonically as we move to right along an isoquant of (4) and increases monotonically as we move to left.

Since $\varepsilon = \varepsilon_k + \varepsilon_\varrho = \beta + s(k,\varrho)[1 - \frac{k\varphi'(k)}{\nu\varphi(k)} - \beta]$, the output elasticity with respect to scale variation is k,ϱ dependent and using (1b) we see that $\varepsilon > 0$. If $\beta = 0$, the effects of the externalities are completely absent in production process and $\varepsilon = \varepsilon_k$. Setting elasticity of capital-labor substitution as $\sigma_0(1+\nu) = 1$, the elasticity of the rental rate of capital with respect to ϱ is given by

$$\varepsilon_{k\varrho} = \varepsilon_{k\varrho}(k,\varrho) = \frac{\varrho\partial_{k\varrho}f(k,\varrho)}{\partial_k f(k,\varrho)} = \frac{\beta(1-s(k,\varrho))}{\sigma_0} \quad \text{(see [19])}. \tag{13}$$

It is worth noting that $\varepsilon_{k\varrho} = (1+\nu)\varepsilon_\varrho$, yet the condition $0 < \beta < 1$ implies that the externalities can be controlled. Since the marginal productivity is an increasing function of the externalities and the capital decreases monotonically with respect to rental rate, (13) shows how to choose the externalities small enough to be compatible with demand for capital. Still, $\nu > 0$ implies $0 < \beta < 1$ which ensures that over business cycle, the capital share is pro-cyclical, on the other hand labor share is countercyclical. Since

[12] $\mathrm{MRS}_{\varrho k}$ indicates how the externalities are transformed into capital assets, sice inelastically supplied by both, factors and production efficiency. In the top-level condition, $\mathrm{MRS}_{\varrho k}$ depicts how the economic agents desire to share the benefits that are generated by technological knowledge.

$$\frac{\varrho \partial_{k\varrho} f(k,\varrho) f(k,\varrho)}{\partial_k f(k,\varrho) \partial_\varrho f(k,\varrho)} = \sigma_0,$$

the production function (4) has the property that the elasticity of the marginal product with respect to output in response to effects of externalities is constant. The value σ_0 lies in interval $(0;\infty)$, which indicates a production of technology allowing for some but non perfect substitutability among the capital factors accumulation and externalities.

3. Optimization Problem

Denoting the consumption per capta by $\mathcal{C}(t)$ and assuming zero growth of the population, the dynamics of the capital per capta can be described by the differential equation

$$k' = f(k(t), \varrho(t)) - k\delta - \mathcal{C}(t) \tag{14}$$

We assume it is possible to find a standard approach to express the relationships among LBD effects caused by productivity, cumulative investment and stock of knowledge. It is well known that the rate of externalities LBD can be proxy of the size of the per-capta capital stock and the level of productivity achieved by a firm or industry depends not only on its own research efforts but also on level of the pool of general knowledge accessible to it[13]. Since investment is function of the production and the replacement decisions are dependent on previous quantities of production, the factors of casuality that are determined by decisions made in the past must be considered. Boucekkine and la Croix in [11] examine the effect of past and future demographic trends on the formation of human and physical capital. They introduce a benchmark model with two delays to analyse the dynamics of effects of the changes of specific demographic parameters on human capital accumulation and economic growth. For instance,

[13]In [5] the authors examine how the rate of investment in human/physical capital and the information flow affects the economic activity. They have argued that at the decision-makers act instantaneously but their actions are implemented force of the forecasting future technological innovations and profit opportunities. Therefore, decision-makers should consider that aggregated productivity that causes economic growth through informational externalities is a delayed-subproduct of a high-investment activity with at least one lag period. Time delays arise as a result of the time processing of information.

gestation period of capital [14], the inertia due to past commitments [15], and the time required to arrange finance [16] (see [4],[8], [9], [17] and [20]). Our benchmark model describes the LBD effects on production process which allows for the possibility of externalities becoming the average stocks of capital and takes into account the memory effect acting on acceleration principle induced by investment and by capital formation due to earnings with education. Assume that the per-capta cumulative gross investment creates dynamic externalities given by

$$\varrho'(t) = \xi(\varrho(t)) - g(k(t), k'(t), k'(t-\tau), k(t-\eta)) \tag{15}$$

where $\xi, g \in C^1$ and there is ϱ_0 so that $\xi(\varrho_0) = A\varrho_0$ [17].

Now, we take into account utility from consumption $\mathcal{C}$ according to the well-known constant relative risk-aversion function such as

$$u(\mathcal{C}) = \frac{\mathcal{C}^{1-\frac{1}{\varepsilon_{\mathcal{C}}}}}{1-\frac{1}{\varepsilon_{\mathcal{C}}}}, \tag{16}$$

with elasticity of intertemporal substitution with consumption at $\varepsilon_{\mathcal{C}} > 0$.

Each representative agent controls the capital accumulation k and considers the evolution of externalities ϱ as exogenous because each of his decision has a negligible impact on the evolution of externalities. We Consider the standard formulation so that the representative agent solves at $t_0 = 0$ the following intertemporal maximization problem:

$$\begin{aligned} &\max_{\mathcal{C}(t),\ u(t)} \{\textstyle\int_0^{+\infty} e^{-\rho t} u(\mathcal{C}(t))dt, \text{ s.t. } k'(t) = f(k(t), \varrho(t)) - \delta_1 k(t) - \mathcal{C}(t) \\ &k(t_0) = k_0 \text{ and } \{\varrho(t)\}_{t\geq 0} \text{ given}\}, \end{aligned} \tag{17}$$

where $\rho > 0$ is the discount factor. From (14), we obtain, by replacing $\mathcal{C}(t)$, the problem of calculus of variation

[14] The actions of growth mechanisms of technological innovation depend on the memory of invention of new production processes and goods.

[15] Investors will require information that are in memory of the capital-formation process because they will be reluctant to invest in activities about which there is little reliable information.

[16] Fundraising Forecast (human/financial) affects capital accumulation either by high cost of such resources or by reallocating them strategically to satisfy of needs of the different capital-producing technologies and depends on the memory of capital formation.

[17] In [7], ξ and g are such that $\varrho(t) = k(t)$, in [6], ξ is null and $g(k(t), k'(t), k'(t-\tau), k(t-\eta)) = -k'(t) + k'(t-\tau) - k(t) + k(t-\tau)$.

$$\max_{k(t)}\{\int_0^{+\infty} e^{-\rho t}u(f(k(t),\varrho(t))-\delta_1 k(t))-k'(t))dt, \quad \text{s.t.}$$
$$(k'(t),k(t))\in\mathcal{S}(\{\varrho(t)\}_{t\geq 0}) \;\; k(t_0)=k_0 \text{ and } \{\varrho(t)\}_{t\geq 0} \;\text{ given}\}, \tag{18}$$

where

$$\mathcal{S}=\mathcal{S}(\{\varrho(t)\}_{t\geq 0})=\{(k(t);k'(t))\in\mathbf{R}\times\mathbf{R} \text{ so that}, k(t)>0,$$
$$f(k(t),\varrho(t))-k(t)\delta_1-k'(t)\geq 0.\}$$

It is easy to see that the set of admissible paths $\mathcal{S}$ is convex. If

$$\lim_{t\to\infty} u'(\mathcal{C}(t))k(t)e^{(-\rho t)}=0, \text{(transversality condition)} \tag{19}$$

holds for all $\varrho(t)\in\mathcal{S}$, then the maximal principle yields

$$\{[\partial_k f(k(t),\varrho(t))-\delta_1]k'(t)+\partial_\varrho f(k(t),\varrho(t))\varrho'(t)-k''(t)\}u''(\mathcal{C}(t))$$
$$+[\partial_k f(k(t),\varrho(t))-\delta_1-\rho]u'(\mathcal{C}(t))=0. \tag{20}$$

that is a necessary condition for the existence of an interior solution to $\mathcal{S}$ with transversability condition (see [22]). The optimization problem (15-17) lead us to study the functional differential equation (18-19) that is a scalar of second order with delayed derivative terms. In the section 4 of [11] the authors address similar problem and assess the asymptotic behaviour of the solutions of the problem (18-19) using, a numerical algorithm for numerical stability assessment for this kind of equations. Our findings allow us to assess directly the asymptotic behaviour of the solutions by means of Hopf-bifurcation theory. It is important to observe that the memory-dependent optimization problem (18-19) can not be transformed into a memory-independent standard problem in sense of [12].

4. Stability

Under condition (19), by general arguments on the existence and uniqueness of solution for a fixed-point problem generated above (see [3]), it can be verified that, at individual level (15) defines a solution of the Euler equation (20) that is in fact a one-parameter family of paths of stock capital parametrized by a given path of externalities $k(t,\{\varrho(s)\}_{s\geq 0})$, solving the corresponding fixed-point problem. In order to avoid the appearance of excessive computation due to application of derivative rules in the face of unfamiliar technical manipulations, in what follows, we will assume that

Assumption 1. *Assume*

(a) σ *and* β *satisfy either* $\sigma(1+\beta)<1$ *or* $\sigma\beta>1$. *Still,* $0<\alpha<\dfrac{1-\sigma\beta}{1-(\beta+1)\sigma}$, $0<\delta<A\nu^{-\frac{1+\nu}{\nu}}$, *and* $\nu^{\nu}=\alpha\sigma+(1-\alpha)(1-\beta\sigma)$.

(b) *If* $P_0=(k_0,0,0,k_0)$, $k_0>0$, *there is* k_0 *so that* $g(P_0)=\nu^{\frac{\nu+1}{\nu}}k_0$.

(c) *If* ϱ_0 *is defined in (15),* $\xi(\varrho_0)=g(P_0)$, $\varphi(k_0)=\nu\sigma$, $\zeta(k_0,\varrho_0)=\nu(1-\sigma\beta)$, $\mathcal{C}_0=f(k_0,\varrho_0)-\delta k_0$, $\partial_x f(k_0,\varrho_0)=\delta+\rho$,

(d) $s_0=s(k_0,\varrho_0)=\nu^{-\nu}\alpha\sigma<1$, $k_0\varphi'(k_0)=\dfrac{\nu}{\alpha A}[A\nu\alpha\sigma-(\rho+\delta)\nu^{\frac{(1+\nu)^2}{\nu}}]$,

(e) $\varepsilon_{\mathcal{C}}\delta k_0\partial_{k\varrho}f(k_0,\varrho_0)=\xi'(\varrho_0)\partial_\varrho f(k_0,\varrho_0))$.

If σ and β satisfy either $\sigma(1+\beta)<1$ or $\sigma\beta>1$, then we can choose α as a parameter on the interval $0<\alpha<\dfrac{1-\sigma\beta}{1-(\beta+1)\sigma}$ and check that the real number $\nu^{\nu}=\alpha\sigma+(1-\alpha)(1-\beta\sigma)$ which is combination convex of the real numbers $1-\sigma\beta$ and σ, is also positive. If $0<\delta<A\nu^{-\frac{1+\nu}{\nu}}$, $\mathcal{C}_0=\Big(A\nu^{-\frac{1+\nu}{\nu}}-\delta\Big)k_0$ is positive.

It follows from (1) that $\xi(\varrho_0)=(\nu\ln\nu+e^{-1})k_0$ defines an equilibrium point (k_0,ϱ_0) for the delayed non-linear system 20. Since ν^{ν} so is defined implies that $f(k_0,\varrho_0)=\dfrac{A}{\nu^{\frac{\nu+1}{\nu}}}k_0$. By using (8): $\varrho_0\partial_{\varrho_0}f(k_0,\varrho_0)=\dfrac{A\beta(\nu-\alpha\sigma)}{\nu^{\frac{2\nu+1}{\nu}}}k_0$. Becuse to $A\epsilon_k=(\rho+\delta)\nu^{\frac{\nu+1}{\nu}}$ along with $\nu\epsilon_{\varrho}=\beta[\nu-\alpha\sigma]$, the ratio of the marginal capital and externalities at the point (k_0,ϱ_0), which is a traditional concept and measures the of rate technical substitution of F defined in (3), is given by

$$\mathrm{MRS}_{\varrho k}=\frac{(\rho+\delta)\nu^{\frac{2\nu+1}{\nu}}}{A\beta(\nu-\alpha\sigma)}.$$

Since $\nu-\alpha\sigma\to 0^+$ if $\nu\to\alpha\sigma^+$, the range of $\mathrm{MRS}_{\varrho k}$ may be finite or infinite. It follows from (7) that in steady state (k_0,ϱ_0) the interest rate is equal to discount rate and wage rate

$$\varpi=\Big(\frac{A}{\nu^{\frac{\nu+1}{\nu}}}-(\delta+\rho)\Big)k_0=\frac{A}{\nu^{\frac{\nu+1}{\nu}}}(1-\varepsilon_k)k_0,$$

By using the change of coordinates $u=k-k_0$, $v=\varrho-\varrho_0$, it follows from (1e) and (15) and Taylor's theorem that, along this steady state local dynamics of the delayed non-linear equation 20 is described by

$$u''(t)-\rho u'(t)+\varepsilon_{\mathcal{C}}(\partial_{kk}f(k_0,\varrho_0)[f(k_0,\varrho_0)-\delta k_0]u(t)=\mathcal{B}(u), \tag{21}$$

where

$$\mathcal{B}(u) = \partial_{x_1} g(P_0)\partial_\varrho f(k_0, \varrho_0)u(t) + \partial_{x_2} g(P_0)u'(t) + bu'(t-\tau) + au(t-\eta),$$

Equation (21) is a version of the van der Pol equation and the damping term in (21) $\rho u'(t)$ can be viewed as representing a linear damping . The effects of this damping term depends on the sign of u'. If on the one hand under special conditions, $\rho u'(t)$ tends to drive the fluctuations of the business cycle to ones of greater amplitudes, on the other hand it tends to damp them. The right hand in (21) is active when $\mathcal{B}(u) \neq 0$ and so, by action of the externalities which are induced by the flow capital-formation (15), the fluctuations of capital formation process could be either attenuated or excited[18].

Now, we define a, b, c and d as

$$\begin{aligned} &a = \partial_{x_4} g(P_0)\partial_\varrho f(k_0, \varrho_0),\ b = \partial_{x_3} g(P_0)\partial_\varrho f(k_0, \varrho_0),\ d = \rho + \partial_{x_2} g(P_0), \\ &c = \varepsilon_{\mathcal{C}} \partial_{kk} f(k_0, \varrho_0)[f(k_0, \varrho_0) - \delta k_0] - \partial_{x_1} g(P_0)\partial_\varrho f(k_0, \varrho_0). \end{aligned} \tag{22}$$

The dynamics of (21) is given by the equation

$$u''(t) - du'(t) + cu(t) = bu'(t-\tau) + au(t-\eta). \tag{23}$$

Remark 2. *For instance, if $\sigma = \eta = 0$, $d + b < 0$ and $\Delta = (d+b)^2 - 4(c - a) < 0$ we can check that the equilibrium point (k_0, ϱ_0) of (20) is asymptotically stable, but with fluctuations. But if $d + b > 0$, (k_0, ϱ_0) is instable.*

When the capital formation does not depend on memory ($\sigma = \eta = 0$) the standard optimal-growth model with one state variable (18-19) close to steady state, is stable, but oscillates. The oscillations are damped by the control represented by LBD externalities. From Remark 2, it turns out that the intertemporal elasticity is lower bounded. On the other hand if $d + b > 0$, this term makes oscillations grow indefinitely.

Because we assume τ, η, a, b, c as positive and d as negative, the description of the dynamics of (20) close to (k_0, ϱ_0) becomes rather complicated from the mathematical standpoint.

Now, we follow the study of the authors in [2] and perform a thorough analysis on the stability switches and Hopf bifurcation of the non-linear model (21) when the time delay τ is taken as the bifurcation parameter.

On account of (8), (9), (13) and the last equality in (1) we have that at (k_0, ϱ_0).

[18] A measure of the stability of an oscillating system is its damping factor. The greater the damping factor the stabler the system. To the extent that there is higher damping in the accumulation-capital system described by (20) the more the fluctuations are attenuated.

As it turns out dynamic properties for the equation 20 close to steady state are more complex to be described due to the infinite-dimensional nature of the phase space. One of the fundamental tools we have is the Poincaré-Andronov-Hopf theorem which predicts a bifurcation for limit cycle if the pair real part of a complex conjugate eigenvalues λ (with non-zero real part) changes sign from negative to positive when τ parameter crosses a critical value τ_0 and the derivative of the real part of eigenvalue λ with respect to delay parameter τ is positive when it passes through zero (see [10] and [18]).

The corresponding characteristic equation of the non-linear system (23) is given by

$$H(\lambda) = \lambda^2 - d\lambda - b\lambda e^{-\lambda\tau} - ae^{-\lambda\eta} + c = 0. \tag{24}$$

Let $\lambda = x + iy$ be a solution of equation 24. Separating real and imaginary parts in (24) we obtain the following equations system for x and y

$$\begin{cases} x^2 - (y^2 - c) - dx - be^{-\tau x}[x\cos\tau y + y\sin y\tau] - ae^{-\eta x}\cos y\eta = 0 \\ 2xy - dy + be^{-\tau x}[x\sin y\tau - y\cos y\tau] + ae^{-\eta x}\sin y\eta = 0. \end{cases} \tag{25}$$

The solutions of the system 25 with null real part are solutions of the system

$$\begin{cases} by\sin y\tau + a\cos y\eta = -y^2 + c \\ -by\cos y\tau + a\sin y\eta = dy. \end{cases} \tag{26}$$

Suppose $y \neq 0$ is a solution of (26), then we must have

$$\sin(\tau - \eta)y = \frac{(-y^2 + c)^2 + d^2y^2 - b^2y^2 - a^2}{2aby}. \tag{27}$$

Theorem 1. *If, $\sigma > 0$, $\eta > 0$ and in (24) either $d = 0$ and $c < a$ or $a > 0$ and $c < 0$, the oscillations of (20) close to (k_0, ϱ_0), caused by action of the LBD-effects which are induced by the flow of capital formation can not be controlled.*

Proof Dynamics of the system (20) close to equilibrium solution (k_0, ϱ_0) relies on detailed information about the behaviour of the eigenvalue of the linear equation associated to it, and so this instability problem can be reduced to the fact that one of the roots of the equation (24) has positive real part. If, in (24) either $d = 0$ and $c < a$ or $a > 0$ and $c < 0$, so it is also not true that all roots of the equation (24) have negative real part, once on the real axis $H(0) = c - a < 0$ and $H(\lambda) \to \infty$ when $\lambda \to \infty$, so, there will be unbounded solutions of (21) and we do not uniform ultimate boundedness. This means that the active control $\mathcal{B}(u)$ in (21) causes unbounded oscillations in the equation

(20) and so the fluctuations of capital-formation process, caused by action of the externalities are not controlled (ver (15)).

The next result shows that the lower bound of the intertemporal elasticity (remark 2) is a necessary condition but it is not sufficient to control the oscillations of the capital-formation process under memory-dependent LBD-effects.

Theorem 2. *Let be ℓ natural number and we define* $\bar{x} = \frac{b+\sqrt{b^2+a}}{2}$, $\mathcal{T} = \frac{\bar{x}}{2} + \frac{b}{2} - \frac{d}{2}$ *and* $\Upsilon = \Upsilon(a,c) = \frac{c^2-a^2}{8c^2+a^2}$. *Assume*

$$\begin{array}{ll} \textbf{a)} & 0 < a < c,\ (-d)\sqrt{2} > \min\{b\ ;\ 2c\} \quad b < 2\sqrt{2\Upsilon(d^2-2c)}. \\ \textbf{b)} & 13(-d) + 3\sqrt{d^2+a} < 8,\ a + c < 4(1-T). \end{array} \tag{28}$$

(i) If $0 < \eta < \tau < \dfrac{\pi}{2}$, *then there is a ℓ natural number such as if* $\bar{x} < \ell^{-1}$, *the steady state of (20)* (k_0, ϱ_0) *is asymptotically stable.*

(ii) If $(-d)\sqrt{2} = b$ *and* $\tau_0 = \dfrac{\pi}{2}$, *then* $y_0 = b^{-1}a$ *and* $0 < \eta_0 < \tau_0$ *that solves* $2a^2 \sin(\tau_0 - \eta)y_0 = (c - y_0^2)^2$ *such that* (τ_0, η_0, y_0) *is solution of the system (26), the real part of* $\lambda'(\tau_0)|_{x=0}$ *is positive and the steady state of (20) equilibrium* (k_0, ϱ_0) *becomes unstable and undergoes a cascade of Hopf bifurcations (see (27)).*

Proof. Let be Γ_2 and $\mathcal{S}$ given by

$$\Gamma_2(x,y) = 2xy - dy + be^{-\sigma x}[x \sin \sigma y - y \cos \sigma y] + ae^{-rx} \sin ry = 0$$

$$\mathcal{S} = \{(x,y) \in R^2\ :\ \pi \le 4\sigma y \le 2\pi \text{ and } 0 < x \le y\}.$$

Assume $(x,y) \in S_6$ Since $d < 0$, $\Gamma_2(x,y) \ge ((-d) - \frac{\sqrt{2}}{2}b)y$, which is positive if $(-d)\sqrt{2} > b$. Hence, there is no solution for (26) that belongs to $\mathcal{S}$. Observe that $\mathcal{S} = S_6$ in Proposition 1 in the Appendix of [2]. The conditions (28a) are the same as in Proposition 1 in the Appendix of [2]. The first inequality in (28b) implies that $\mathcal{T} < 1$. The second inequality in (28b) implies that $\bar{y} = \frac{\mathcal{T}+\sqrt{\mathcal{T}^2+c+a}}{2} < 2$. Now we can choose a ℓ natural number such as $\ell^{-1} < \epsilon_0$, where ϵ_0 is defined in the Proposition 1 in the Appendix of [2] and the proof of theorem 2(i) is then complete. In conditions (ii), if we use $by = \theta$, we can see that (27) is equivalent to $2a^2 \sin(\sigma_0 - \eta)b^{-1}\theta = (c - b^{-2}\theta)^2$. By taking $\theta = a$ it turns out that (τ_0, η_0, y_0) solves (27), which is a solution of the System (26). By using implicit derivative in (24) it turns out that $\lambda'(\tau_0)|_{x=0} > 0$ (see details in Appendix of [2]).

5. Concluding Remarks

This paper studies dynamic characteristics of feed-back effects of consumption and economic growth generated by externalities LBD that are characterized in the memory of the process of accumulation of both stock of knowledge and physical capital. It is considered that a delay takes place caused by gestation period of capital and inertia both due to past commitments of investments and others caused by the time required to arranges finances. It is used optimality principle to obtain the existence of stead-state and bifurcation theory to describe the dynamic characteristics of the system close to this steady-state. The implicit theorem determines lower and higher value for the elasticity of intertemporal substitution in consumption and in-between them a critical value that is a bifurcation point. This three values are delay-dependent.

Focusing on the ideas emphasized by authors in [7], it is assumed that the consumer maximizes the utility of consumption according to a constant relative risk-aversion function and it is possible to prove that the memory-effect which acts on acceleration-principle of the induced investment and on the capital formation due to earnings with learning/education, and it can generate either stable or unstable fluctuations Around steady state, if the capital formation oscillates but is stable, a measure of stability can be the ratio relative magnitude of two consecutive oscillations. It can be verified that the greater the damping factor the stabler the system. Here the damping factor $(-d)$ is higher bound for parameters a, b and c (see theorem 2).

The conditions (28) and (i) of the theorem 2 define in a smooth manifold $\mathcal{K}$ the stability region in the parameter space. In (24) we can consider $H(\lambda(\tau, c(\epsilon_{\mathcal{C}}))) = 0$ that are given by theorem 2(ii). Applied on manifold $\mathcal{K}$, the implicit theorem gives us the elasticity of intertemporal substitution in consumption as a real and smooth function defined for $0 \leq \tau \leq \frac{\pi}{2}$. The conditions (28) and (i) impose lower $(\epsilon^*_{\mathcal{C}})$ and higher $(\tilde{\epsilon_{\mathcal{C}}})$ bound for $\epsilon_{\mathcal{C}}(\tau)$ that is, $\epsilon^*_{\mathcal{C}} < \epsilon^*_{\mathcal{C}}(\tau) < \tilde{\epsilon_{\mathcal{C}}}$ for $0 < \tau \leq \tau_0$. In addition, the theorem 2 shows that the elasticity of intertemporal substitution in consumption $\epsilon_{\mathcal{C}}(\tau)$ can be chosen in such way that if $0 < \tau < \tau_0$, then $\epsilon^*_{\mathcal{C}} < \epsilon^*_{\mathcal{C}}(\tau) < \bar{\epsilon_{\mathcal{C}}} \leq \tilde{\epsilon_{\mathcal{C}}}$ and so the steady-state (k_0, ϱ_0) is asymptotically stable. This means that in the neigborhood of the steady state the oscillations of the variational problem (20) can be controlled and that $\bar{\epsilon_{\mathcal{C}}}$ is a critical value. This finding shows how the phenomena delay influence the dynamic charcteristics of the economic system. Under this conditions the damping force dominates and makes trajectories approach the steady-state for small disturbances.

If (28) and (ii) of the theorem 2 are fulfilled we are able to localize a non-null purely imaginary root of equation 25 and to show that when the delay τ crosses the critical value τ_0 there are two simple roots of equation 24 crossing transversally the imaginary axis from left to right, whereas all others have negative real part. We can also verify that (τ_0, η_0) determines a sequence $\{(\tau_{0_\ell}, \eta_{0_p})\}_{(\ell,p)\in N^2}$ of critical values that lies in a smooth manifold $\mathcal{K}$ that represents the cascade of Hopf bifurcations. It can be checked that $\epsilon'_{\mathcal{C}}(\tau_0) > 0$. Then, the critical value τ_0 determines the critical value $\bar{\epsilon}_{\mathcal{C}} = \epsilon_{\mathcal{C}}(\tau_0)$. The elasticity of intertemporal substitution in consumption can be chosen in such way that each one of this critical values, near the equilibrium point (k_0, ϱ_0), is associated to a non-constant periodic solution of the variational problem (20). Since $\epsilon_{\mathcal{C}}(\tau_0)$ determines a sequence of characteristic roots for the linear part of the problem (20), the critical value $\epsilon_{\mathcal{C}}(\tau_0)$ represents the cascade of periodic solutions. This finding makes it clear how the delay phenomena and anti-damping force dominate, lead fluctuations to the business-cyle, and it can make trajectories converge to the other limit cycle which may be stable or not. Under such circumstances the capital-formation process can either have a stable fixed point sorrounded by an unstable cycle that appears because of a subcritical Hopf bifurcation; or a stable cycle loses its stability and a stable cycle appears because of supercritical Hopf bifurcation when the parameters gets near to a critical value. To prove this, we have involve ourselfes with normal-form theory, in order to detemine the direction of the bifurcation and the stability of periodic orbits bifurcate from steady-state with are hard computations.

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