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THE EFFECT OF HEAVY SMOKERS ON THE DYNAMICS OF A SMOKING MODEL

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Abstract: In this paper we present a non-linear mathematical model which analyzes the spread of smoking in a population. The population is divided into five classes: potential smokers, occasional smokers, heavy smokers, temporary quitters and permanent quitters. We study the effect of considering the class of occasional smokers and the impact of adding this class to the smoking model in [1] on the stability of its equilibria. This model is similar to the model in [2] where we studied the effect of occasional smokers on potential smokers, but here we're going to consider the effect of heavy smokers on potential smokers and it's impact on the stability of the model. Numerical results are also given to support our results and to compair the two models.

Key Words: smoking model, smoking generation number, linearization, next generation matrix, Routh-Herwitz, Liapunov function, local stability, global stability

1. Introduction

There is a lot of studies that has been done on the epidemics of smoking and it's hazards. The World Health Organization estimates that tobacco causes approximately 5 million deaths annually worldwide, and this number is expected

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to double by 2025. The reason for that high number is that Tobacco use is a major cause of many of the world's top killer diseases including cardiovascular disease, chronic lung disease and lung cancer. Smoking is often the hidden cause of many killing diseases. In Saudi Arabia, the prevalence of current smoking ranges from 2.4-52.3% (median = 17.5%) depending on the age group. The results of a Saudi modern study predicted an increase of smokers number in the country to 10 million smokers by 2020. The current number of smokers in Saudi Arabia is approximately 6 million, and they spend around 21 billion Saudi Riyal on smoking annually. Clearly smoking is a prevalent problem among Saudis that requires intervention for eradication. Persistent education of the health hazards related to smoking is recommended particularly at early ages in order to prevent initiation of smoking [4, 18]. Tobacco use is considered a disease that can spread through social contact in a way very similar to the spread of infectious diseases.

Like many infectious diseases, mathematical models can be used to understand the spread of smoking and to predict the impact of smokers on the community in order to help reducing the number of smokers. Castillo-Garsow et al [8] presented a general epidemiological model to describe the dynamics of Tobacco use and they considered the effect of peer presure, relapse, counselling and treetment. In their model the population was devided into non-smokers, smokers and smokers who quit smoking. Later, this mathematical model was refined by Sharomi and Gumel [15], they introduced a new class Q_t of smokers who temporarily quit smoking. They concluded that the smoking-free equilibrium is globally-asymptotically stable whenever a certain threshold, known as the smokers-generation number, is less than unity, and unstable if this threshold is greater than unity. The public health implication of this result is that the number of smokers in the community will be effectively controlled (or eliminated) at equilibrium point if the threshold is made to be less than unity. Such a control is not feasible if the threshold exceeds unity. Later, Lahrouz et al [12] proved the global stability of the unique smoking-present equilibrium state of the mathematical model developed by Sharomi and Gumel. Zaman [19] derived and analyzed a smoking model taking into account the occasional smokers compartment, and later [20] he extended the model to consider the possibility of quitters becoming smokers again. Erturk et al [9] introduced fractional derivatives into the model and studied it numerically. Zeb et al [21] presented a new giving up smoking model based on the model in [19] for which the interaction term is the square root of potential and occasional smokers. Van Voorn and Kooi [17] presented a three compartment smoking model which was studied using brute force simulations for the short term dynamics and bifurcation analysis for the long-term dynamics. In 2013 [1], we adopted the model developed and studied in [12, 15] and considered the effect of peer pressure on temporarily quitters. By this we mean the effect of smokers on temporarily quitters which is considered one of the main causes of their relapse. In 2014 [2], we introduced a new model by dividing the smokers into two subclasses: occasional smokers and heavy smokers, and the impact of these two subclasses on the existence and stability of equilibrium points. In that paper we studied the effect of occasional smokers on potential smokers.

In this paper, we study a model similar to that in [2], but we consider the effect of heavy smokers instead of occasional smokers. The aim of this work, is to analyze the model by using stability theory of non-linear differential equations and supporting the results with numerical simulation and to compair the two models. The paper is organized as follows: In section 2, we explain the formulation of the model. In section 3, The equilibria of the model are found. The stability of the equilibria is investigated in section 4. In section 5, we present numerical simulations to support our results. Finally, we end the paper with a conclusion and a compairsion between this model and the model in [2].

2. The Mathematical Model

The only difference between this model and the one in [2] is that the potential smokers here start to smoke under the influence of smokers (not occasional smokers as before)

Let the total population size at time t be denoted by $N(t)$. We divide the population $N(t)$ into five subclasses, potential smokers (non-smoker) $P(t)$, occasional (Light) smokers $L(t)$, heavy smokers $S(t)$, smokers who temporary quit smoking $Q_t(t)$ and smokers who permanently quit smoking $Q_p(t)$ such that $N(t) = P(t) + L(t) + S(t) + Q_t(t) + Q_p(t)$.Consider the following mathematical model:

$$
\begin{aligned}
\frac{dP}{dt} =&\mu - \mu P - \beta_1 P, \\
\frac{dL}{dt} =& - \mu L + \beta_1 PS - \beta_2 LS, \\
\frac{dS}{dt} =& - (\mu + \gamma)S + \beta_2 LS + \alpha Q_t, \\
\frac{dQ_t}{dt} =& - (\mu + \alpha)Q_t + \gamma(1 - \sigma)S, \\
\frac{dQ_p}{dt} =& - \mu Q_p + \sigma\gamma S,
\end{aligned}
\tag{1}
$$

where μ, β_1, β_2, γ, α and σ are positive constants defined as: β_1 is the contact rate between potential smokers and occasional smokers, β_2 is the contact rate between occasional smokers and smokers, μ is the rate of natural death, α is the rate at which temporary quitters who revert back to smoking, γ is the rate of quitting smoking, $(1 - \sigma)$ is the fraction of smokers who temporarily quit smoking (at a rate γ), σ is the remaining fraction of smokers who permanently quit smoking (at a rate γ).

We assume that the class of potential smokers is increased by the recruitment of individuals at a rate μ. In system (1), the total population is supposed constant and $P(t), L(t), S(t), Q_t(t)$,$Q_p(t)$ are respectively the proportions of potential smokers, smokers, temporarily quitters and permanent quitters at time t. Then

$$P(t) + L(t) + S(t) + Q_t(t) + Q_p(t) = 1.$$

Since the variable Q_p of system (1) does not appear in the first four equations, we will only consider the subsystem:

$$
\begin{aligned}
\frac{dP}{dt} =&\mu - \mu P - \beta_1 PS, \\
\frac{dL}{dt} =& - \mu L + \beta_1 PS - \beta_2 LS, \\
\frac{dS}{dt} =& - (\mu + \gamma)S + \beta_2 LS + \alpha Q_t, \\
\frac{dQ_t}{dt} =& - (\mu + \alpha)Q_t + \gamma(1 - \sigma)S.
\end{aligned}
\tag{2}
$$

The considered region for system (2) is:

$$\Gamma = \{(P, L, S, Q_t) : P + L + S + Q_t \leq 1, P > 0, L \geq 0, S \geq 0, Q_t \geq 0\}$$

As before, Γ is positively invariant.

3. Equilibria of the Model

The model also has the smoking-free equilibrium $E_0 = (1,0,0,0)$, and the smoking-present equilibrium E^*.

To find E^* we set the right-hand sides of the eqs.(2) to zero, we get:

$$\begin{aligned} P^* &= \frac{\mu}{\mu+\beta_1 S^*}, \\ L^* &= \frac{\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu}{\beta_2(\mu+\alpha)}, \\ Q_t^* &= \frac{\gamma(1-\sigma)S^*}{\mu+\alpha}. \end{aligned}$$

We also have the following equation

$$-\mu L^* + \beta_1 P^* S^* - \beta_2 S^* L^* = 0. \tag{3}$$

Substituting for $P^*$and L^* and multiplying by $\mu+\beta_1 S^*$ in (3) we receive

$$\begin{aligned} &-\beta_1\left(\frac{\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu}{(\mu+\alpha)}\right){S^*}^2 \\ &-\left(\frac{\mu(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)}{\mu+\alpha}+\frac{\beta_1\mu(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)}{\beta_2(\mu+\alpha)}-\beta_1\mu\right)S^* \\ &-\mu^2\left(\frac{\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu}{\beta_2(\mu+\alpha)}\right)=0. \end{aligned}$$

So, we have the following 2^{nd} order equation in S^*

$$\begin{aligned} {S^*}^2 + \frac{\mu\beta_2(\mu+\alpha)(\mu-\beta_1)+\mu(\alpha\sigma\gamma+\gamma\mu)+\mu\beta_1(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)}{\beta_1\beta_2(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)}S^* \\ + \frac{\mu^2}{\beta_1\beta_2} = 0. \end{aligned} \tag{4}$$

Eq.(4) has solutions of the form:

$$S_{1,2}^* = \frac{1}{2}(-A_1 \pm \sqrt{A_1^2 - 4A_2}),$$

where A_1 and A_2 respectively are:

$$A_1 = \frac{\mu\beta_2(\mu+\alpha)(\mu-\beta_1)+\mu\beta_2(\alpha\sigma\gamma+\gamma\mu)+\mu\beta_1(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)}{\beta_1\beta_2(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)},$$

$$A_2 = \frac{\mu^2}{\beta_1\beta_2}.$$

If $\mu \geq \beta_1$, then $A_1 > 0$. So we have no postive solution.
We can re-write A_1 as:

$$A_1 = \frac{\mu(\beta_1+\beta_2)}{\beta_1\beta_2}\left[1 - \frac{\beta_1\beta_2(\mu+\alpha)}{(\beta_1+\beta_2)(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)}\right].$$

Therefore

$$\begin{aligned}
A_1^2 - 4A_2 =& \frac{\mu^2(\beta_1+\beta_2)^2}{\beta_1^2\beta_2^2}\left[1 - \frac{\beta_1\beta_2(\mu+\alpha)}{(\beta_1+\beta_2)(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)}\right]^2 - \frac{4\mu^2}{\beta_1\beta_2} \\
=& \frac{\mu^2(\beta_1+\beta_2)^2}{\beta_1^2\beta_2^2} - \frac{2\mu^2(\beta_1+\beta_2)(\mu+\alpha)}{\beta_1\beta_2(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)} \\
&+ \frac{\mu^2(\mu+\alpha)^2}{(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)^2} - \frac{4\mu^2}{\beta_1\beta_2} \\
=& \frac{\mu^2(\beta_1-\beta_2)^2}{\beta_1^2\beta_2^2} - \frac{2\mu^2(\beta_1+\beta_2)(\mu+\alpha)}{\beta_1\beta_2(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)} \\
&+ \frac{2\mu^3\alpha}{(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)^2} \\
&+ \frac{2\mu^4}{(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)^2} - \frac{\mu^4}{(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)^2} \\
&+ \frac{\mu^2\alpha^2}{(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)^2} \\
=& \frac{\mu^2(\beta_1-\beta_2)^2}{\beta_1^2\beta_2^2} + \frac{2\mu^2(\mu+\alpha)(\beta_1+\beta_2)}{\beta_1\beta_2(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)} \\
&+ (\frac{\beta_1\beta_2\mu}{(\beta_1+\beta_2)(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)} - 1) \\
&+ \frac{\mu^2}{(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)^2}(\alpha^2-\mu^2).
\end{aligned}$$

Let

$$R_2 = \frac{\beta_1\beta_2\mu}{(\beta_1+\beta_2)(\alpha\sigma\gamma+\mu(\mu+\alpha)+\gamma\mu)}.$$

If $R_2 \geq 1$ and $\alpha \geq \mu$, then $A_1 < 0$ and $A_1^2 - 4A_2 > 0$. Hence we have two postive solutions $S^*_{1,2}$.

We can summurize the results above in the following theorem

Theorem 1. *System (2) always has the smoking-free equilibrium point* $E_0 = (1,0,0,0)$. *As for the existence of a smoking-present equilibrium point, we have two cases:*

(i) If $\mu \geq \beta_1$, *we have no positive equilibrium point.*

(ii) If $R_2 \geq 1$ *and* $\alpha \geq \mu$, *we have two positive equilibrium point* $E^*_{1,2}$.

4. Stability of Equilibria

4.1. Local Stability

First, we investigate the local stability of E_0 which can be stated in the following theorem:

Theorem 2. (Local stability of E_0) *The smoking-free equilibrium point* E_0 *of system (4) is always locally asymptotically stable.*

Proof. The Jacobian matrix at the equilibrium $E_0 = (1,0,0,0)$ gives:

$$J(E_0) = \begin{bmatrix} -\mu & 0 & -\beta_1 & 0 \\ 0 & -\mu & \beta_1 & 0 \\ 0 & 0 & -(\mu+\gamma) & \alpha \\ 0 & 0 & \gamma(1-\sigma) & -(\mu+\alpha) \end{bmatrix}.$$

The eigenvalues are given by $\lambda_1, \lambda_2 = -\mu < 0$, and λ_3, λ_4 satisfy the equation

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$\begin{aligned} a_1 &= 2\mu + \alpha + \gamma > 0, \\ a_2 &= \alpha\gamma\sigma + \mu(\mu+\alpha) + \gamma\mu > 0. \end{aligned}$$

Hence, by the Routh–Hurwitz criterion, all eigenvalues have negative real parts and hence E_0 is always locally asymptotically stable. □

4.2. Global Stability

We can investigate the global stability of E_0 by using Liapunov function

$$V = L + S + Q_t,$$

to derive the following theorem:

Theorem 3. (Global stability of E_0) *If* $\beta_1 \leq \mu$, *then* E_0 *is globally asymptotically stable in* Γ.

The proof is similar to the proof of Theorem 5 in [2].

For the stability of the positive equilibria E_1^* we will only use numerical simulations to get the results.

5. Numerical Simulations

In this section, we illustrate some numerical solutions of system (2) for different values of the parameters, and show that these solutions are in agreement with the qualitative behavior of the solutions.

We use the following parameters: $\beta_1 = 0.23$, $\beta_2 = 0.3$, $\mu = 0.04$, $\gamma = 0.2$, $\alpha = 0.25$ and $\sigma = 0.4$, and we choose different initial values such that

$$P + L + S + Q_t + Q_p = 1,$$

as follows:

$$\begin{aligned}
&1 - P(0) = 0.60301, \quad L(0) = 0.24000, \quad S(0) = 0.10628,\\
&Q_t(0) = 0.03260, \quad Q_p(0) = 0.01811,\\
&2 - P(0) = 0.55000, \quad L(0) = 0.20000, \quad S(0) = 0.17272,\\
&Q_t(0) = 0.06700, \quad Q_p(0) = 0.01028,\\
&3 - P(0) = 0.50000, \quad L(0) = 0.15000, \quad S(0) = 0.26200,\\
&Q_t(0) = 0.08066, \quad Q_p(0) = 0.00734,\\
&4 - P(0) = 0.45900, \quad L(0) = 0.10000, \quad S(0) = 0.21900,\\
&Q_t(0) = 0.21800, \quad Q_p(0) = 0.00400.
\end{aligned}$$

In Figure 1, we use the same initial values and the parameters as presented above. Figure 1(a) shows that the number of potential smokers increases and approaches the total population 1. Figure 1(b), (c) and (d) show that the number of the occasional smokers, the smokers and the temporary quitters decreases and approaches *zero*. In Figure 1(e), the number of permanent quitters increases at first, after that it decreases and approaches zero. We see from these figures that for any initial value, the solution curves tend to the equilibrium E_0. Hence, system (2) is locally asymptotically stable about E_0 for the above set of parameters.

In Figure 2, we will prove the local stability of E_1^* by using $\beta_1 = 0.7$, $\beta_2 = 0.55$, $\mu = 0.04$, $\gamma = 0.1$, $\alpha = 0.06$ and $\sigma = 0.4$ for $R_2 = 1.0267 >$

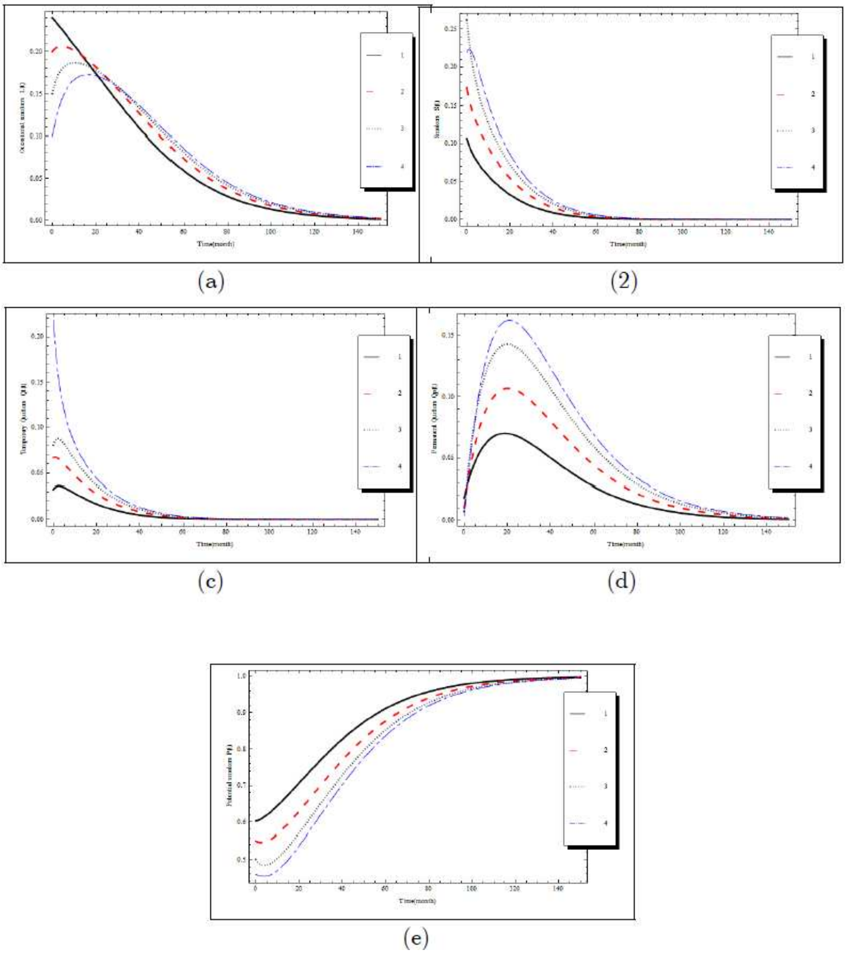


Figure 1: Time plots of system (2) with different initial conditions for any parameters

. (a) Potential smokers; (b) Occasional smokers; (c) Smokers; (d) Temporary quitters; (e) Permanent quitters.

1and $\alpha \geq \mu$ and the same initial values as above. We see from these figures that for any initial value, the solution curves tend to the equilibrium $E_1^* = (0.19912, 0.18909, 0.23723, 0.14234, 0.23723)$, when $R_2 \geq 1$ and $\alpha \geq \mu$. Hence, system (2) is locally asymptotically stable about E_1^* for the above set of parameters.

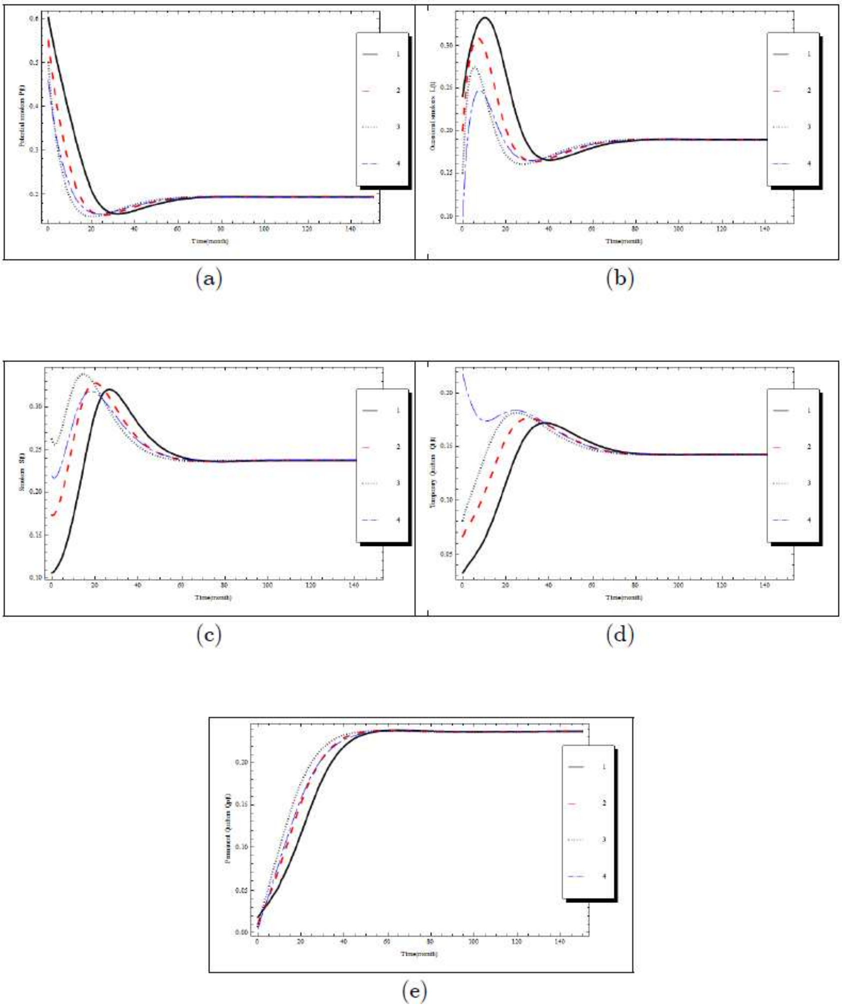
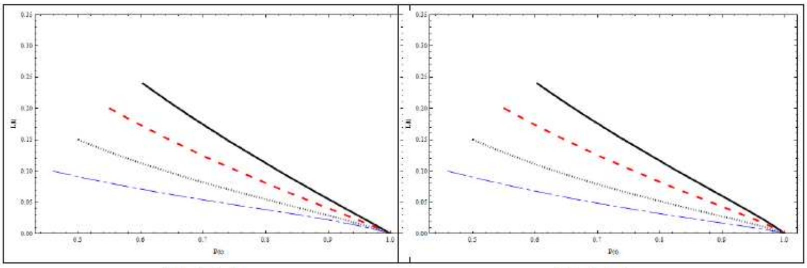


Figure 2: Time plots of system (2) with different initial conditions for any parameters
. (a) Potential smokers; (b) Occasional smokers; (c) Smokers; (d) Temporary quitters; (e) Permanent quitters.

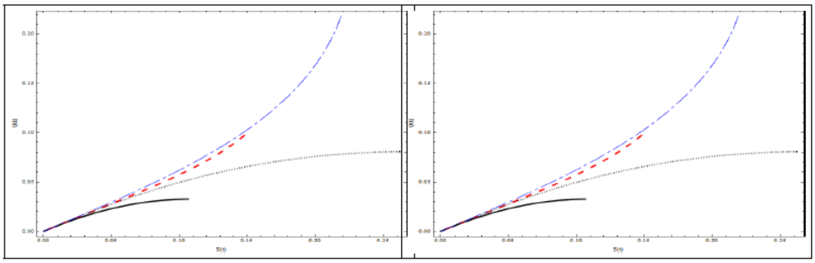
6. Discussion and Conclusions

In this paper, we presented a non-linear model which describes the overall smoking population dynamics when the population is assumed to remain constant and when the smokers are divided into two subclasses: occasional smokers and



Model 1 Model 2

Figure 3: Occasional smokers L vs. potential smokers P.



Model 1 Model 2

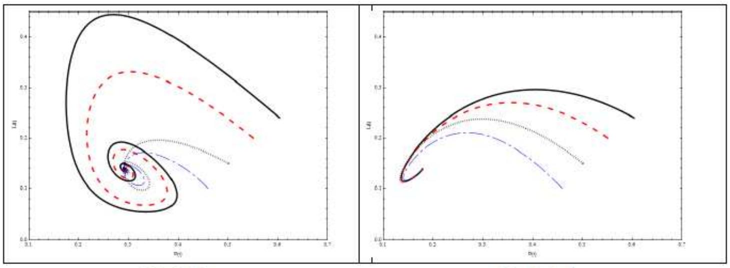
Figure 4: Temporary quitters Q_t vs. smokers S

heavy smokers. We introduced the stability analysis theory for nonlinear systems and used it to analyze the mathematical smoking models and to study both the local and global behavior of smoking dynamics. We conclude that E_0 is always locally asymptotically stable. On the other hand, one of the smoking present equilibrium points E_1^* is locally asymptotically stable if $R_2 \geq 1$ and $\alpha \geq \mu$. A Liapunov function was used to show that E_0 is globally asymptotically stable when the contact rate between potential smokers and smokers is less than or equal to the natural death rate ($\beta_1 \leq \mu$). This means that the number of smokers can be reduced by reducing the contact rate β_1 to be less than the natural death rate μ.

Under the same parameters and initial conditions used in the numerical simulations we introduce the following figures to compare the this model to ur model in [2].

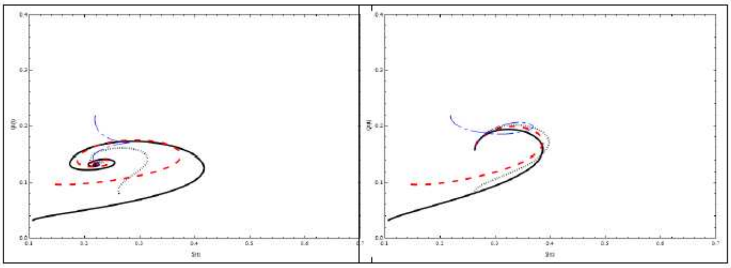
Near E_0: In fig. 3 and 4, we notice that there is no difference in the behavior of solutions of the two models near E_0.

Near E^* and E_1^*: We will use the following parameters $\beta_1 = 0.7$, $\beta_2 = 0.75$,



Model 1 Model 2

Figure 5: Occasional smokers L vs. potential smokers P



Model 1 Model 2

Figure 6: Temporary quitters Q_t vs. smokers S

$\mu = 0.04$, $\gamma = 0.1$, $\alpha = 0.06$ and $\sigma = 0.4$ for the two models, and use the same initial conditions as above. In fig. 5, we notice that P^* is greater in model 1 than in model 2 and L^* is the same in both models. Fig. 6 shows that S^* and Q_t^* are greater in model 2 than in model 1. So, model 1 is better than model 2 in the sense that if smoking persists in the community, the number of smokers is less in model 1.

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