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LÉVY-BASED HEATH-JARROW-MORTON INTEREST RATE DERIVATIVES: CHANGE OF TIME METHOD AND PIDES

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Abstract: In this paper, we show how to calculate the price of zero-coupon bonds Heath-Jarrow-Morton (HJM) Lévy model of forward interest rate $f(t,T)$ using the change of time method. We also derive partial integro-differential equations (PIDEs) for the values of swaps, caps, floors and options on them, swaptions, captions and floortions, respectively. We apply the change of time method to price the interest rate derivatives for the forward interest rates $f(t,T)$ described by the stochastic differential equation driven by α-stable Lévy processes.

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1. Introduction

Interest rate models are mainly used to price and hedge bonds and bond options. One of the biggest assumptions is that interest rates r are constant or, at least, known functions of time $r \equiv r(t)$. In reality this is far from the case. Many other securities that are influenced by interest rates have long duration. Their analysis in the presence of unpredictable interest rates is of crucial practical importance.

In this paper we will look at models for describing the stochastic behavior of interest rates. Typically, the classical models are often based on assumptions that returns in the bond market are approximately normally distributed (Gaussian). For comprehensive introduction to interest rate modeling we refer to [4], [8], [26], [35], [2] and [48].

Empirical studies show that the Gaussian assumptions of the classical models do not hold in general. We will describe the Lévy-based models first introduced in [25]. See also[46].

The main drawback of the Vasicek, the Cox-Ingersoll-Ross and other one-factor models lies in the fact that prices are explicit functions of the instantaneous 'spot' interest rate so that these models are unable to take the whole yield curve observed on the market into account in the price structure. Some authors have returned to a two-dimensional analysis to improve the models in terms of discrepancies between short and long rates. See, for example, [9], [52], [14]. These more complex models do not lead to explicit formula and require the solution of partial differential equations. The general concept of multi-factor affine-yield models is developed in [20] and [21]. The reduction of affine-yield models to canonical versions is due to [16]. [41] provides a detailed study of the one-dimensional CIR equation when the parameters are time-varying. Although affine-yield models have simple bond price formulas, the prices for fixed income derivatives are more complicated. However, numerical solutions of partial differential equations can be avoided by Fourier transform analysis (see [22]). There are many other common short rate models, such as in [5], [6], and [40]. An empirical comparison of various short rate models is provided in [11].

[32] proposes a discrete-time model describing the behavior of the whole yield curve. The continuous-time model is based on the same idea and has been introduced by Heath, Jarrow and Morton (HJM) [30]. They considered forward interest rates. [26] examines the issue of making the yield curves generated by the HJM model consistent with the scheme used to generate the initial yield curve. [38] considers an HJM-type model but for interest rate futures rather than forward interest rates.

The use of a log-normal simple interest rate to price caps and floors was worked out in [42]. This idea was embedded in a full forward LIBOR term-structure model in [7]. Recently, a variation on forward LIBOR models has been developed for swaps markets (see [37]). Term-structure models with jumps have been studied [3], [17], [18], [28], [29] and [53]. Three relatively recent books by authors with practical experience in term-structure modeling are [45], [8] and [48].

In this paper, we show how to calculate the price of zero-coupon bonds for many models of $r(t)$ using the change of time method, see [56] and [57]. These models include, in particular, the Ornstein-Uhlenbeck [44], Vasićek [54], Cox-Ingersoll-Ross [15], continuous-time GARCH, Ho-Lee [32], Hull-White [31] and Heath-Jarrow-Morton [30] models. In the same way, we can calculate the values of swaps, caps, floors and options on them, swaptions, captions and floortions, respectively. We also apply the change of time method to price the interest rate derivatives for the interest rates $r(t)$ described by various stochastic differential equations driven by α-stable Lévy processes, see Sato [51], Applebaum [1], Schoutens [55]. The change of time method is convenient for presentation of a solution of a SDE. Sometimes one can not explicitly present the solution and uses numerical approaches. For example, there was no explicit solution for the CIR SDE (7). In [56] the solution for this CIR SDE (7) was presented for the first time using the change of time method. By applying the change of time method we can use the presentation for the solution to calculate many financial derivatives, for example variance and volatility swaps, options for mean-reverting models, etc. (see [56] and [57]). The paper [58] incorporates the change of time method for pricing of interest rate derivatives for many Lévy-based stochastic interest rate one-factor and multi-factor models. Also, PIDEs derived for values of swaps, caps and floors and options on them, swaptions, captions and floortions.

The rest of the paper is organized as follows. In Section 2 we give an overview of the basics of bond pricing. One-factor and multi-factor stochastic HJM Lévy-based interest rate models are introduced in Section 3. The change of time method in Lévy settings is described in Section 4. Solution HJM Lévy model considered in Section 3 using the change of time method is also stated in Section 4. Bond pricing in Lévy setting using both the change of time method and partial integro-differential equations is studied in Section 5. Various classes of interest rate derivatives, such as bond options, swaps, caps and floors, and options on them, swaptions, captions and floortions, respectively, are introduced in Section 6. Pricing of Lévy HJM bond options is studied in Section 7. Pricing of swaps, caps and floors for Lévy HJM interest rate models is considered in

Section 8. Pricing of swaptions, captions and floortions in Lévy setting is studied in Section 9. Section 10 concludes the paper. Appendix A (Section 11) contains one-factor and multi-factor Gaussian interest rate models gives the solutions to these models using the change of time method.

2. Basics of Bond Pricing

2.1. Zero-Coupon Bonds

We begin with the subject of bond pricing. We do this under the assumption of a deterministic interest rate. This simplification allows us to discuss the effect of coupons on the prices of bonds and the appearance of the yield curve. A bond is a contract, paid for up-front, that yields a known amount on a known date in the future, the maturity date, $t = T$. The bond may also pay a known cash dividend (the coupon) at fixed times during the life of the contract. If there is no coupon the bond is known as a zero-coupon bond. In this way, we call zero-coupon bond a security paying \$1 at a maturity date T and we note $P(t,T)$ is the value of this security at time t. We have $P(T,T) = 1$ and in the world where the future is certain $P(t,T) = \exp(-\int_t^T r(s)ds)$, where $r(t)$ is the instantaneous interest rate.

2.2. Bond Pricing with Known Interest Rates

Let $V(t,T)$ be the value of the bond contract, $t < T$. If the interest rate $r(t)$ and coupon payment $K(t)$ are known functions of time, the bond price is also a function of time only: $V = V(t,T)$. If this bond pays the owner Z at time $t = T$ then we know that $V(T,T) = Z$. We now derive an equation for the value of the bond at a time before maturity, $t < T$.

Suppose we hold one bond. Then arbitrage considerations lead us to the following equation:

$$\frac{dV}{dt} + K(t) = r(t)V. \tag{1}$$

The right-hand side is the return we would have received had we converted our bond into cash at time t.

2.3. Yield Curve

Interest rates are not deterministic. For short-dated derivative products such as options, the errors associated with assuming a deterministic or even constant

rate are small, typically 2%. In dealing with products with a longer lifespan we must address the problem of random interest rates. The first step is to decide on a suitable measure for future values of interest rates, one that enables traders to communicate effectively about the same quantity. In the previous section we have seen a definition, (4), that gives an interest rate from bond price data but this relies on bond prices being differentiable with respect to the maturity date.

The yield curve is another measure of future values of interest rates. With the value of zero-coupon bonds $V(t,T)$ taken from real data, define

$$Y(t,T) = -\frac{\log(V(t,T)/V(T,T))}{T-t}, \tag{2}$$

where t is the current time.

The yield curve is the plot of Y against time to maturity $T-t$. The dependence of the yield curve on the time to maturity is called the term structure of interest rates.

2.4. The Short Rate

The short rate $r(t) := Y(t,t) = \lim_{T\to t} Y(t,T)$ is the rate on instantaneous borrowing and lending. Historically, it was the short rate which was modeled as the basic process. In practice, this rate is stochastic and can fluctuate over time. Note that the short rate is actually a theoretical entity which does not exist in real life and can not be directly observed. A sum of 1 invested in the short rate at time zero and continuously rolled over, i.e., instantaneously reinvested, is called the money-market account. Its value $S_0(t) = \exp[\int_0^t r(s)ds]$. If r is deterministic and constant, $S_0(t)$ reduces to the classical bank account: $S_0(t) = B(t) = \exp[rt]$.

2.5. The Instantaneous Forward Rate

We define the instantaneous forward rate to be

$$f(t,T) := -\frac{\partial \log P(t,T)}{\partial T}. \tag{3}$$

The function $f(t,T)$ corresponds to the rate we can contract for at time t on a riskless loan that begins at time T and is returned an instant later. Since

$$P(t,T) = \exp\left[-\int_t^T f(t,s)ds\right], \tag{4}$$

zero-coupon bond prices and forward rates represent equivalent information. Note that the short rate $r(t)$ is contained in this forward rate structure since $r(t) = f(t,t)$.

2.6. Stochastic Interest Rates

For an uncertain future, one must think of the interest rate $r(t)$ in terms of a random or stochastic process (short rate). We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ and filtration $\mathcal{F}_t$ is the natural filtration of a standard Brownian motion $(W_t)_{0\le t\le T}$ and $\mathcal{F}_T = \mathcal{F}$.

In view of our uncertainty about the future of the interest rate, it is natural to model it as a random variable. We should specify that r is the interest rate received by the shortest possible deposit. The interest rate for the shortest possible deposit is commonly called the spot rate.

We consider a model for the interest rate r that is governed by a SDE of the form

$$dr(t) = a(r,t)dt + b(r,t)dW(t), \tag{5}$$

where W is a standard Wiener process, or

$$dr(t) = a(r,t)dt + b(r,t)dL(t), \tag{6}$$

where L is a Lévy process, and a and b are 'good' functions.

A basic model for the behavior of the stochastic process r is the CIR model ([15]). It is based on the CIR process:

$$dr(t) = k(\theta - r(t))dt + \gamma\sqrt{r}dW(t). \tag{7}$$

Under this model, the interest rate is mean reverting: it fluctuates around a long-term mean θ. In the literature, this model was generalized in different ways. Well-known alternatives are the Vasicek and the Ho-Lee models, together with their extensions. We refer to [4].

We consider a 'riskless' asset $S_t^0 = \exp(\int_0^t r(s)ds)$, where $r(t)$ is an adapted process, such that $\int_0^T |r(s)|ds < +\infty$, almost surely. The risky assets are the zero-coupon bonds with maturity less than or equal to the horizon T. For each $u \le T$, we define an adapted process $P(t,u), 0 \le t \le u$, satisfying $P(u,u) = 1$, giving the price of the zero-coupon bond with maturity u as a function of time. We note that $P(t,u) = E^*[\exp(-\int_t^u r(s)ds)|\mathcal{F}_t]$, where E^* is an expectation with respect to the risk-neutral probability P^*.

We note, that for each maturity u, there is an adapted process $\sigma(t,u), 0 \leq t \leq u$, such that, on $[0,u]$,

$$\frac{dP(t,u)}{P(t,u)} = (r(t) - \sigma(t,u)q(t))dt + \sigma(t,u)dW(t), \tag{8}$$

where $q(t)$ is an adapted process corresponding to 'risk premium' and $W(t)$ is a Wiener process under physical measure P. For example, for the Vasićek model $q(t) = -\lambda, \lambda \in R$ and for the CIR model $q(t) = -\alpha\sqrt{r(t)}, \alpha \in R$.

The last formula is to be related with the equality $dS_0(t) = r(t)S_0(t)dt$, satisfied by the so-called riskless asset. It is the term in $dW(t)$ which makes the bond riskier.

Under risk-neutral probability P^*, the process $W^*(t)$ defined by $W^*(t) := W(t) - \int_0^t q(s)ds$ is a standard Brownian motion (Girsanov theorem), and we have

$$\frac{dP(t,u)}{P(t,u)} = r(t)dt + \sigma(t,u)dW^*(t). \tag{9}$$

(See [34])

3. Lévy-based HJM Stochastic Interest Rate Model (SIRMs)

Throughout the paper we set $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in R_+}, P)$ for a filtered probability space, Jacod and Shiryaev (1987).

In this section we give an overview of one-factor and multi-factor Lévy interest rate models.

3.1. One-Factor HJM Lévy SIRMs

Definition 1. By *Lévy process* we define a process with stationary and independent increments, [51], [1], [55].

Definiton 2. Let $\alpha \in (0,2]$. An *α-stable Lévy process* L such that L_1 (or equivalently any L_t) has a strictly α-stable distribution (i.e., $L_1 \equiv S_\alpha(\sigma,\beta,\nu)$) for some $\alpha \in (0,2] \setminus \{1\}$, $\sigma \in R_+$, $\beta \in [-1,1]$, $\nu = 0$ or $\alpha = 1$, $\sigma \in R_+$, $\beta = 0$, $\nu \in R$). (See [51]). We call L a *symmetric α-stable Lévy process* if the distribution of L_1 is even symmetric α-stable (i.e., $L_1 \equiv S_\alpha(\sigma,0,0)$ for some $\alpha \in (0,2]$, $\sigma \in R_+$.) A process L is called $(T_t)_{t\in R_+}$-adapted if L is constant on $[T_{t-}, T_t]$ for any $t \in R_+$.

$L(t)$ below is a symmetric α-stable Lévy process with triplet $(\gamma, \sigma, \nu(dy))$. Similar to Section 12.1 definitions of various processes defined via SDEs driven

by Brownian motions, we define below various processes via SDE driven by α-stable Lévy processes.

The Heath, Jarrow and Morton Process Driven by α-stable Lévy Motion: Define the forward interest rate $f(t,s)$, for $t \le s$, characterized by the following equality $P(t,u) = \exp[-\int_t^u f(t,s)ds]$ for any maturity u. $f(t,s)$ represents the instanteneous interest rate at time s as 'anticipated' by the market at time t. It is natural to set $f(t,t) = r(t)$. The process $f(t,u)_{0\le t\le u}$ satisfies an equation

$$f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dL(v),$$

where the processes a and b are continuous.

3.2. Multi-Factor HJM Lévy-based SIRMs

Multi-factor HJM Lévy-based models driven by α-stable Lévy motions can be obtained using various combinations of the above-mentioned processes, see Subsection 3.3. We give one example of a two-factor HJM Lévy-based model driven by α-stable Lévy motions.

$$\begin{cases} f(t,u) &= f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dL_1(v) \\ db(t) &= \xi b(t)dt + \eta b(t)dL^2(t), \end{cases} \tag{10}$$

where L^1, L^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Also, we can consider various combinations of models, presented above, i.e., mixed models containing Brownian and Lévy motions. For example,

$$\begin{cases} f(t,u) &= f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dL(v) \\ db(t) &= \xi b(t)dt + \eta b(t)dW(t), \end{cases} \tag{11}$$

where Brownian motion $W(t)$ and Lévy process $L(t)$ may be correlated.

Remark. Corresponding one-factor and multi-factor Gaussian interest rate models may be found in Appendix A.

4. Change of Time Method

Definition 3. A *time change* is a right-continuous increasing $[0,+\infty]$-valued process $(T_t)_{t\in R_+}$ such that T_t is a stopping time for any $t \in R_+$. By $\hat{\mathcal{F}}_t := \mathcal{F}_{T_t}$ we define the time-changed filtration $(\hat{\mathcal{F}}_t)_{t\in R_+}$. The *inverse time change* $(\hat{T}_t)_{t\in R_+}$ is defined as $\hat{T}_t := \inf\{s \in R_+ : T_s > t\}$. (See [33]).

4.1. Change of Time Method (CTM) for SDE driven by Lévy Motion

We denote by $L^{\alpha}_{a.s.}$ the family of all real measurable $\mathcal{F}_t$-adapted processes a on $\Omega \times [0, +\infty)$ such that for every $T > 0$, $\int_0^T |a(t,\omega)|^{\alpha} dt < +\infty$ a.s. We consider the following SDE driven by a Lévy motion:

$$dX(t) = a(t, X(t))dL(t), \tag{16}$$

with the same reason as above for the SDE with a Brownian motion as the driving process.

We are going to use the following result to get the solutions of SDEs driven by Lévy processes.

Theorem. (Rosiński and Woyczyński [50], Theorem 3.1., p. 277) *Let $a \in L^{\alpha}_{a.s.}$ be such that $T(u) := \int_0^u |a|^{\alpha} dt \to +\infty$ a.s. as $u \to +\infty$. If $\hat{T}(t) := \inf\{u : T(u) > t\}$ and $\hat{\mathcal{F}}_t = \mathcal{F}_{\hat{T}(t)}$, then the time-changed stochastic integral $\hat{L}(t) = \int_0^{\hat{T}(t)} a dL(t)$ is an $\hat{\mathcal{F}}_t - \alpha$-stable Lévy process, where $L(t)$ is $\mathcal{F}_t$-adapted and $\mathcal{F}_t$-α-stable Lévy process. Consequently, a.s. for each $t > 0$ $\int_0^t a dL = \hat{L}(T(t))$, i.e., the stochastic integral with respect to a α-stable Lévy process is nothing but another α-stable Lévy process with randomly changed time scale.*

Remark. CTM for Gaussian SIRM is considered in Appendix A.

4.2. Solution of One-Factor Lévy SIRMs Using CTM

Below we give the solutions to some one-factor Lévy SIRM decribed by SDE driven by α-stable Lévy process. $L(t)$ below is a symmetric α-stable Lévy process, and $\hat{L}$ is a $(\hat{T}_t)_{t\in R_+}$-adapted symmetric α-stable Levy processes on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t\in R_+}, P)$.

The Heath, Jarrow and Morton Driven by α-stable Lévy Motion: $f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dL(v)$. Solution $f(t,u) = f(0,u) + \hat{L}(\hat{T}_t) + \int_0^t a(v,u)dv$, where $\hat{T}_t = \int_0^t b^{\alpha}(f(0,u) + \hat{L}(\hat{T}_s) + \int_0^s a(v,u)dv)ds$.

Remark. Solutions of one-factor Gaussian SIRMs may be found in Appendix A.

5. Bond Pricing

Pricing a bond is harder than pricing an option even in the Gaussian case, since there is no underlying asset with which to hedge (you can not buy an

interest rate of, say, 15%). The only alternative is to hedge in the Gaussian case is the case with bonds of different maturity dates. To calculate the value of a European options with maturity θ on the zero-coupon bond with maturity equal to T we could proceed as follows. If it is a call with strike price K, the value of the option at time θ is $\max(P(\theta,T)-K),0)$ and it seems reasonable to hedge this call with a portfolio of riskless asset S_t^0 and zero-coupon bond $P(\theta,T)$ with maturity T. Using the classical arguments (and using the money-market account as numeraire), the price at time 0 of a European call option on a bond is given by

$$V(\theta,T,r)=E^*\left[\frac{1}{S_0(t)}\max(P(\theta,T)-K,0)\right], \tag{18}$$

where E^* is the expectation under the risk-neutral measure,

$$S_0(t)=\exp[\int_0^t r(s)ds],$$

and $P(\theta,T)$ is defined in Section 2.6.

5.1. Gaussian Bond Pricing for One-Factor SIRMs via CTM

The solution of the SDE (8) or (9) can be written in the form

$$P(t,T)=P(0,T)\exp\left[\int_0^T r(s)ds\right]\frac{\exp[\int_0^t\sigma(s,T)d\hat{W}(s)]}{E(\exp[\int_0^t\sigma(s,T)d\hat{W}(s)])}$$

$$=P(0,T)\exp\left[\int_0^T r(s)ds\right]\exp\left[\int_0^t\sigma(s,T)d\hat{W}(s)-\frac{1}{2}\int_0^t\sigma(s,T)^2ds\right], \tag{19}$$

where $r(t)$ has one of the representations in Section 5.1.

We see that the log returns under risk-neutral measure approximately follow a Normally distributed random variable.

Remark. Empirical studies (see, for example, [46]) show that the normality assumption does not reflect reality. Empirically observed log returns of bonds turn out to have a leptokurtic distribution.

5.2. Gaussian Bond Pricing for One-Factor SIRMs via PDE

Consider $V(t,T,r)$-bond price at time t, where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t)=a(r,t)dt+b(r,t)dW(t). \tag{20}$$

For example, for GBM, $a = \mu r$ and $b = \sigma r$, for OU process, $a = -\mu r$ and $b = \sigma$, and so on (see Section 3.1).

The zero-coupon bond pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial r^2} + (a - \lambda b)\frac{\partial V}{\partial r} - rV = 0, \tag{21}$$

where the function λ is often called the market price of risk. The final condition is

$$V(T, T, r) = Z.$$

This PDE may be solved approximately by standard numerical methods, see, for example, [59].

Remark. To price options on bonds with coupons, the reader is referred to [36].

5.3. Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform

We model the zero-coupon bond price with the following process (see [25], [46]):

$$P(t,T) = P(0,T)\exp\left[\int_0^T r(s)ds\right]\frac{\exp[\int_0^t \sigma(s,T)dL(s)]}{E(\exp[\int_0^t \sigma(s,T)dL(s)])},$$

where $r(t)$ has one of the representations in Section 5.3.

Eberlein and Raible (1999) derived the bond price process in the form

$$P(t,T) = P(0,T)\exp\left[\int_0^T r(s)ds\right]\frac{\exp[\int_0^t \sigma(s,T)dL(s)]}{\exp[\int_0^t \theta(\sigma(s,T))dL(s)]},$$

where $\theta(u) := \log(E[\exp(uL(1))])$ denotes the logarithm of the moment-generating function of the Lévy process at time 1. For example, in the classical Gaussian model we choose $\theta(u) = u^2/2$ and $L(s) = W(s)$. We note that we know the expressions for $r(t)$ in the above formula for many SIRMs, see Section 5.3.

Except when $L(t)$ is a Poisson or a Brownian motion, our Lévy market model is an incomplete model. This means that there are many different equivalent martingale measures to choose. In general this leads to many different possible prices for European options or bond options, etc.

One of the ways to price a bond is to use the Esscher transform equivalent martingale measure for the P^* in (19). Following [27], we can, by using the so-called Esscher transform, find in some cases at least one equivalent martingale

measure P^*. Let $f(t,x)$ be the density of our model's (real world, i.e. under P) distribution of $L(t)$. For some real number $\theta \in \{\theta \in R | \int_{-\infty}^{+\infty} \exp(\theta y) f(t,y) dy < +\infty\}$ we can define a new density

$$f^{(\theta)}(t,x) = \frac{\exp(\theta x) f(t,x)}{\int_{-\infty}^{+\infty} \exp(\theta y) f(t,y) dy}.$$

In order to assume finiteness of the expectation in the denominator above, in the case of general Lévy processes, we assume that

$$\int_{\{|x|>1\}} \exp[vx] \nu(dx) < \infty, \quad \text{for } |v| < (1+\epsilon)M,$$

where $\epsilon > 0$ and M is such that $0 \leq \sigma(s,T) \leq M$ (a.s.) (see (8) and (9)) for $0 \leq s \leq T$ and $\nu(dx)$ is the Lévy measure of L_1. Typical choices of this Lévy process are the variance gamma, the normal inverse Gaussian, the generalized hyperbolic, the Meixner or CGMY processes (see Schoutens (2003)). Another way to price a bond is to consider the characteristic function (or Fourier transform), if it is known, of the risk-neutral log returns (see [10]). We note, that if we know the explicit expression for $r(t)$ (see Section 5.3), then we can find the characteristic function of the risk-neutral bond price.

5.4. Lévy HJM Bond Pricing via PIDE

One more way to price a bond is to consider the solution of a partial integro-differential equation (PIDE) with boundary condition, all in terms of the triplet of Lévy characteristics $[\gamma, \sigma, \nu^{P^*}(dy)]$ of the Lévy process under the risk-neutral measure P^*.

Consider $V(t,T,r)$-bond price at time t, where the forward interest rate $f(t,T)$ follows the following SDE (in general form)

$$df(t,T) = a(t,T)dt + b(f(t,T))dL(t), \tag{22}$$

where $L(t)$ is a Lévy process.

The zero-coupon bond pricing eqaution is

$$\begin{array}{rcl} \frac{\partial V}{\partial t} & + & \frac{1}{2} b^2 \sigma^2 \frac{\partial^2 V}{\partial r^2} + (a + \gamma b - \lambda b \sigma) \frac{\partial V}{\partial r} \\ & + & \int_{-\infty}^{+\infty} [V(t, r+by) - V(t,r) - by \frac{\partial V(t,r)}{\partial r}] \nu(dy) - rV = 0, \end{array} \tag{23}$$

where λ is the *market price of risk*. The final condition is

$$V(T,T,r) = Z.$$

This PIDE is the analogue of the famous Black-Scholes PDE and follows from the Feynman-Kac formula for Lévy processes.

Since there is no closed solution in general we must resort to approximate methods. We may pose the PIDE in a form that becomes amenable to a solution using the finite difference method. A number of schemes are discussed in [23]. He considers a) implicit and explicit methods, b) implicit-explicit Runge-Kutta methods, c) operator splitting (method of fractional steps) and d) splitting in conjunction with predictor-corrector methods. See also [12] and [13].

Remark. The European call option price for the stock price that follows exponential Lévy process was derived in [43] and [46].

6. Interest Rate Derivatives

There are a large number of different interest rate derivative products. Here, we consider options, swaps, caps and floors. Having valued swaps, caps and floors we can value options on these instruments: swaptions, captions, floortions.

A bond option is the option on a bond and is identical to an equity option except that the underlying asset is a bond.

Both European and American versions exist.

An interest rate swap is an agreement between two parties to exchange the interest rate payments on a certain amount, the principal, for a certain length of time. (One party A, pays the other party B, a fixed rate of interest in return for a variable interest rate payment. For example, A pays 10% of $\$1,000,000$ p.a. to B and B pays r (variable) of the same amount to A. Say this agreement is to last for three years).

A cap is a loan at the floating interest rate but with the provision that the interest rate charged is guaranteed not to exceed a specified value, the cap, which is denoted by r^*.

A floor is similar to a cap except that the interest rate does not go below r^*.

Swaption is an option on a swap. Caption is an option on a cap. Floortion is an option on a floor.

7. Pricing of Lévy Bond Options

7.1. Pricing of Lévy HJM Bond Options

Let the forward interest rate $f(t,T)$ follow the following SDE (in general form)

$$df(t,T) = a(t,T)dt + b(f(t,T))dL(t), \tag{27}$$

where $L(t)$ is a Lévy process.

Consider the European call bond option, with exercise price K and expiry date T, on a zero-coupon bond with maturity date $T_B \geq T$.

To find the value of the call option on a bond (to buy a bond) we proceed with the following steps:

1. Find the value of the bond: $V_B(r,t;T_B)$, that satisfies the following PIDE:

$$\begin{aligned} \frac{\partial V_B}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_B}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_B}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[V_B(t,r+by) - V_B(t,r) - by\frac{\partial V_B(t,r)}{\partial r}]\nu(dy) - rV_B = 0 \end{aligned} \tag{28}$$

with the final condition

$$V_B(r,T_B;T_B) = Z.$$

2. Let $C_B(r,t)$ be the value of the call option on this bond. Since C_B also depends on the random walk $r(t)$, it must satisfy equation (27) too:

$$\begin{aligned} \frac{\partial C_B}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_B}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_B}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[C_B(t,r+by) - C_B(t,r) - by\frac{\partial C_B(t,r)}{\partial r}]\nu(dy) - rC_B = 0 \end{aligned} \tag{29}$$

with the final condition

$$C_B(r,T) = \max(V_B(r,T;T_B) - K, 0). \tag{30}$$

Remark. One numerical approach to solve this PIDE would be to use various finite difference methods, see [24].

8. Pricing of Swaps, Caps and Floors

8.1. Pricing of Swaps, Caps and Floors for Lévy HJM SIRMs

Consider $V(r,t)$-bond price at time t, where the forward interest rate $f(t,T)$ follows the following SDE (in general form)

$$df(t,T) = a(t,T)dt + b(f(t,T))dL(t), \tag{35}$$

where $L(t)$ is a Lévy process.

Pricing of Swaps. We consider valuing such swaps in general. Suppose that A pays the interest on an amount Z to B at a fixed rate r^* and B pays interest to A at the floating rate r. These payments continue until time T_S. Denote the value of this swap to A by $ZV_S(r,t)$. We note, that in a time-step dt A receives $(r-r^*)Zdt$. If we think of this payment as being similar to a coupon payment on a simple bond then we find that:

$$\begin{array}{rcl} \frac{\partial V_S}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_S}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[V_S(t,r+by) - V_S(t,r) - by\frac{\partial V_S(t,r)}{\partial r}]\nu(dy) \\ & - & rV_S + (r - r^*) = 0 \end{array} \tag{36}$$

with the final condition

$$V_S(r,T_S) = 0.$$

We note that r can be greater or less than r^* and so $V_S(r,t)$ need not be positive.

Pricing of Caps. The loan of Z is to be paid back at time T_C. The value of the capped loan, $ZV_C(r,t)$ satisfies the following PIDE:

$$\begin{array}{rcl} \frac{\partial V_C}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_C}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[V_C(t,r+by) - V_C(t,r) - by\frac{\partial V_C(t,r)}{\partial r}]\nu(dy) \\ & - & rV_C + \min(r, r^*) = 0 \end{array} \tag{37}$$

with the final condition

$$V_C(r,T_C) = 1.$$

Pricing of Floors. The value of the floored loan, $ZV_F(r,t)$, satisfies the following PIDE:

$$\begin{array}{rcl} \frac{\partial V_F}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_F}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_F}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[V_F(t,r+by) - V_F(t,r) - by\frac{\partial V_F(t,r)}{\partial r}]\nu(dy) \\ & - & rV_F + \max(r, r^*) = 0 \end{array} \tag{38}$$

with the final condition

$$V_F(r, T_F) = 1,$$

where T_F is an expiry time for the floor.

Remark. One approach to solve these PIDEs would be to use various finite difference methods, see [24].

9. Pricing of Swaptions, Captions and Floortions

9.1. Pricing of Swaptions, Captions and Floortions for Lévy HJM SIRMs

Consider $V(r,t)$-bond price at time t, where the forward interest rate $f(t,T)$ follows the following SDE (in general form)

$$df(t,T) = a(t,T)dt + b(f(t,T))dL(t), \tag{50}$$

where $L(t)$ is a Lévy process.

Pricing of Swaptions. Consider a European swap call pption, that is an option to buy this swap (a call swaption) for an amount K at time $T < T_S$, where T_S is an expiry time for swap with value $V_S(r,t)$, $t \le T_S$. Thus this value V_S satisfies the following PIDE (see (36)):

$$\begin{array}{rcl} \frac{\partial V_S}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_S}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[V_S(t, r+by) - V_S(t,r) - by\frac{\partial V_S(t,r)}{\partial r}]\nu(dy) \\ & - & rV_S + (r - r^*) = 0 \end{array} \tag{51}$$

with the final condition

$$V_S(r, T_S) = 0.$$

Then the value $C_S(r,t)$ of this call swap option (call swaption) satisfies the following PIDE:

$$\begin{array}{rcl} \frac{\partial C_S}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_S}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[C_S(t, r+by) - C_S(t,r) - by\frac{\partial C_S(t,r)}{\partial r}]\nu(dy) - rC_S = 0 \end{array} \tag{52}$$

with the final condition

$$C_S(r,T) = \max(V_S(r,T) - K, 0). \tag{53}$$

We solve for the value of the swap first and then use this value as the final data for the value of the swaption.

Pricing of Captions. Consider a European cap call option, that is an option to buy a cap (a call caption) for an amount K at time $T < T_C$, where T_C is an expiry time for the cap with value $V_C(r,t)$, $t \leq T_C$. Thus, this value V_C satisfies the following PIDE (see (37)):

$$\begin{array}{rcl} \frac{\partial V_C}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_C}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[V_C(t, r+by) - V_C(t,r) - by\frac{\partial V_C(t,r)}{\partial r}]\nu(dy) \\ & - & rV_C + \min(r, r^*) = 0 \end{array} \tag{54}$$

with the final condition

$$V_C(r, T_C) = 1.$$

Then the value $C_C(r,t)$ of this call cap option (call caption) satisfies the following PIDE:

$$\begin{array}{rcl} \frac{\partial C_C}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_C}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[C_C(t, r+by) - C_C(t,r) - by\frac{\partial C_C(t,r)}{\partial r}]\nu(dy) - rC_C = 0 \end{array} \tag{55}$$

with the final condition

$$C_C(r, T) = \max(V_C(r,T) - K, 0). \tag{56}$$

We solve for the value of the cap first and then use this value as the final data for the value of the caption.

Pricing of Floortions. Consider a European floor call option, that is an option to buy this floor (a call floortion) for an amount K at time $T < T_F$, where T_F is an expiry time for the floor with value $V_F(r,t)$, $t \leq T_F$. Thus this value V_F satisfies the following PIDE (see (38)):

$$\begin{array}{rcl} \frac{\partial V_F}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_F}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_F}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[V_F(t, r+by) - V_F(t,r) - by\frac{\partial V_F(t,r)}{\partial r}]\nu(dy) \\ & - & rV_F + \min(r, r^*) = 0 \end{array} \tag{57}$$

with the final condition

$$V_F(r, T_F) = 1.$$

Then the value $C_F(r,t)$ of this call floor option (call floortion) satisfies the following PIDE:

$$\begin{array}{rcl} \frac{\partial C_F}{\partial t} & + & \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_F}{\partial r^2} + (a + \gamma - \lambda b\sigma)\frac{\partial C_F}{\partial r} \\ & + & \int_{-\infty}^{+\infty}[C_F(t, r+y) - C_F(t,r) - y\frac{\partial C_F(t,r)}{\partial r}]\nu(dy) - rC_F = 0 \end{array} \tag{58}$$

with the final condition

$$C_F(r,T) = \max(V_F(r,T) - K, 0). \tag{59}$$

We solve for the value of the floor first and then use this value as the final data for the value of the floortion.

Remark. One approach to solve these PIDEs would be to use various finite difference methods, see [24].

10. Conclusion

We discussed how to calculate the price of zero-coupon bonds for Lévy-based HJM one-factor models of forward interest rate $f(t,T)$ using the change of time method. We also derived PIDEs for the values of swaps, caps, floors and options on them, swaptions, captions and floortions, respectively.

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A. Appendix: One-Factor and Multi-Factor Gaussian HJM Interest Rate Models

A.1. One-Factor Gaussian HJM Model

The Heath, Jarrow and Morton (1987) Model: Define the forward interest rate $f(t,s)$, for $t \leq s$, characterized by the following equality

$$P(t,u) = \exp[-\int_t^u f(t,s)ds]$$

for any maturity u. $f(t,s)$ represents the instanteneous interest rate at time s as anticipated by the market at time t. It is natural to set $f(t,t) = r(t)$. The process $f(t,u)_{0\leq t\leq u}$ satisfies an equation

$$f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dW(v),$$

where the processes a and b are continuous. We note, that the last SDE may be written in the following form:

$$df(t,u) = b(f(t,u))(\int_t^u b(f(t,s)))ds + b(f(t,u))d\hat{W}(t),$$

where

$$\hat{W}(t) = W(t) - \int_0^t q(s)ds$$

and

$$q(t) = \int_t^u b(f(t,s))ds - \frac{a(t,u)}{b(f(t,u))}.$$

A.2. Multi-Factor Gaussian SIRMs

Multi-factor models driven by Brownian motions can be obtained using various combinations of the above-mentioned processes. We give one example of the two-factor continuous-time GARCH SIRM:

$$\begin{cases} df(t,u) &= b(f(t,u))(\int_t^u b(f(t,s)))ds + b(f(t,u))d\hat{W}_1(t), \\ db(t) &= \xi b(t)dt + \eta b(t)dW_2(t), \end{cases}$$

where W^1, W^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

A.3. Change of Time Method (CTM) for SDE driven by Brownian Motion

We consider the following SDE driven by a Brownian motion:

$$dX(t) = a(t, X(t))dW(t), \tag{12}$$

where $W(t)$ is a Brownian motion and $a(t, X)$ is continuous and measurable by t and X is a function on $[0, +\infty) \times R$.

The reason to consider this equation is the following one: if we solve the equation, then we can solve a more general equation with a drift $\beta(t, X)$ by the drift transformation method or Girsanov transformation (see Ikeda and Watanabe (1981), Chapter 4, Section 4]). We use the following result to get the solution of the previous equation using the change of time method.

Theorem. (Ikeda and Watanabe (1981), Chapter IV, Theorem 4.3) *Let $\hat{W}(t)$ be an one-dimensional $\mathcal{F}_t$-Wiener process with $\hat{W}(0) = 0$ given on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and let $X(0)$ be an $\mathcal{F}_0$-adapted random variable. Define a continuous process $V = V(t)$ by the equality*

$$V(t) = X(0) + \hat{W}(t). \tag{13}$$

Let T_t be the change of time process (see Section 2.3):

$$T_t = \int_0^t a^{-2}(T_s, X(0) + \hat{W}(s))ds. \tag{14}$$

If

$$X(t) := V(\hat{T}_t) = X(0) + \hat{W}(\hat{T}_t) \tag{15}$$

and $\hat{\mathcal{F}}_t := \mathcal{F}_{\hat{T}_t}$, then there exists $\hat{\mathcal{F}}_t$-adapted Wiener process $W = W(t)$ such that $(X(t), W(t))$ is a solution of (12) on the probability space $(\Omega, \mathcal{F}, \hat{\mathcal{F}}_t, P)$, where $\hat{T}_t$ is the inverse to the T_t time change.

A.4. Solution of One-Factor Gaussian SIRMs Using CTM

We use the change of time method (see Ikeda and Watanabe (1981)) to get the solutions to the following equations (see Swishchuk (2007)). $W(t)$ below is an standard Brownian motion, and $\hat{W}$ is a $(\hat{T}_t)_{t\in R_+}$-adapted standard Brownian motion on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t\in R_+}, P)$.

The Heath, Jarrow and Morton Process:

$$f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dW(v).$$

Solution

$$f(t,u) = f(0,u) + \hat{W}(\hat{T}_t) + \int_0^t a(v,u)dv,$$

where

$$\hat{T}_t = \int_0^t b^2(f(0,u) + \hat{W}(\hat{T}_s) + \int_0^s a(v,u)dv)ds.$$

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