International Journal of Differential Equations and Applications
Volume 15 No. 1 2016, 12-32
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v15i1.2749

MULTILINEAR MCSHANE INTEGRAL IN $\mathbb{R}^n$

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Abstract: This work studies multilinear McShane-Stieltjes integrals of vector-valued functions defined on compact intervals in $\mathbb{R}^n$.

AMS Subject Classification: 46G10, 28B05
Key Words: multilinear integral, Stieltjes integral, McShane integral, vector-valued functions, several variables

1. Introduction

The multilinear integrals are Stieltjes-type integrals in which the number of integrands or the number of integrators (or both) can be greater than one, e.g. $\int_B \mathcal{A}(f_1, dg_1, f_2, dg_2)$. Trilinear, bilinear and ordinary integrals can be regarded as special cases of the multilinear integrals. Stieltjes sums and convergence of such sums can be defined in different ways which lead to different types of multilinear integrals including the following: Riemann, Moore-Pollard, McShane, and Henstock-Kurzweil.

Multilinear integrals for vector-valued functions of one variable can be found in [1], [2] and [3]. Riemann, Moore-Pollard and Henstock-Kurzweil multilinear Stieltjes integrals for vector-valued functions of several variables have been considered in [4]. The present work can be regarded as a continuation of [4]

Received: October 17, 2015

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and focuses on the McShane type which is of special interest since McShane's approach is a generalization of the Riemann integral and, specifically, it is well known that the ordinary McShane integral $\int_a^b f(x)dx$, where $f(x)$ is a real-valued function and a, $b \in \mathbb{R}$, is equivalent to the Lebesgue integral. (See [7] for the ordinary McShane integral.)

We denote the McShane-Stieltjes (MS) multilinear integral by

$$(MS)\int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q), \tag{1.1}$$

for vector-valued functions defined on a closed bounded interval B in $\mathbb{R}^n$. To simplify the notation we consider multilinear integrals in the case where the integrands f_k are placed in the beginning and integrators g_l at the end of the expression (1.1) but all the results are valid for any order of integrands and integrators.

We define McShane multilinear integrals in a way similar to the way we defined Henstock-Kurzweil multilinear integral in [4]. The difference between the constructions of these integrals lies the way we form tagged partitions. Unlike Henstock-Kurzweil-type tagged partititions, tags in a McShane-type tagged partition are not necessarily located in the associated subblocks. We use Stieltjes quasi volumes, a bounded multilinear operator $\mathcal{A}$ and tagged partitions of McShane-type, to define the McShane-Stieltjes multilinear integral (1.1) as the limit of McShane-Stieltjes sums.

The existence of the MS-integral is proved for the case when the integrands are continuous and the integrators are of bounded semivariation. Two convergence theorems (4.2 and 4.3) for the MS-integral are given.

2. Definitions and Terminology

We begin with some definitions and terminology that are used throughout this work. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two points in $\mathbb{R}^n$. Here we use the *maximum norm*, given by $|x| = max(|x_1|, \ldots, |x_n|)$ and the induced metric $dist(x, y) = |x - y|$. Let E and F be subsets of $\mathbb{R}^n$. The extended real number $diam(E) = \sup\{|x - y| : x, y \in E\}$ and real number $dist(A, B) = inf\{|x - y| : x \in E, y \in F\}$ are called, respectively, the *diameter* of E and *distance* between E and F. For $x_0 \in \mathbb{R}^n$ and $r > 0$ we define the following cubes: $U(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and $U[x_0, r] = \{x \in \mathbb{R}^n : |x - x_0| \leq r\}$. The interior of a set E we denote by $int(E)$.

An *interval* I in $\mathbb{R}^n$ is the Cartesian product $\prod_{i=1}^n I_i$ where I_i are intervals in $\mathbb{R}$. A collection of intervals is called *nonoverlapping* if their interiors are disjoint.

Definition 2.1. By an *n-dimensional block* (or *n-dimensional closed interval*), or just *block*, we mean a set of points

$$B = \{(x_1, \dots, x_n) : a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n\},$$

where $-\infty < a_1 \leq b_1 < \infty, \dots, -\infty < a_n \leq b_n < \infty$. We shall also occasionally write $B = [a, b]$ where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. (Note that blocks are always compact.)

The *volume* of B is the real nonnegative number

$$v(B) = \prod_{i=1}^{n} (b_i - a_i)$$

Any point $\gamma = (c_1, \dots, c_n)$, where for each c_j we have $c_j = a_n$ or $c_j = b_n$, is called a *vertex* of B. Note that B has 2^n vertices.

The number $|B| = max\{b_1 - a_1, \dots, b_n - a_n\}$ will be called the *size* of the block B. A block B' is a *subblock* of a block B if $B' \subseteq B$.

Definition 2.2. Let B be an interval. We say that $\Pi = \{B_1, \dots, B_r\}$, consisting of a finite collection of nonoverlapping subintervals of B, forms a *partition* of B if $B = \cup_{i=1}^{r} B_i$.

Definition 2.3. If a partition Π' is obtained by further subdividing the blocks of the partition Π we say that Π' is a *refinement* of Π.

Definition 2.4. (*Stieltjes quasi volume.*) Let $B = [a, b]$ with $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Let $g : B \to X$ be a vector-valued function defined on B with values in a normed linear space X. Let $\gamma = (c_1, \dots, c_n)$ where $c_j = a_n$ or $c_k = b_n$ and $\nu(\gamma)$ denotes the number of terms in γ for which $c_k = a_n$. We define, in a common way (see for example [6] or [8]), a *Stieltjes quasi volume* $\sigma_g(B)$ of the block B associated to the function g by

$$\sigma_g(B) = \sum_{\gamma} (-1)^{\nu(\gamma)} g(\gamma)$$

where the summation is taken over all vertices $\gamma = (c_1, \dots, c_n)$, where $c_j = a_n$ or $c_j = b_n$, of the block B.

Definition 2.5. Let $g : B \to X$ be a vector-valued function defined on B with values in a normed space X. The quantity

$$V_B(\sigma) = \sup_{\Pi} \sum_{i=1}^{r} |\sigma(B_i)|,$$

where the supremum is taken over all partitions $\Pi = \{B_1, \dots, B_r\}$ of B, is called the *variation of the Stieltjes quasi volume* σ in the block B. We say that the Stieltjes quasi volume σ is of *bounded variation* on B if $V_B(\sigma) < \infty$.

Definition 2.6. A vector-valued function $C \mapsto F(C)$ defined on intervals is *interval-additive* if for any partition of C consisting of the subintervals $C_i,\ i = 1, \dots, r$, we have $F(C) = \sum_{i=1}^{r} F(C_i)$.

Remark 2.1. It is easy to prove that $\sigma_g(B)$ is an interval-additive function.

Remark 2.2. The following properties follow from the definition of the Stieltjes quasi volume:

1. If B is a degenerate interval, that is $a_j = b_j$ for some j, then $\sigma_g(B) = 0$.
2. If $h = f + g$ then $\sigma_h(B) = \sigma_f(B) + \sigma_g(B)$.
3. If $h = \lambda g$, where λ is a real or complex number, then $\sigma_h(B) = \lambda \sigma_g(B)$.

3. McShane-Stieltjes Multilinear Integrals

The difference in the procedure when introducing the McShane-type integrals and the Henstock-type integrals, lies in the demands on the tags. Unlike the Henstock (and Riemann) case, the tags of a McShane-type partition are not required to belong to the associated subblocks.

Definition 3.1. Let B be an n-dimensional block. Let $\Pi = \{B_1, \dots, B_r\}$ be a partition of B, and let

$$D_I = \{(\mathfrak{S}_i, B_i) : i = 1, \dots, r\}$$

where $\mathfrak{S}_i = (s_{i,1}, \dots, s_{i,p})$ and $s_{i,k} \in B$. We say that D_I is a *McShane-tagged partition of type I*, or simply M_p*-partition* of the block B and refer to the points $s_{i,k},\ k = 1, \dots, p,\ i = 1, \dots, r$ as the *tags* of D_I.

Remark 3.1. An M_p-partition has the following properties:

1. A tagged partition D_I has p tags $s_{i,1}, \dots, s_{i,p}$ associated to the block B_i. All of these tags belong to B but, as mentioned above, need not belong to B_i.
2. Some (or all) tags $s_{i,k},\ k = 1, \dots, p,\ i = 1, \dots, r$ can coincide.

Definition 3.2. A tagged partition which has (only) a single point s_i associated to each block B_i will be denoted D_{II}. Thus

$$D_{II} = \{(s_i, B_i) : i = 1, \ldots, r\}.$$

D_{II} will then be called a *McShane-tagged partition of type II* or simply an *M_1-partition.*

We now formalize what it means that a tag is sufficiently close to an associated subblock.

Definition 3.3. Let δ be a positive function (gauge) defined on the block B, that is $\delta : B \to \mathbb{R}_+$. We say that an M_p-partition D_I of the block B is *δ-fine* if

$$B_i \subseteq U(s_{i,k}, \delta(s_{i,k})),\ k = 1, \ldots, p,\ i = 1, \ldots, r.$$

Changing $s_{i,k}$ to s_i in the above definition we get the δ-fine partition of type II. Thus,

$$D_{II} = \{(s_i, B_i) : i = 1, \ldots, r\}$$

is *δ-fine* if

$$B_i \subseteq U(s_i, \delta(s_i)),\ i = 1, \ldots, r.$$

Remark 3.2. Let $D_{II} = \{(s_i, B_i) : i = 1, \ldots, r\}$ be a δ-fine partition of type II. Then simply taking $s_{i,1} = s_{i,2} = \cdots = s_{i,p} = s_i$ we get a δ-fine partition of type I: $D_I = \{(s_{i,1} \ldots s_{i,p}), B_i),\ i = 1, \ldots, r\}$.

The following lemma is a variant of Cousin's Lemma (see for example [7]) and can be proved in a similar way as Lemma 4.1 in [4].

Lemma 3.1. *Let B be an n-dimensional block. For any gauge δ on B there exists at least one δ-fine M_p-partition D_I of B.*

McShane-Stieltjes sums. Throughout the rest of this section, $X_k,\ k = 1, \ldots, p$, $Y_l,\ l = 1, \ldots, q$ and Z will be linear normed spaces over the same field $\mathbb{R}$ or $\mathbb{C}$. Further, Z will be a Banach space,

$$\mathcal{A} : X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \to Z$$

will be a bounded multilinear operator and

$$f_k : B \to X_k,\ k = 1, \ldots, p,$$

$$g_l : B \to Y_l,\ l = 1, \ldots, q$$

will be vector-valued functions defined on a block $B \subset \mathbb{R}^n$.

We define two types of McShane-Stieltjes sums, I and II, which differ in number of associated points in each subblock.

Type I. Let $D_I = \{(\mathfrak{S}_i, B_i) : i = 1, \ldots, r\}$ be an M_p-partition of B where, for each i, $\mathfrak{S}_i = (s_{i,1}, \ldots, s_{i,p})$, $s_{i,k} \in B$, is a sequence of p points associated to the interval B_i. We substitute the associated points $s_{i,1} \ldots s_{i,p}$ into $f_1, \ldots, f_p$, and form the *Stieltjes sum of the first type* by

$$S_I(D_I) = \sum_{i=1}^{r} \mathcal{A}[f_1(s_{i,1}), \ldots, f_p(s_{i,p}), \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)].$$

Type II. With each block B_i we associate one point s_i and form the Stieltjes sum of the second type by

$$S_{II}(D_{II}) = \sum_{i=1}^{r} \mathcal{A}[f_1(s_i), \ldots, f_p(s_i), \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)].$$

In the above sums $\sigma_{g_l}(B_i)$ denotes the quasi volume of the block B_i associated to the function g_l.

Note again that in the above definition we do not require $s_{i,k} \in B_i$ or $s_i \in B_i$. In other words $s_{i,k}$ and s_i may lie outside B_i.

Definition 3.4. *(McShane-Stieltjes Multilinear Integral)* We say that the type I multilinear Stieltjes integral of $f_1, \ldots, f_p$ with respect to $\mathcal{A}$ and $g_1, \ldots, g_q$, exists in the *McShane* sense on B, if there exists a vector $J_I \in Z$ with the following property:

For every $\epsilon > 0$, there is a gauge δ on B such that for any δ-fine M_p-partition D_I we have

$$|J_I - S_I(D_I)| < \epsilon. \tag{3.1}$$

We then write

$$J_I = (MS_I) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q)$$

and occasionally we abbreviate this to $J_I = (MS_I) \int_B \mathcal{A}(F, dG)$.

Lemma 3.2. *The integral in Definition 3.4 is uniquely determined.*

Proof. Assume that $(MS_I) \int_B \mathcal{A}(F, dG)$ has two values, J_I and J_I'. Then for an arbitrary $\epsilon > 0$ there exist δ_1 and δ_2 satisfying the inequality in (3.1) Definition 3.4. However, then the gauge $\delta_3 = \min(\delta_1, \delta_2)$ is also δ_i-fine $(i = 1, 2)$. Lemma 3.1 then ensures the existence of a δ_3-fine partition D for which we have

$$|J_I - J_I'| \leq |J_I - S_I(D)| + |S_I(D) - J_I'| < 2\epsilon$$

which completes the proof. □

Definition 3.5. By changing type I Stieltjes sums to type II sums in the above definition we get the definition of the type II multilinear McShane-Stieltjes integral

$$J_{II} = (MS_{II}) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q).$$

Remark 3.3. It is clear that the existence of the integral J_I implies the existence of J_{II}. Throughout the rest of this article we deal only with McShane-Stieltjes integrals of type I. Therefore, instead of $(MS_I) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q)$ and $S_I(D_I)$, we just write (MS) $\int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q)$ and $S(D)$.

Lemma 3.3. *(Cauchy Criterion) The integral*

$$J = (MS) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q)$$

exists if and only if for every $\epsilon > 0$ there exists a gauge δ on B, such that for any δ-fine M_p-partitions $D^{'}$ and $D^{''}$ of the block B we have

$$|S(D^{'}) - S(D^{''})| < \epsilon.$$

Proof. This lemma can be proved exactly as the corresponding lemma for Henstock-Kurzweil integrals, see [4]. □

The Cauchy test is simplified considerably if we have interval additivity, then we can have the same subblocks of the two partitions (however with differing tags) as we show in the next lemma.

Lemma 3.4. *The integral*

$$J = (MS) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q)$$

exists if the two following conditions hold:

1. *(Interval additivity) If $\{C_1, \ldots, C_m\}$ is a partition of a block $C \subseteq B$ and $s_1, \ldots, s_p \in C$ then*

$$\mathcal{A}[f_1(s_1), \ldots, f_p(s_p), \sigma_{g_1}(C), \ldots, \sigma_{g_q}(C)] = \sum_{i=1}^{m} \mathcal{A}[f_1(s_1), \ldots, f_p(s_p), \sigma_{g_1}(C_i), \ldots, \sigma_{g_q}(C_i)].$$

2. *For every $\epsilon > 0$ there exists a gauge δ on B, such that for any δ-fine M_p-partitions $D^{'} = \{(\mathfrak{S}_1, B_1), \ldots, (\mathfrak{S}_r, B_r)\}$ and $D^{''} = \{(\mathfrak{T}_1, B_1), \ldots, (\mathfrak{T}_r, B_r)\}$ we have*

$$|S(D^{'}) - S(D^{''})| < \epsilon.$$

Proof. Let $\epsilon > 0$ be arbitrary and suppose δ satisfies the second condition of the lemma. Choose two arbitrary δ-fine partitions of B:

$$P = \{(\mathfrak{U}_1, B_1'), \ldots, (\mathfrak{U}_s, B_s')\} \text{ , where } \mathfrak{U}_i = (u_{i,1}, \ldots, u_{i,p}), \text{ and}$$

$$Q = \{(\mathfrak{V}_1, B_1''), \ldots, (\mathfrak{V}_t, B_t'')\}, \text{ where } \mathfrak{V}_j = (v_{j,1}, \ldots, v_{j,p}).$$

Now set

$$A_{i,j} = B_i' \cap B_j'', \quad \mathfrak{W}_{i,j} = \mathfrak{U}_i, \quad \mathfrak{X}_{i,j} = \mathfrak{V}_j$$

for $i = 1, \ldots, s$ and $j = 1, \ldots, t$. Then, since the tags of a McShane-tagged partition do not have to reside inside their associated blocks,

$$D' = \{(\mathfrak{W}_{i,j}, A_{i,j}) : \quad i = 1, \ldots, s, \quad j = 1, \ldots, t\}, \text{ and}$$

$$D'' = \{(\mathfrak{X}_{i,j}, A_{i,j}) : \quad i = 1, \ldots, s; \quad j = 1, \ldots, t\}$$

are also M_p-partitions of B that are δ-fine. Hence the estimate $|S(D') - S(D''))\})| < \epsilon$ holds according to the second prerequisite of this lemma. Now $S(D')$ is given by

$$\sum_{i=1}^{s} \sum_{j=1}^{t} \mathcal{A}[f_1(s_{i,j,1}), \ldots, f_p(s_{i,j,p}), \sigma_{g_1}(A_{i,j}), \ldots, \sigma_{g_q}(A_{i,j})] \tag{3.2}$$

where $s_{i,j,1} \ldots, s_{i,j,p}$ denotes the tags associated with block $A_{i,j}$ in D'. However, by the construction of D', we have

$$(s_{i,j,1} \ldots, s_{i,j,p}) = (u_{i,1}, \ldots, u_{i,p}).$$

Note that all terms in the sum in (3.2) for which $A_{i,j}$ has empty interior vanish. Expression (3.2) is therefore equal to

$$\sum_{i=1}^{s} \sum_{j=1}^{t} \mathcal{A}[f_1(u_{i,1}), \ldots, f_p(u_{i,p}), \sigma_{g_1}(A_{i,j}), \ldots, \sigma_{g_q}(A_{i,j})]$$

$$= \sum_{i=1}^{s} \mathcal{A}[f_1(u_{i,1}), \ldots, f_p(u_{i,p}), \sigma_{g_1}(B_i'), \ldots, \sigma_{g_q}(B_i')] = S(P),$$

where we have used the interval additivity of $\mathcal{A}$ with respect to $\sigma_{g_1}, \ldots, \sigma_{g_q}$. Similarly one proves $S(D'') = S(Q)$. Hence

$$|S(P) - S(Q)| = |S(D') - S(D'')| < \epsilon$$

and the integral exists by the Cauchy Criterion (Lemma 3.3). □

We formulate a proposition regarding the standard properties of the multilinear McShane-Stieltjes integral. The proof follows directly from the definition and can be carried out in a way analogous to the proof of the corresponding proposition for multilinear Henstock-Kurzweil Stieltjes integrals in $\mathbb{R}^n$ in [4].

Proposition 3.1. *The following properties hold for the MS-integral.*

a) *Let $f_k = f_{k,a} + f_{k,b}$ for some $k, 1 \leq k \leq p$. If the integrals*

$$J_a = (MS) \int_B \mathcal{A}(f_1, \ldots, f_{k,a}, \ldots, f_p, dg_1, \ldots, dg_q)$$

and

$$J_b = (MS) \int_B \mathcal{A}(f_1, \ldots, f_{k,b}, \ldots, f_p, dg_1, \ldots, dg_q)$$

exist then the integral

$$J = (MS) \int_B \mathcal{A}(f_1, \ldots, f_k, \ldots, f_p, dg_1, \ldots, dg_q)$$

also exists and $J = J_a + J_b$.

b) *Let $g_l = g_{l,a} + g_{l,b}$ for some $l, 1 \leq l \leq q$. If the integrals*

$$J_a = (MS) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_{l,a}, \ldots, dg_q)$$

and

$$J_b = (MS) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_{l,b}, \ldots, dg_q)$$

exist then the integral

$$J = (MS) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_l, \ldots, dg_q)$$

also exists and $J = J_a + J_b$.

c) *Let* λ *be a real (or complex) number. If the integral*

$$(MS)\int_B \mathcal{A}(f_1,\dots,f_p,dg_1,\dots,dg_q)$$

exists then

c_1) *the integral*

$$(MS)\int_B \mathcal{A}(f_1,\dots,\lambda f_k,\dots,f_p,dg_1,\dots,dg_q)$$

also exists and is equal to

$$= \lambda\cdot(MS)\int_B \mathcal{A}(f_1,\dots,f_k,\dots,f_p,dg_1,\dots,dg_q)$$

and

c_2) *the integral*

$$(MS)\int_B \mathcal{A}(f_1,\dots,f_p,dg_1,\dots,d(\lambda g_l),\dots,dg_q)$$

exists and is equal to

$$= \lambda\cdot(MS)\int_B \mathcal{A}(f_1,\dots,f_p,dg_1,\dots,dg_l,\dots,dg_q).$$

d) *Let*

$$\mathcal{A}_1,\mathcal{A}_2 : X_1\times\cdots\times X_p\times Y_1\cdots\times Y_q\to Z$$

be bounded multilinear operators and let λ_1,λ_2 *be real (or complex) numbers. If the integrals*

$$(MS)\int_B \mathcal{A}_1(F,dG) \qquad \text{and} \qquad (MS)\int_B \mathcal{A}_2(F,dG)$$

exist then $(MS)\int_B(\lambda_1\mathcal{A}_1+\lambda_2\mathcal{A}_2)(F,dG)$ *exists and is equal to*

$$\lambda_1\cdot(MS)\int_B \mathcal{A}_1(F,dG)+\lambda_2\cdot(MS)\int_B \mathcal{A}_2(F,dG).$$

Remark 3.4. All properties *a)-d)* in Proposition 3.1 also hold for the type II integrals.

4. Semivariation and Convergence Theorems

Definition 4.1. Let $\mathcal{A} : X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \to Z$ be a multilinear operator. We say that $g_1, \ldots, g_q$ are of *bounded semivariation on* B *with respect to* $\mathcal{A}$ if there exists a constant M_s such that for any partition $\Pi = \{B_1, \ldots, B_r\}$ of B and arbitrary vectors $x_{i,k} \in X_j$, $i = 1, \ldots, r$, $k = 1, \ldots, p$ the following holds

$$|\sum_{i=1}^{r} \mathcal{A}[x_{i,1}, \ldots, x_{i,p}, \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)]| \le M_s \cdot K_1 \cdot K_2 \cdots K_p$$

where

$$K_k = max\{|x_{i,k}| : i = 1, \ldots, r\}.$$

Let $G = (g_1, \ldots, g_q)$. The *semivariation of* $g_1, \ldots, g_q$ *with respect to* $\mathcal{A}$ *on* B, denoted by $SV(G, \mathcal{A}, B)$, is then defined as

$$\sup\{|\sum_{i=1}^{r} \mathcal{A}[x_{i,1}, \ldots, x_{i,p}, \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)]|\},$$

where $P = \{B_i\}$ varies over all partitions of B and $x_{i,k}$ varies over all $x_{i,k} \in X_k$, with $|x_{i,k}| \le 1$ $(k = 1, \ldots, p)$.

For arbitrary vectors $x_{i,k} \in X_k$, using the definition of semivaritation, we get

$$\begin{aligned}
&|\sum_{i=1}^{r} \mathcal{A}[x_{i,1}, \ldots, x_{i,p}, \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)]| \\
&\quad = K_1 \cdot K_2 \cdots K_p |\sum_{i=1}^{r} \mathcal{A}[\frac{x_{i,1}}{K_1}, \ldots, \frac{x_{i,p}}{K_p}, \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)]| \\
&\quad \le SV(G, \mathcal{A}, B) \cdot K_1 \cdot K_2 \cdots K_p.
\end{aligned}$$

Thus, $g_1, \ldots, g_q$ are of bounded semivariation on B with respect to $\mathcal{A}$ if and only if $SV(G, \mathcal{A}, B) < \infty$.

Remark 4.1. Clearly $SV(G, \mathcal{A}, C) \le SV(G, \mathcal{A}, B)$ for any subblock $C \subseteq B$.

The following theorem shows that bounded functions $g_1, \ldots, g_q$ are of bounded semivariation if one of their Stieltjes quasi volumes has bounded variation.

Proposition 4.1. *Let X_k, $k = 1,\dots,p$, Y_l, $l = 1,\dots,q$ be linear normed spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ and let Z be a Banach space over the same field. Let $\mathcal{A} : X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \to Z$ be a bounded multilinear operator. Let $g_l : B \to Y_l$, $l = 1,\dots,q$ be functions defined on B. Assume that the following hold:*

1. *The functions $g_1,\dots,g_q$ are bounded on B.*
2. *At least one of the Stieltjes quasi volumes σ_{g_l} is of bounded variation on B.*

Then the functions $g_1,\dots,g_q$ are of bounded semivariation on B with respect to $\mathcal{A}$.

Proof. Assume that the Stieltjes quasi volume σ_{g_q} is of bounded variation on B, that is, $V_B(\sigma_{g_q}) < \infty$. Since the functions $g_1,\dots,g_q$ are bounded on B the same is valid for their quasi volumes. Thus, there is a constant $M < \infty$ such that $|\sigma_{g_l}| \le M$, for $l = 1,\dots,q$.

From definition (4.1) we immediately get

$$SV(G,\mathcal{A},B) \le \sup\{\sum_{i=1}^{r} |\mathcal{A}||x_{i,1}|\cdots|x_{i,p}||\sigma_{g_1}(B_i)|\cdots|\sigma_{g_q}(B_i)|\}$$

where the supremum is taken over all partitions $\{B_i\}$ of B and all $x_{i,k} \in X_k$, with $|x_{i,k}| \le 1$ $(k = 1,\dots,p)$. This is then less than

$$\sup\{\sum_{i=1}^{r} |\mathcal{A}| \cdot 1 \cdots 1 \cdot M \cdots M|\sigma_{g_q}(B_i)|\}$$
$$= |\mathcal{A}|M^{q-1} \sup\{\sum_{i=1}^{r} |\sigma_{g_q}(B_i)|\} = |\mathcal{A}|M^{q-1}V_B(\sigma_{g_q}) < \infty. \quad \square$$

Proposition 4.2. *Let X_k, $k = 1,\dots,p$, Y_l, $l = 1,\dots,q$ be linear normed spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ and let Z be a Banach space over the same field. Let $\mathcal{A} : X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \to Z$ be a bounded multilinear operator. Let $f_k : B \to X_k, k = 1,\dots,p$ and $g_l : B \to Y_l$, $l = 1,\dots,q$ be functions defined on B. Assume that the following hold:*

1. *The functions $f_k : B \to X_k$ are bounded on B.*
2. *The functions $g_1,\dots,g_q$ are of bounded semivariation on B with respect to $\mathcal{A}$.*

3. *The integral* $J = (MS)\int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q)$ *exists.*

Then

$$J = (MS)\int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q) \leq K_1 \cdots K_p \cdot SV(G, A, B),$$

where $K_k = \sup\{|f_k(x)| : x \in B\}$.

Proof. The assertion follows immediately from

$$|\sum_{i=1}^{r} \mathcal{A}[f_1(s_1), \ldots, f_p(s_p), \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)]| \\ \leq K_1 \cdots K_p \cdot SV(G, \mathcal{A}, B). \quad \square$$

In the following theorem we prove the existence of the MS-integral for the case when the integrands are continuous and the integrators are of bounded semivariation.

Theorem 4.1. *Let* $X_k,\ k = 1, \ldots, p,\ Y_l,\ l = 1, \ldots, q$ *be linear normed spaces over the same field* $\mathbb{R}$ *or* $\mathbb{C}$ *and let* Z *be a Banach space over the same field. Let* $\mathcal{A} : X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \to Z$ *be a bounded multilinear operator. Let* $f_k : B \to X_k, k = 1, \ldots, p$ *and* $g_l : B \to Y_l,\ l = 1, \ldots, q$ *be functions defined on* B*. Assume that the following hold:*

1. *(Interval additivity.) If* $\{C_1, \ldots, C_m\}$ *is a partition of a block* $C \subseteq B$ *and* $s_1, \ldots, s_p \in C$ *then*

$$\mathcal{A}[f_1(s_1), \ldots, f_p(s_p), \sigma_{g_1}(C), \ldots, \sigma_{g_q}(C)] \\ = \sum_{i=1}^{m} \mathcal{A}[f_1(s_1), \ldots, f_p(s_p), \sigma_{g_1}(C_i), \ldots, \sigma_{g_q}(C_i)].$$

2. *The functions* $g_1, \ldots, g_q$ *are of bounded semivariation on* B *with respect to* $\mathcal{A}$*.*

3. *The functions* $f_1, \ldots, f_p$ *are continuous on* B*.*

Then

a) *The integral*

$$J = (MS)\int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q)$$

exists.

b) *Given $\epsilon > 0$. There exists $\delta(\epsilon) > 0$ such that, for any $\delta(\epsilon)$-fine tagged M_p-partition D of the block B and associated Stieltjes sum $S(D)$, we have*

$$|J - S(D)| \leq \epsilon \cdot M \cdot SV(G, \mathcal{A}, B) \tag{4.1}$$

where $M = 1$ if $p = 1$, otherwise, if $p \geq 2$, we have

$$M = K_2 \cdots K_p + \cdots + K_1 \cdots K_{p-1}$$

where

$$K_k = \sup\{|f_k(s)| : s \in B\}, k = 1, \ldots, p.$$

Proof. We prove this for $p \geq 2$. *a)* Since B is a compact set, the functions $f_1, \ldots, f_p$ are uniformly continuous and bounded on B. Given $\epsilon > 0$ we can therefore find δ_1 such that for $k = 1, \ldots, p$

$$\{x, y \in B \text{ and } |x - y| < \delta_1\} \Rightarrow |f_k(x) - f_k(y)| < \epsilon.$$

We shall prove the existence of the integral using Lemma (3.4) and therefore it is enough to consider two tagged partitions of B with the same subblocks. Let $\delta(\epsilon) = \delta_1/2$. Choose two arbitrary $\delta(\epsilon)$-fine M_p-partitions

$$D^{'} = \{(\mathfrak{S}_1, B_1), \ldots, (\mathfrak{S}_r, B_r)\}, \text{ where } \mathfrak{S}_i = (s_{i,1}, \ldots, s_{i,p}),$$

and

$$D^{''} = \{(\mathfrak{T}_1, B_1), \ldots, (\mathfrak{T}_r, B_r)\}, \text{ where } \mathfrak{T}_i = (t_{i,1}, \ldots, t_{i,p}),$$

and consider the difference between their Stieltjes sums,

$$\begin{aligned} S(D^{'}) - S(D^{''}) = &\sum_{i=1}^{r} \mathcal{A}[f_1(s_{i,1}), \ldots, f_p(s_{i,p}), \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)] \\ &- \sum_{i=1}^{r} \mathcal{A}[f_1(t_{i,1}), \ldots, f_p(t_{i,p}), \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)]. \end{aligned}$$

Denote $\sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)$ by σ_i. Then write $S(D^{'}) - S(D^{''})$ using a telescoping sum $S_1 + S_2 + \cdots + S_p$, where

$$\begin{aligned} S_1 = &\sum_{i=1}^{r} \mathcal{A}[f_1(s_{i,1}), \ldots, f_p(s_{i,p}), \sigma_i] \\ &- \sum_{i=1}^{r} \mathcal{A}[f_1(t_{i,1}), f_2(s_{i,2}), \ldots, f_p(s_{i,p}), \sigma_i] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \mathcal{A}[f_1(s_{i,1}) - f_1(t_{i,1}), f_2(s_{i,2}), \dots, f_p(s_{i,p}), \sigma_i], \\
S_2 &= \sum_{i=1}^{r} \mathcal{A}[f_1(t_{i,1}), f_2(s_{i,2}), \dots, f_p(s_{i,p}), \sigma_i] \\
&\quad - \sum_{i=1}^{r} \mathcal{A}[f_1(t_{i,1}), f_2(t_{i,2}), \dots, f_p(s_{i,p}), \sigma_i] \\
&= \sum_{i=1}^{r} \mathcal{A}[f_1(t_{i,1}), f_2(s_{i,2}) - f_2(t_{i,2}), \dots, f_p(s_{i,p}), \sigma_i], \\
&\ \vdots \\
S_p &= \sum_{i=1}^{r} \mathcal{A}[f_1(t_{i,1}), f_2(t_{i,2}), \dots, f_p(s_{i,p}), \sigma_i] \\
&\quad - \sum_{i=1}^{r} \mathcal{A}[f_1(t_{i,1}), f_2(t_{i,2}), \dots, f_p(t_{i,p}), \sigma_i] \\
&= \sum_{i=1}^{r} \mathcal{A}[f_1(t_{i,1}), f_2(t_{i,2}), \dots, f_p(s_{i,p}) - f_p(t_{i,p}), \sigma_i].
\end{aligned}
$$

Using the second assumption (bounded semivariation) of this theorem we have

$$
\begin{gathered}
|S_1| \leq \epsilon \cdot K_2 \cdots K_p \cdot SV(G, \mathcal{A}, B) \\
\vdots \\
|S_p| \leq \epsilon \cdot K_1 \cdots K_{p-1} \cdot SV(G, \mathcal{A}, B).
\end{gathered}
$$

where $K_k = \sup\{|f_k(s)|, s \in B\}, k = 1, \dots, p$. Thus

$$
\begin{aligned}
|S(D^{'}) - S(D^{''})| =& |S_1 + S_2 + \cdots + S_p| \\
\leq& |S_1| + |S_2| + \cdots + |S_p| \leq \epsilon \cdot M \cdot SV(G, \mathcal{A}, B),
\end{aligned} \tag{4.2}
$$

where

$$
M = K_2 \cdots K_p + \cdots + K_1 \cdots K_{p-1}.
$$

Now Lemma 3.4 and (4.2) imply the existence of the integral

$$
J = (MS) \int_B \mathcal{A}(f_1, \dots, f_p, dg_1, \dots, dg_q).
$$

b) Given $\epsilon > 0$. In the same way as in *a)* we choose $\delta = \delta_1/2$ such that for $k = 1, \dots, p$

$$\{x, y \in B \text{ and } |x - y| < \delta_1\} \Rightarrow |f_k(x) - f_k(y)| < \epsilon.$$

Then we define $\delta(\epsilon) = \delta_1/2$. We shall show that for any $\delta(\epsilon)$-fine M_p-partition D of the block B and associated Stieltjes sum $S(D)$, we have $|J - S(D)| \leq \epsilon \cdot M \cdot SV(G, \mathcal{A}, B)$.

If E is another $\delta(\epsilon)$-fine tagged M_p-partition then, according to (4.2), we have

$$|S(D) - S(E)| \leq \epsilon \cdot M \cdot SV(G, \mathcal{A}, B). \tag{4.3}$$

Let $\eta > 0$. Since the integral J exists, we can find $\delta(\eta) > 0$, such that, if E is $\delta(\eta)$-fine, we have

$$|J - S(E)| < \eta. \tag{4.4}$$

We can assume that $\delta(\eta) \leq \delta(\epsilon)$, otherwise we can take a new gauge $\min(\delta(\eta), \delta(\epsilon))$. Thus any $\delta(\eta)$-fine tagged partition E is also $\delta(\epsilon)$-fine. Then, using (4.3) and (4.4) we have

$$|J - S(D)| \leq |J - S(E)| + |S(E) - S(D)| < \eta + \epsilon \cdot M \cdot SV(G, \mathcal{A}, B).$$

Since $\eta > 0$ was arbitrary we have

$$|J - S(D)| \leq \epsilon \cdot M \cdot SV(G, \mathcal{A}, B). \qquad \square$$

Remark 4.2. Note that $\delta(\epsilon)$ in Theorem 4.1 *b)* depends on ϵ and the functions f_k but $\delta(\epsilon)$ does not depend on the integrators g_l. We shall use this fact in Theorem 4.3.

The next two theorems are convergence theorems. The second theorem, in the case of ordinary Riemann-Stieltjes integral, is sometimes called Helly's theorem. (See for example [8].)

Theorem 4.2. *Let X_k, $k = 1, \dots, p$, Y_l, $l = 1, \dots, q$ be linear normed spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ and let Z be a Banach space over the same field. Let $\mathcal{A} : X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \to Z$ be a bounded multilinear operator. Let $f_k : B \to X_k, k = 1, \dots, p$ and $g_l : B \to Y_l$, $l = 1, \dots, q$ be functions defined on B. Assume that the following hold:*

1. *The functions $g_1, \dots, g_q$ are of bounded semivariation on B with respect to $\mathcal{A}$.*

2. *$f_k^m : B \to X_k$ are sequences of bounded functions which converge uniformly to $f_k, k = 1, \dots, p$, as $m \to \infty$.*

3. The following MS-integrals exist

$$J = (MS)\int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q),$$

$$J_m = (MS)\int_B \mathcal{A}(f_1^m, \ldots, f_p^m, dg_1, \ldots, dg_q),$$

Then

$$\lim_{m\to\infty} J_m = J$$

Proof. We shall prove this in the case of two integrands, f_1,f_2. The general proof can be carried out using a telescoping sum, as in the proof of the previous theorem.

Let $\epsilon > 0$ be arbitrary. As f_1^m and f_2^m converge uniformly to f_1 and f_2 respectively, we can choose N so that $|f_k^m(s) - f_k(s)| < \epsilon$ $(k = 1, 2)$ for all $s \in B$ whenever $m > N$. Additionally, uniform convergence together with boundedness, implies the existence of a bound M_f on B for all integrands.

Let m be an arbitrary integer greater than N.

As the integrals J and J_m exist there are gauges δ_1 and δ_2 such that

$$|S(P) - J| < \epsilon \quad \text{and} \quad |S^m(Q) - J_m| < \epsilon$$

whenever P is a δ_1-fine M_p-partition and Q is a δ_2-fine M_p-partition.

Here $S(P)$ denotes the McShane-Stieltjes sum belonging to the M_p-partition P for $\{f_1, f_2\}$. Similarly, $S^m(Q)$ denotes the sum for $\{f_1^m, f_2^m\}$. Let $\delta = min(\delta_1, \delta_2)$. Clearly any δ-fine M_p-partition must be both δ_1-fine and δ_2-fine so let $D = \{((s_{i,1}, s_{i,2}), B_i) : i = 1, \ldots, r\}$ be such a δ-fine M_p-partition. Then $|S(D) - J| < \epsilon$ and $|S^m(D) - J_m| < \epsilon$.

Consequently

$$\begin{aligned} |J - J_m| \le& |J - S(D)| + |S^m(D) - S(D)| + |J_m - S^m(D)| \\ <& \epsilon + |S^m(D) - S(D)| + \epsilon. \end{aligned} \tag{4.5}$$

To simplify notation we again denote $\sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)$ by σ_i. Then we have

$$\begin{aligned} |S^m(D) - S(D)| =& \left| \sum_{i=1}^{r} \left(\mathcal{A}[f_1^m(s_{i,1}), f_2^m(s_{i,2}), \sigma_i] - \mathcal{A}[f_1(s_{i,1}), f_2(s_{i,2}), \sigma_i]\right) \right| \\ =& \left| \sum_{i=1}^{r} \left(\mathcal{A}[f_1^m(s_{i,1}), f_2^m(s_{i,2}), \sigma_i] - \mathcal{A}[f_1(s_{i,1}), f_2^m(s_{i,2}), \sigma_i]\right) \right. \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{r}(\mathcal{A}[f_1(s_{i,1}),f_2^m(s_{i,2}),\sigma_i]-\mathcal{A}[f_1(s_{i,1}),f_2(s_{i,2}),\sigma_i])\,|\\
\le&|\sum_{i=1}^{r}\mathcal{A}[f_1^m(s_{i,1})-f_1(s_{i,1}),f_2^m(s_{i,2}),\sigma_i]|\\
&+|\sum_{i=1}^{r}\mathcal{A}[f_1(s_{i,1}),f_2^m(s_{i,2})-f_2(s_{i,2}),\sigma_i]|\\
\le&\epsilon\cdot M_f\cdot SV(G,\mathcal{A},B)+M_f\cdot\epsilon\cdot SV(G,\mathcal{A},B).
\end{aligned}
$$

This estimate, together with (4.5), yields

$$|J-J_m|<\epsilon+2\epsilon\cdot M_f\cdot SV(G,\mathcal{A},B)+\epsilon$$

for all integers $m>N$. As ϵ was arbitrary, the proof is complete.

□

Theorem 4.3. *Let $X_k,\ k=1,\dots,p,\ Y_l,\ l=1,\dots,q$ be linear normed spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ and let Z be a Banach space over the same field. Let $\mathcal{A}:X_1\times\cdots\times X_p\times Y_1\times\cdots\times Y_q\to Z$ be a bounded multilinear operator. Let $f_k:B\to X_k, k=1,\dots,p$ and $g_l^m:B\to Y_l,\ l=1,\dots,q,\quad m=1,2,\dots$ be functions defined on B. Assume that the following hold:*

1. *The functions $f_k:B\to X_k$ are continuous.*
2. *For each $l=1,\dots,q$ and $x\in B$, $\lim_{m\to\infty}g_l^m(x)=g_l(x)$.*
3. *For each m, the functions $g_1^m,\dots,g_q^m$ are of bounded semivariation $SV(G^m,\mathcal{A},B)$. Furthermore, there exists a constant $L<\infty$ such that*

$$SV(G^m,\mathcal{A},B)\le L$$

for all m. (G^m denotes $(g_1^m,\dots,g_q^m)$.)

4. *(Interval additivity) If $\{C_1,\dots,C_r\}$ is a partition of a block $C\subseteq B$ and $s_1,\dots,s_p\in C$ then*

$$
\begin{aligned}
&\mathcal{A}[f_1(s_1),\dots,f_p(s_p),\sigma_{g_1^m}(C),\dots,\sigma_{g_q^m}(C)]\\
&\qquad=\sum_{i=1}^{r}\mathcal{A}[f_1(s_1),\dots,f_p(s_p),\sigma_{g_1^m}(C_i),\dots,\sigma_{g_q^m}(C_i)].
\end{aligned}
$$

Then the following MS-integrals

$$J = (MS) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1, \ldots, dg_q),$$

$$J_m = (MS) \int_B \mathcal{A}(f_1, \ldots, f_p, dg_1^m, \ldots, dg_q^m)$$

exist and and

$$\lim_{m\to\infty} J_m = J.$$

Proof. Let $\{B_i\}$ be a partition of B. Condition 3 written out for arbitrary vectors $x_{i,k} \in X_k$, with $|x_{i,k}| \leq 1$, reads

$$|\sum_{i=1}^{r} \mathcal{A}[x_{i,1}, \ldots, x_{i,p}, \sigma_{g_1^m}(B_i), \ldots, \sigma_{g_q^m}(B_i)]| \leq SV(G^m, \mathcal{A}, B) \leq L$$

$(m = 1, 2, \ldots)$ for all j. Since $\sigma_{g_j^m}(B_i)$ are linear combinations of expressions of the type $g_j^m(\gamma)$ where γ are vertices of the blocks of the subblocks of the partition, and as the linear operator $\mathcal{A}$ is continuous (since it is bounded), we can simply let $m \to \infty$ to obtain

$$|\sum_{i=1}^{r} \mathcal{A}[x_{i,1}, \ldots, x_{i,p}, \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)]| \leq L.$$

Hence

$$SV(G, \mathcal{A}, B) = \sup |\sum_{i=1}^{r} \mathcal{A}[x_{i,1}, \ldots, x_{i,p}, \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)]| \leq L.$$

Similarly, if $m \to \infty$ in condition 4, we get the interval additivity for $\sigma_{g_1}, \ldots, \sigma_{g_q}$:

$$\mathcal{A}[f_1(s_1), \ldots, f_p(s_p), \sigma_{g_1}(C), \ldots, \sigma_{g_q}(C)] = \sum_{i=1}^{r} \mathcal{A}[f_1(s_1), \ldots, f_p(s_p), \sigma_{g_1}(C_i), \ldots, \sigma_{g_q}(C_i)].$$

The integrals J_m and J exist since the integrands $f_k : B \to X_k$ are continuous functions and the integrators are of bounded semivariation with respect to $\mathcal{A}$ (Theorem 4.1).

It remains to show that $\lim_{m\to\infty} J_m = J$. As B is compact, given $\epsilon > 0$ we can find δ_1 such that, for $k = 1, \ldots, p$,

$$\{x, y \in B \text{ and } |x - y| < \delta_1\} \Rightarrow |f_k(x) - f_k(y)| < \epsilon.$$

Let $\delta = \delta_1/2$ and let $D = \{(s_i, B_i),\ i = 1, \ldots, r\}$ be a fixed δ-fine partition. We extend the notation for Stieltjes sums slightly and write

$$S(D, G) = \sum_{i=1}^{r} \mathcal{A}[f_1(s_i), \ldots, f_p(s_i), \sigma_{g_1}(B_i), \ldots, \sigma_{g_q}(B_i)].$$

Then we have

$$J = (J - S(D, G)) + S(D, G), \tag{4.6}$$

where, according to inequality (4.1) in Theorem (4.1) and condition 3 of this theorem, we obtain

$$|J - S(D, G)| < \epsilon \cdot M \cdot SV(G, \mathcal{A}, B) \leq \epsilon \cdot M \cdot L. \tag{4.7}$$

Similarly, with

$$S(D, G^m) = \sum_{i=1}^{r} \mathcal{A}[f_1(s_i), \ldots, f_p(s_i), \sigma_{g_1^m}(B_i), \ldots, \sigma_{g_q^m}(B_i)]$$

we have

$$J_m = (J_m - S(D, G^m)) + S(D, G^m), \tag{4.8}$$

where

$$|J_m - S(D, G^m)| < \epsilon \cdot M \cdot SV(G^m, \mathcal{A}, B) \leq \epsilon \cdot M \cdot L. \tag{4.9}$$

Now from (4.6) - (4.9) we get

$$\begin{aligned} |J - J_m| \leq & |J - S(D, G)| + |J_m - S(D, G^m)| + |S(D, G) - S(D, G^m)| \\ < & 2\epsilon \cdot M \cdot L + |S(D, G) - S(D, G^m)|. \end{aligned} \tag{4.10}$$

Since D is fixed $S(D, G) - S(D, G^m)$ is a finite sum formed with a continuous linear operator operating on values of vertices of blocks of D substituted into $g_1^m, \ldots, g_q^m$. This means that the convergence $g_l^m \to g_l$ as $m \to \infty$ implies $|S(D, G) - S(D, G^m)| \to 0$ as $m \to \infty$. Hence from (4.10) we have

$$\limsup_{m\to\infty} |J - J_m| \leq 2\epsilon M L,$$

which gives

$$\lim_{m\to\infty} |J - J_m| = 0.$$

□

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