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UNIQUENESS OF N-POINT BOUNDARY VALUE PROBLEMS

Curtis Kunkel
University of Tennessee at Martin
Martin, TN 38238, USA

Abstract: For the n^{th}-order differential equation, $y^{(n)} = f(x, y, \ldots, y^{(n-1)})$, it is established that the uniqueness of solutions of $(n-1)$-point boundary value problems implies uniqueness of solutions of n-point boundary value problems.

1. Introduction

In this paper, we will be concerned with the uniqueness of solutions of certain boundary value problems for the n^{th}-order differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}). \tag{1}$$

We will assume throughout that $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and that solutions of initial value problems for (1) are unique and extend to (a, b).

For (1), there have been numerous uniqueness results obtained for solutions of boundary value problems. For uniqueness results concerning conjugate boundary value problems and right focal boundary value problems, see [11], [4] and [7], [1], respectively. One motivation of studying uniqueness of solutions of boundary value problems for (1) is that sometimes uniqueness of solutions of (1) implies their existence. For existence results concerning conjugate boundary value problems and right focal boundary value problems, see [13], [5] and [8], [2], respectively.

In this paper, we will consider the uniqueness of solutions of certain $(n-1)$-point boundary value problems implying the uniqueness of n-point boundary

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value problems for (1). Specifically, we will consider a generalization of [3], with slightly more relaxed boundary conditions. Earlier results were given by [12] in which he showed that 2-point disconjugacy implied 3-point disconjugacy in the case that $n = 3$. Later, analogues were given in [9], in that he proved 2-point right disfocality implied 3-point right disfocality, again when $n = 3$. For a similar result "between" conjugate type and right focal type for boundary value problems for third order ordinary differential equations, see [6]. Also, in [9], it was shown that $(n-1)$-point right disfocality, along with the existence of solutions of certain $(n-2)$-point right focal boundary value problems, implied that (1) is n-point right disfocal. Moreover, it is shown in [10] that (1) is n-point disconjugate, if (1) is $(n-1)$-point disconjugate and all 2-point conjugate boundary value problems for (1) have unique solutions.

Consistent with [3], we begin with presenting conditions sufficient for the uniqueness of solutions of (1) satisfying the n-point conditions

$$\begin{aligned} y(x_{i_0}) &= y_{i_0}, \quad i_0 \in \{1, 2, \ldots, n\}, \\ y'(x_i) &= y_i, \quad i \in \{1, 2, \ldots, n\} \setminus \{x_{i_0}\}, \end{aligned} \tag{2}$$

where $a < x_1 < \cdots < x_n < b$. Again, we consider our results as a generalization of [3], which was technically an extension of [6].

In addition to the above assumptions, we will assume further conditions from the following:

(A) Given any $a < x_1 < \cdots < x_{n-1} < b$, if there are solutions $y(x)$ and $z(x)$ of (1) such that $y(x_{i_0}) = z(x_{i_0})$ for $i_0 \in \{1, 2, \ldots, n-1\}$, $y'(x_i) = z'(x_i)$ for $i \in \{1, 2, \ldots, n-1\} \setminus \{i_0\}$, and $y''(x_j) = z''(x_j)$ for some $j \in \{1, 2, \ldots, n-1\} \setminus \{i_0\}$, it follows that $y(x) \equiv z(x)$.

(B) For $i_0 \in \{1, 2, \ldots, n-1\}$, there exists $r_0 \in \{1, 2, \ldots, n-1\} \setminus \{i_0\}$, such that, given any $a < x_1 < \cdots < x_{n-1} < b$ and $y_1, y_2, \ldots, y_n \in \mathbb{R}$, there exists a solution $y(x)$ of (1) satisfying

$$\begin{aligned} y(x_{i_0}) &= y_{i_0}, \\ y'(x_i) &= y_i, \quad i \in \{1, 2, \ldots, n-1\} \setminus \{i_0\}, \\ y''(x_{r_0}) &= y_n. \end{aligned}$$

(C) For $i_0 \in \{1, 2, \ldots, n-2\}$, there exists $s_0 \in \{1, 2, \ldots, n-2\} \setminus \{i_0\}$, such that, given any $a < x_1 < \cdots < x_{n-2} < b$ and any $y_1, y_2, \ldots, y_n \in \mathbb{R}$, there exists a unique solution $y(x)$ of (1) satisfying

$$\begin{aligned} y(x_{i_0}) &= y_{i_0}, \\ y'(x_i) &= y_i, \quad i \in \{1, 2, \ldots, n-2\} \setminus \{i_0, s_0+1, s_0+2\}, \\ y''(x_{s_0}) &= y_{n-1}, \\ y'''(x_{s_0}) &= y_n. \end{aligned}$$

In the second section, we state a theorem concerning continuous dependence on boundary conditions of solutions of (1), when solutions of that type of boundary value problem are unique. Also, we establish a type of compactness condition on solutions of (1), when there exists a uniform bound on the first derivative of solutions.

Then, in the final section, we show that hypotheses (A), (B), and (C) imply that solutions of (1), (2) are unique, when they exist.

2. Continuous Dependence and Convergence

The following two theorems will be used extensively in the proof of the main result in the subsequent section. The first theorem establishes that, under hypothesis (A), solutions of boundary value problems of that type depend continuously on boundary conditions. Its proof is a standard application of the Brouwer Invariance of Domain Theorem; see [7] or [11] for a typical argument.

Theorem 2.1. *Assume that with respect to (1), condition (A) is satisfied. Let $z(x)$ be an arbitrary but fixed solution of (1). Let $i_0 \in \{1, 2, \ldots, n-1\}$. Then for any fixed $j_0 \in \{1, 2, \ldots, n-1\} \setminus \{i_0\}$, for any $a < c < x_1 < \cdots < x_{n-1} < d < b$, and any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x_i - t_i| < \delta$ for $1 \le i \le n-1$, $|z(x_{i_0}) - y_{i_0}| < \delta$, $|z'(x_i) - y_i| < \delta$ for $i \in \{1, 2, \ldots, n-1\} \setminus \{i_0\}$, and $|z''(x_{j_0}) - y_n| < \delta$, imply that (1) has a solution $y(x)$ with $y(t_{i_0}) = y_{i_0}$, $y'(t_i) = y_i$ for $i \in \{1, 2, \ldots, n-1\} \setminus \{i_0\}$, $y''(t_{j_0}) = y_n$, and*

$$|z^{(i)}(x) - y^{(i)}(x)| < \varepsilon, \quad 0 \le i \le n-1,$$

on the interval $[c, d]$.

There is a completely analogous theorem concerning continuous dependence on boundary conditions for boundary value problems of the type that is in hypothesis (C).

The next theorem establishes a compactness condition on solutions of (1), under hypothesis (A) and (B).

Theorem 2.2. *Assume that with respect to (1), conditions (A) and (B) are satisfied. If* $\{y_n(x)\}$ *is a sequence of solutions of (1) such that, for* $\alpha_1 \in \mathbb{R}, x_{i_0} \in (a,b)$*, for some* $a < c < d < b$*, and for some* $M > 0, y_m(x_{i_0}) = y_{m+1}(x_{i_0}) = \alpha_1$ *and* $|y'_m(x)| \le M$ *on* $[c,d]$*, for every positive integer* m*, then there exists a subsequence* $\{y_{m_k}(x)\}$*, such that* $\{y^{(i)}_{m_k}(x)\}$ *converges uniformly on each compact subset of* $(a,b), 0 \le i \le n-1$.

Proof. Assume without loss of generality that r_0 is an even integer. Let $c = \rho_0 < \rho_1 < \cdots < \rho_{2n-5} = d$. Then, on each of the disjoint subintervals, $[\rho_0,\rho_1],[\rho_2,\rho_3],\ldots,[\rho_{2n-6},\rho_{2n-5}]$, we have that $|y'_m(x)| \le M$, for each m. Then, for every m, there exists $q_m \in (\rho_{r_0}, \rho_{r_0+1})$ such that

$$|y''_m(q_m)| \le \frac{2M}{\rho_{r_0+1} - \rho_{r_0}}.$$

We have the bounded sequences $\{y_m(x_{i_0})\}$, $\{y'_m(\rho_0)\}$, $\{y'_m(\rho_2)\},\ldots,\{y'_m(\rho_{r_0-2})\}$, $\{y'_m(q_m)\}$, $\{y''_m(q_m)\}$, $\{y'_m(\rho_{r_0+2})\},\ldots,\{y'_m(\rho_{2n-6})\}$, and
$\{q_m\} \subset (\rho_{r_0}, \rho_{r_0+1})$. Then there exists a subsequence $\{m_k\} \subseteq \{m\}$, a point $q'_{r_0} \in [\rho_{r_0}, \rho_{r_0+1}]$, and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \beta_1 \in \mathbb{R}$ such that

$$\begin{array}{rcll} q_{m_k} & \to & q'_{r_0}, & \\ y'_{m_k}(\rho_{2i}) & \to & \alpha_{i+2}, & 0 \le i \le n-3, i \ne \frac{r_0}{2}, \\ y'_{m_k}(q_{m_k}) & \to & \alpha_{r_0}, & \\ y''_{m_k}(q_{m_k}) & \to & \beta_1. & \end{array}$$

Also, $y_{m_k}(x_{i_0}) = \alpha_1$, for each k. Now, let $y(x)$ be the solution of (1) given by (B) satisfying $y(x_{i_0}) = \alpha_1, y'(\rho_0) = \alpha_2, y'(\rho_2) = \alpha_3, \ldots,$ $y'(\rho_{r_0-2}) = \alpha_{\frac{r_0}{2}+1}$, $y'(q'_{r_0}) = \alpha_{r_0}, y''(q'_{r_0}) = \beta_1, y'(\rho_{r_0+2}) = \alpha_{\frac{r_0}{2}+3}, \ldots,$ $y'(\rho_{2n-6}) = \alpha_{n-1}$. By Theorem 2.1, $\{y^{(i)}_{m_k}(x)\}$ converges uniformly to $y^{(i)}(x)$ on compact subsets of $(a,b), 0 \le i \le n-1$. □

3. $(n-1)$-Point Uniqueness Implies n-Point Uniqueness

The following theorem establishes uniqueness of solutions of (1), (2) under hypotheses (A), (B), and (C). In particular, because of the role of (A), we think of this as a uniqueness result for n-point boundary value problems.

Theorem 3.1. *Assume that with respect to (1), conditions (A), (B), and (C) are satisfied. Then, there exists at most one solution of (1), (2). More* *precisely, given any $a < x_1 < \cdots < x_n < b$, if there are solutions $y(x)$ and $z(x)$ of (1) such that*

$$\begin{aligned} y(x_{i_0}) &= z(x_{i_0}), \quad i_0 \in \{1, 2, \ldots, n\}, \\ y'(x_i) &= z'(x_i), \quad i \in \{1, 2, \ldots, n\} \setminus \{i_0\}, \end{aligned}$$

it follows that $y(x) \equiv z(x)$.

Proof. Assume, for the purpose of establishing a contradiction, that there exist $a < x_1 < \cdots < x_n < b$, and solutions $y(x)$ and $z(x)$ of (1), with $y(x) \not\equiv z(x)$, such that

$$\begin{aligned} y(x_{i_0}) &= z(x_{i_0}), \quad i_0 \in \{1, 2, \ldots, n\}, \\ y'(x_i) &= z'(x_i), \quad i \in \{1, 2, \ldots, n\} \setminus \{i_0\}. \end{aligned}$$

By (A), $y''(x_i) \neq z''(x_i), i \in \{1, 2, \ldots, n\} \setminus \{i_0\}$. We may assume for some $s_0 \in \{1, 2, \ldots, n-3\} \setminus \{i_0\}$ that $y''(x_{s_0}) > z''(x_{s_0})$. Then $y''(x_{s_0+1}) < z''(x_{s_0+1})$, $y''(x_{s_0+2}) > z''(x_{s_0+2})$, $y'(x) > z'(x)$ on (x_{s_0}, x_{s_0+1}), and $y'(x) < z'(x)$ on (x_{s_0+1}, x_{s_0+2}).

For every $\varepsilon > 0$, let $y_\varepsilon(x)$ be the solution of (1), given by (C), satisfying

$$\begin{aligned} y_\varepsilon(x_{i_0}) &= y(x_{i_0}) = z(x_{i_0}), \\ y'_\varepsilon(x_i) &= y'(x_i) = z'(x_i), \quad i \in \{1, 2, \ldots, n\} \setminus \{i_0, s_0 + 1, s_0 + 2\}, \\ y''_\varepsilon(x_{s_0}) &= y''(x_{s_0}), \\ y'''_\varepsilon(x_{s_0}) &= y'''(x_{s_0}) + \varepsilon. \end{aligned}$$

Now, $y''(x_{s_0}) > z''(x_{s_0})$, and $y''_\varepsilon(x_{s_0}) = y''(x_{s_0})$, forcing $y''_\varepsilon(x_{s_0}) > z''(x_{s_0})$. Moreover, by (A), for $0 \leq \varepsilon_1 < \varepsilon_2$, we have the monotonicity property

$$y'_{\varepsilon_1}(x) < y'_{\varepsilon_2}(x) \text{ on } (x_{s_0}, x_{s_0+3}). \tag{3}$$

In particular, $y'_\varepsilon(x) > y'(x)$ on (x_{s_0}, x_{s_0+3}), for $\varepsilon > 0$. Define

$$H = \{\varepsilon \geq 0 | \text{ for some } \tau_\varepsilon \in [x_{s_0+1}, x_{s_0+2}], y'_\varepsilon(\tau_\varepsilon) \leq z'(\tau_\varepsilon)\}.$$

Clearly $H \neq \emptyset$, since $\varepsilon = 0$ belongs to H. We claim that H is unbounded above. To verify this, assume there exists n_0 which is an upper bound of H. Then $y'_{n_0}(x) > z'(x)$ on $[x_{s_0+1}, x_{s_0+2}]$. Then there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 = \sup H$.

Consider the solution of (1), $y_{\varepsilon_0}(x)$. There are two cases. Either, (i) $y'_{\varepsilon_0}(x) > z'(x)$ on $[x_{s_0+1}, x_{s_0+2}]$, or (ii) $y'_{\varepsilon_0}(x) \leq z'(x)$ at some points of (x_{s_0+1}, x_{s_0+2}).

Case (i). Suppose $y'_{\varepsilon_0}(x) > z'(x)$ on $[x_{s_0+1}, x_{s_0+2}]$. By the remark after Theorem 2.1, for condition (C), there exists $0 < \varepsilon_1 < \varepsilon_0$, such that $y'_{\varepsilon_1}(x) > z'(x)$

on $[x_{s_0+1}, x_{s_0+2}]$. Hence, ε_1 is an upper bound of H and $\varepsilon_1 < \varepsilon_0$, contradicting the fact that $\varepsilon_0 = \sup H$.

Case (ii). Suppose $y'_{\varepsilon_0}(x) \le z'(x)$ at some points in (x_{s_0+1}, x_{s_0+2}). Now, if there exists $x_{s_0+1} < \rho < x_{s_0+2}$, so that $y'_{\varepsilon_0}(\rho) < z'(\rho)$, then by continuity, $y'_{\varepsilon_0}(x) < z'(x)$ on $[\rho - \delta, \rho + \delta]$, for some $\delta > 0$. Again, by the remark after Theorem 2.1, there exists $\varepsilon_0 < \varepsilon_2$ such that $y'_{\varepsilon_2}(x) < z'(x)$ on $[\rho - \delta/2, \rho + \delta/2]$. Thus, ε_0 is not an upper bound of H, which is a contradiction. Hence, there must exist $x_{s_0+1} < \tau < x_{s_0+2}$ such that $y'_{\varepsilon_0}(\tau) = z'(\tau)$, and because $y'_{\varepsilon_0}(x) \ge z'(x)$ on (x_{s_0+1}, x_{s_0+2}), we have that $y''_{\varepsilon_0}(\tau) = z''(\tau)$. Also, $y_{\varepsilon_0}(x_{i_0}) = y(x_{i_0}) = z(x_{i_0})$, and $y'_{\varepsilon_0}(x_i) = y'(x_i) = z'(x_i), i \in \{1, 2, \ldots, n\} \setminus \{i_0, s_0 + 1, s_0 + 2\}$, and so by (A), $y_{\varepsilon_0}(x) \equiv z(x)$. But $y''_{\varepsilon_0}(x_{s_0}) = y''(x_{s_0}) > z''(x_{s_0})$, which is a contradiction.

We conclude that H is not bounded above, and so $H = [0, +\infty)$. Thus, for every $\varepsilon \ge 0$,

$$y'_\varepsilon(x_\varepsilon) \le z'(x_\varepsilon), \text{ for some } x_{s_0+1} < x_\varepsilon < x_{s_0+2}.$$

Now, for every positive integer m, let $y_m(x)$ be the solution of (1) on (a, b) satisfying the boundary conditions

$$\begin{aligned} y_m(x_{i_0}) &= y(x_{i_0}), \\ y'_m(x_i) &= y'(x_i), \quad i \in \{1, 2, \ldots, n\} \setminus \{i_0, s_0 + 1, s_0 + 2\}, \\ y''_m(x_{s_0}) &= y''(x_{s_0}), \\ y'''_m(x_{s_0}) &= y'''(x_{s_0}) + m. \end{aligned}$$

The monotonicity condition (3) gives, for each m,

$$y'(x) < y'_m(x) < y'_{m+1}(x) \text{ on } (x_{s_0}, x_{s_0+3}). \tag{4}$$

For each $m \ge 1$, let $E_m = \{x \in (x_{s_0+1}, x_{s_0+2}) | y'_m(x) \le z'(x)\}$. Since $H = [0, +\infty), E_m \ne \emptyset$, for each m. Moreover, by (4), we have the nesting $E_{m+1} \subset E_m \subset \cdots \subset (x_{s_0+1}, x_{s_0+2})$, for each m, and by the continuity of y'_m and z', each E_m is compact. It follows from the Cantor Intersection Theorem that

$$\bigcap_m E_m \equiv E \ne \emptyset.$$

Well, either $\cap_m E_m \equiv E$ consists of a single point or more than one point. If $t_1 < t_2$ belongs to E, then by an argument similar to the one used showing H to be unbounded above, it follows that $[t_1, t_2] \subseteq E$. Then, by the definition of E_m for each $m, y'_m(x) \le z'(x)$ on $[t_1, t_2]$, and by the monotonicity condition (4) on $\{y'_m(x)\}$, it follows that $y'(x) < y'_m(x) \le z'(x)$ on $[t_1, t_2]$. Thus, there exists $M > 0$ such that $|y_m'(x)| \le M$ on $[t_1, t_2]$, for each m. By Theorem 2.2, there exists a subsequence $\{y_{m_k}(x)\}$ such that $\{y_{m_k}^{(i)}(x)\}$ converges uniformly on compact subsets of $(a, b), 0 \le i \le n-1$. But $y_{m_k}'''(x_{s_0}) = y'''(x_{s_0}) + m_k$ which diverges to $+\infty$, a contradiction. Hence

$$\bigcap_m E_m \equiv E = \{x_0\},$$

where $x_{s_0+1} < x_0 < x_{s_0+2}$, and $y'(x_0) < y_m'(x_0) < y_{m+1}'(x_0) \le z'(x_0)$, for every m. Thus,

$$y'(x_0) < \lim_{m\to\infty} y_m'(x_0) \equiv y_0 \le z'(x_0).$$

There are two cases to consider. Either, (i) $\lim_{m\to\infty} y_m'(x_0) = y_0 = z'(x_0)$, or (ii) $\lim_{m\to\infty} y_m'(x_0) = y_0 < z'(x_0)$.

Case (i). Suppose $\lim_{m\to\infty} y_m'(x_0) = y_0 = z'(x_0)$. Given $\varepsilon > 0$, let $z(x, \varepsilon)$ be the solution of (1) satisfying

$$\begin{aligned}
z(x_{i_0}, \varepsilon) &= z(x_{i_0}) = y(x_{i_0}),\\
z'(x_i, \varepsilon) &= z'(x_i) = y'(x_i),\\
&\qquad\qquad i \in \{1, 2, \ldots, n\} \setminus \{i_0, s_0+1, s_0+2\},\\
z''(x_{s_0}, \varepsilon) &= z''(x_{s_0}),\\
z'''(x_{s_0}, \varepsilon) &= z'''(x_{s_0}) - \varepsilon.
\end{aligned}$$

Then by (A), $z'(x, \varepsilon) < z'(x)$ on (x_{s_0}, x_{s_0+3}). By Theorem 2.1, there exists $\varepsilon_0 > 0$, such that $y'(x) < z'(x, \varepsilon_0) < z'(x)$ on (t_1, t_2), where $x_{s_0+1} < t_1 < x_0 < t_2 < x_{s_0+2}$, and $y'(t_1) = z'(t_1, \varepsilon_0), y'(t_2) = z'(t_2, \varepsilon_0)$. Note in particular that $z'(x_0, \varepsilon_0) < z'(x_0)$. With $y_m(x)$ as above, we define sets $F_m = \{x \in (t_1, t_2) | y_m'(x) \le z'(x, \varepsilon_0)\}$. Using our previous argument, but with $z'(x)$ replaced by $z'(x, \varepsilon_0)$ on $[t_1, t_2]$, each $F_m \ne \emptyset$ and is also compact. Moreover, we also obtain $\bigcap_m F_m \ne \emptyset$ and from $F_m \subseteq E_m$, for every m,

$$\bigcap_m F_m \subseteq \bigcap_m E_m = \{x_0\}.$$

This implies $\bigcap_m F_m = \{x_0\}$. Consequently,

$$y_0 = \lim_{m\to\infty} y_m'(x_0) \le z'(x_0, \varepsilon_0) < z'(x_0) = y_0,$$

a contradiction.

Case (ii). Now, suppose $\lim_{m\to\infty} y'_m(x_0) = y_0 < z'(x_0)$. For $0 \le \lambda \le 1$, let $z(x,\lambda)$ be the solution of (1) given by (C), such that

$$\begin{aligned} z(x_{i_0},\lambda) &= \lambda y(x_{i_0}) + (1-\lambda) z(x_{i_0}),\\ z'(x_i,\lambda) &= \lambda y'(x_i) + (1-\lambda) z'(x_i),\\ &\qquad\qquad i \in \{1,2,\ldots,n\} \setminus \{i_0, s_0+1, s_0+2\},\\ z''(x_{s_0},\lambda) &= \lambda y''(x_{s_0}) + (1-\lambda) z''(x_{s_0}),\\ z'''(x_{s_0},\lambda) &= \lambda y'''(x_{s_0}) + (1-\lambda) z'''(x_{s_0}). \end{aligned}$$

Observe that $z(x,0) = z(x)$ and $z(x,1) = y(x)$. The set

$$\begin{aligned} L = \{&(z(x_{i_0},\lambda), z'(x_{j_1},\lambda), z'(x_{j_2},\lambda), \ldots, z'(x_{s_0},\lambda), z''(x_{s_0},\lambda),\\ &z'''(x_{s_0},\lambda), z'(x_{s_0+3},\lambda), \ldots, z'(x_{j_{n-3}},\lambda)) | 0 \le \lambda \le 1\}, \end{aligned}$$

where $\{j_1, j_2, \ldots, j_{n-3}\} \equiv \{1,2,\ldots,n\} \setminus \{i_0, s_0+1, s_0+2\}$, is a line segment in $\mathbb{R}^n$. In fact, $h : [0,1] \to L$, defined by $h(\lambda) = (z(x_{i_0},\lambda),\ z'(x_{j_1},\lambda),\ z'(x_{j_2},\lambda),\ \ldots, z'(x_{s_0},\lambda),\ z''(x_{s_0},\lambda),\ z'''(x_{s_0},\lambda),\ z'(x_{s_0+3},\lambda),\ \ldots,\ z'(x_{j_{n-3}},\lambda))$, is one-to-one, continuous, and onto. Now, define $g : L \to \mathbb{R}$ by

$$\begin{aligned} g(z(x_{i_0},\lambda), z'(x_{j_1},\lambda), z'(x_{j_2},\lambda), \ldots, z'(x_{s_0},\lambda), z''(x_{s_0},\lambda), z'''(x_{s_0},\lambda),\\ z'(x_{s_0+3},\lambda), \ldots, z'(x_{j_{n-3}},\lambda)) = z'(x_0,\lambda). \end{aligned}$$

By the remark after Theorem 2.1, g is continuous. Thus, $g \circ h : [0,1] \to \mathbb{R}$ is a continuous function. Now,

$$g \circ h(0) = z'(x_0,0) = z'(x_0) > y_0 > y'(x_0) = z'(x_0,1) = g \circ h(1),$$

and so, by the Intermediate Value Theorem, there exists $0 < \lambda_0 < 1$ such that $g \circ h(\lambda_0) = y_0$. But $g \circ h(\lambda_0) = z'(x_0,\lambda_0)$. So we have a solution $z(x,\lambda_0)$ of (1) such that $z'(x_0,\lambda_0) = y_0$. This is the same situation as in case (i), except $z'(x_0) = y_0$ is now replaced by $z'(x_0,\lambda_0) = y_0$, and this again leads to a contradiction.

Having reached a contradiction in all cases, we conclude that our initial assumption was false. Therefore, there exists at most one solution of (1), (2). □

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