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EXISTENCE OF SOLUTIONS FOR A BOUNDARY VALUE PROBLEM OF FRACTIONAL ORDER ON THE HALF-LINE VIA MONOTONE THEORY

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Abstract: In the paper we discuss existence and uniqueness of a weak solution of a fractional problem on the half-line via the Minty-Browder theorem.

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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary order. In this paper we study the fractional boundary value problem

$$\begin{cases} D_-^\alpha(D_+^\alpha u(t)) + u(t) = f(t, u(t)), & t \in (0, +\infty), \\ u(0) = u(+\infty) = 0, \end{cases} \tag{1.1}$$

where $\frac{1}{2} < \alpha < 1$ and $f : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function.

First we recall fractional integral and derivatives operators.

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Definition 1.1. ([5], [8], [7]) Let u be a function defined on $(0,+\infty)$. The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a function u denoted by $I_+^\alpha u$ and $I_-^\alpha u$, respectively, are defined by

$$I_+^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds, \quad t \in (0,+\infty),$$

and

$$I_-^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (s-t)^{\alpha-1} u(s)ds, \quad t \in (0,+\infty),$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$; here $\Gamma(\alpha)$ is the gamma function.

Definition 1.2. ([5], [8], [7]) Let u be a function defined on $(0,+\infty)$. For $n-1 \leq \alpha < n \, (n \in \mathbb{N}^*)$, the left and right Riemann-Liouville fractional derivatives of order α for a function u denoted by $D_+^\alpha u$ and $D_-^\alpha u$, respectively, are defined by

$$\begin{aligned} D_+^\alpha u(t) &= \frac{d^n}{dt^n} I_+^{n-\alpha} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)ds, \ t \in (0,+\infty), \end{aligned}$$

and

$$\begin{aligned} D_-^\alpha u(t) &= (-1)^n \frac{d^n}{dt^n} I_-^{n-\alpha} u(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^{+\infty} (s-t)^{n-\alpha-1} u(s)ds, \ t \in (0,+\infty), \end{aligned}$$

provided that the right-hand side is pointwise defined.

In particular, for $\alpha = n$, $D_+^\alpha u(t) = D^n u(t)$ and $D_-^\alpha u(t) = (-1)^n D^n u(t), t \in (0,+\infty)$.

Proposition 1.1. *[5] If $D_+^\alpha u, D_-^\alpha u \in L^1(0,+\infty)$ and $n-1 \leq \alpha < n$, then*

$$I_+^\alpha D_+^\alpha u(t) = u(t) + \sum_{j=1}^n c_j (t-a)^{\alpha-j}$$

with $c_j = \frac{D_{a+}^{\alpha-j} u(a)}{\Gamma(\alpha-j+1)} \in \mathbb{R}, \quad j = 1,2,...,n$ and

$$I_-^\alpha D_-^\alpha u(t) = u(t) + \sum_{j=1}^n c'_j (b-t)^{\alpha-j}$$

with $c'_j = \frac{(-1)^{n-j} D_{b^-}^{\alpha-j} u(b)}{\Gamma(\alpha-j+1)} \in \mathbb{R}, \quad j = 1, 2, ..., n.$

Now we introduce a new space which is suitable for the study of our fractional BVP. Let

$$E_0^\alpha(0,+\infty) = \Big\{ u \in L^2(0,+\infty), D_+^\alpha u \in L^2(0,+\infty), u(0) = u(\infty) = 0 \Big\},$$

with the natural norm

$$\|u\|_\alpha = \Big(\int_0^{+\infty} |u(t)|^2 dt + \int_0^{+\infty} |D_+^\alpha u(t)|^2 dt \Big)^{\frac{1}{2}}, \forall u \in E_0^\alpha(0,+\infty). \tag{1.2}$$

Let the space $C_p([0,+\infty))$ be defined by

$$C_p([0,+\infty)) = \Big\{ u \in C([0,+\infty)), \mathbb{R}) : \lim_{t\to+\infty} p(t)u(t) \text{ exists} \Big\}$$

and endowed with the norm

$$\|u\|_{\infty,p} = \sup_{t\in[0,+\infty)} p(t)|u(t)|,$$

where the function $p : [0,+\infty) \longrightarrow (0,+\infty)$ is continuous and satisfies

$$\lim_{t\longrightarrow+\infty} p(t) t^{\alpha-\frac{1}{2}} = 0.$$

We put

$$M = \frac{1}{\sqrt{2\alpha-1}.\Gamma(\alpha)} \cdot \sup_{t>0} p(t) t^{\alpha-\frac{1}{2}}.$$

Throughout this paper we assume p satisfies these conditions. Using the same idea as in [5] one can easily prove the following proposition.

Proposition 1.2. *[5] If* $u \in L^2(0,+\infty), D_+^\alpha u \in L^2(0,+\infty)$ *with* $u(0) = u(+\infty) = 0$ *and* $v \in C_0^\infty([0,+\infty))$, *then*

$$\int_0^{+\infty} D_+^\alpha u(t) v(t) dt = \int_0^{+\infty} u(t) D_-^\alpha v(t) dt.$$

Using Proposition 1.2, we now define a weak solution of problem (1.1).

Definition 1.3. A weak solution of the fractional boundary value problem (1.1) is given by a solution of the following variational formula

$$\int_0^{+\infty} \Big[D_+^\alpha u(t) D_+^\alpha v(t) + u(t)v(t) - f(t,u(t))v(t) \Big] dt = 0, \quad \text{for all} \quad v \in E_0^\alpha(0,+\infty).$$

Now we recall some information from the literature needed in this paper.

Definition 1.4. [9] Let X be a Banach space. An operator $A : X \longrightarrow X^*$ which satisfies

$$\langle Au - Av, u - v\rangle \geq 0 \tag{1.3}$$

for any $u, v \in X$ is called a monotone operator. An operator A is called strictly monotone if for $u \neq v$ strict inequality holds in (1.3). An operator A is called strongly monotone if there exists $C > 0$ such that

$$\langle Au - Av, u - v\rangle \geq C\|u - v\|^2$$

for any $u, v \in X$. It is clear that a strongly monotone is strictly monotone.

Definition 1.5. [9] Let $A : X \longrightarrow X^*$ be an operator on the real Banach space X.

(a) A is said to be demicontinuous if

$$u_n \longrightarrow u \text{ as } n \longrightarrow +\infty \quad \textit{implies } Au_n \rightharpoonup Au \text{ as } n \longrightarrow +\infty.$$

(b) A is said to be hemicontinous if the real function

$$t \mapsto \langle A(u + tv), w\rangle \text{ is continuous on } [0, 1] \text{ for all } u, v, w \in X.$$

(c) A is said to be coercive if

$$\lim_{\|u\| \longrightarrow +\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty\cdot$$

Remark 1.1. [4] It is easy to see that for monotone operator $A : X \longrightarrow X^*$ with $Dom(A) = X$, demicontinuity and hemicontinuity are equivalent.

Finally in this section we recall the Minty-Browder Theorem.

Theorem 1.3. *[6](Minty-Browder) Let X be a reflexive real Banach space. Let $A : X \longrightarrow X^*$ be an operator which is bounded, hemicontinous, coercive and monotone on the space X. Then, the equation $Au = f$ has at least one solution $u \in X$ for each $f \in X^*$. If A is strictly monotone then the solution is unique.*

2. Main Result

We begin with the space $E_0^\alpha(0,+\infty)$.

Proposition 2.1. *$E_0^\alpha(0,+\infty)$ is a Banach space.*

Proof. Let $(u_n)_{n\geq 1}$ be a Cauchy sequence in $E_0^\alpha(0,+\infty)$. Then $(u_n)_{n\geq 1}$, $(D_+^\alpha u_n)_{n\geq 1}$ are Cauchy sequences in $L^2(0,+\infty)$. From (1.2) we have $\|u_n - u_m\|_\alpha \longrightarrow 0$ as $n,m \longrightarrow +\infty$ which implies that

$$\|u_n - u_m\|_{L^2} \longrightarrow 0, \ \ \|D_+^\alpha u_n - D_+^\alpha u_m\|_{L^2} \longrightarrow 0$$

as $n,m \longrightarrow +\infty$. Since $L^2(0,+\infty)$ is a Banach space, there exist functions $u_1, u_2 \in L^2(0,+\infty)$, such that $u_n \longrightarrow u_1$, $\ D_+^\alpha u_n \longrightarrow u_2$ in $L^2(0,+\infty)$ as $n \longrightarrow +\infty$. We now show that $D_+^\alpha u_1 = u_2$. From Proposition 1.2, we have

$$\int_0^{+\infty} D_+^\alpha u_n(t)\varphi(t)dt = \int_0^{+\infty} u_n(t) D_-^\alpha \varphi(t)dt, \ \forall\, \varphi \in C_0^\infty([0,+\infty))$$

and then by using the definition of the inner product in $L^2(0,+\infty)$, we obtain that

$$\int_0^{+\infty} u_2(t)\varphi(t)dt = \int_0^{+\infty} u_1(t) D_-^\alpha \varphi(t)dt, \ \forall\, \varphi \in C_0^\infty([0,+\infty)),$$

and so $D_+^\alpha u_1 = u_2$. Thus $\lim\limits_{n\longrightarrow+\infty} \|u_n - u_1\|_\alpha = 0$, and so $E_0^\alpha(0,+\infty)$ is a Banach space. □

Lemma 2.2. *The operator*

$$\begin{aligned} T : E_0^\alpha(0,+\infty) &\longrightarrow T(E_0^\alpha(0,+\infty)) \subset L^2(0,+\infty)\times L^2(0,+\infty) = L_2^2(0,+\infty) \\ u &\longrightarrow T(u) = (u, D_+^\alpha u) \end{aligned}$$

is an isometric isomorphic mapping.

Proof. It is clear that T is a linear operator and we now show that T conserves norms, i.e

$$\forall u \in E_0^\alpha(0,+\infty) : \|Tu\|_{L_2^2} = \|u\|_\alpha.$$

Indeed, we have

$$\begin{aligned} &\|(u, D_+^\alpha u)\|_{L_2^2} = \|u\|_\alpha \\ \Leftrightarrow\ \ &\|u\|_{L^2} + \|D_+^\alpha u\|_{L^2} = \|u\|_\alpha. \end{aligned}$$

□

Proposition 2.3. *$E_0^\alpha(0,+\infty)$ is a reflexive space.*

Proof. Since, $L^2((0,+\infty),\mathbb{R})$ is a reflexive Banach space, the cartesian space

$$L_2^2((0,+\infty),\mathbb{R}) = L^2((0,+\infty),\mathbb{R}) \times L^2((0,+\infty),\mathbb{R})$$

is also a reflexive Banach space with respect to the norm

$$\|u\|_{L_2^2} = \sum_{i=1}^{2} \|u_i\|_{L^2} \text{ where } u = (u_1,u_2) \in L_2^2((0,+\infty),\mathbb{R}).$$

Then

$$\begin{aligned} T : E_0^\alpha(0,+\infty) &\longrightarrow T(E_0^\alpha(0,+\infty)) \subset L_2^2(0,+\infty) \\ u &\longrightarrow T(u) = (u, D_+^\alpha u) \end{aligned}$$

is an isometric isomorphic, so $T(E_0^\alpha(0,+\infty))$ is a closed subspace of $L_2^2(0,+\infty)$ and by [[2], Theorem 4.10.5] then $T(E_0^\alpha(0,+\infty))$ is reflexive. Consequently $E_0^\alpha(0,+\infty)$ is also reflexive (see [[2], Lemma 4.10.4]). □

Proposition 2.4. *$E_0^\alpha(0,+\infty)$ is a separable space.*

Proof. Since, $L^2((0,+\infty),\mathbb{R})$ is a separable Banach space, the cartesian space

$$L_2^2((0,+\infty),\mathbb{R}) = L^2((0,+\infty),\mathbb{R}) \times L^2((0,+\infty),\mathbb{R})$$

is also a separable Banach space with respect to the norm

$$\|u\|_{L_2^2} = \sum_{i=1}^{2} \|u_i\|_{L^2} \text{ where } u = (u_1,u_2) \in L_2^2((0,+\infty),\mathbb{R}).$$

Then, the space $T(E_0^\alpha(0,+\infty)) \subset L_2^2$ is also separable (see [1], Proposition III.22). Moreover, the operator

$$\begin{aligned} T : E_0^\alpha(0,+\infty) &\longrightarrow T(E_0^\alpha(0,+\infty)) \subset L_2^2(0,+\infty) \\ u &\longrightarrow T(u) = (u, D_+^\alpha u) \end{aligned}$$

is an isometric isomorphic, so $E_0^\alpha(0,+\infty)$ is a separable space. □

Lemma 2.5. *For all $u \in E_0^\alpha(0,+\infty)$, we have that $E_0^\alpha(0,+\infty)$ embeds continuously in $C_p([0,+\infty))$, i.e.,*

$$\exists\, M_0 > 0,\ \|u\|_{\infty,p} \le M_0 \|u\|_\alpha.$$

Proof. For all $u \in E_0^\alpha(0, +\infty)$, and $t > 0$,

$$u(t) = I_+^\alpha(D_+^\alpha u(t)),$$

so

$$p(t)u(t) = p(t)I_+^\alpha(D_+^\alpha u(t))$$

which implies from the Cauchy-Schwartz inequality

$$\begin{aligned}
\left|p(t)I_+^\alpha(D_+^\alpha u(t))\right| &= \frac{p(t)}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1}\, D_+^\alpha u(s)ds\right| \\
&\leq \frac{p(t)}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{2(\alpha-1)}ds\Big)^{\frac{1}{2}}\Big(\int_0^t |D_+^\alpha u(s)|^2 ds\Big)^{\frac{1}{2}} \\
&\leq \frac{p(t)}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{2(\alpha-1)}ds\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty} |u(s)|^2 ds \\
&+ \int_0^{+\infty} |D_+^\alpha u(s)|^2 ds\Big)^{\frac{1}{2}} \\
&= \frac{\|u\|_\alpha}{\sqrt{2\alpha-1}.\Gamma(\alpha)} p(t)t^{\alpha-\frac{1}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
\|u\|_{\infty,p} &= \sup_{t\in[0,+\infty)} |p(t)u(t)| \\
&= \sup_{t\in[0,+\infty)} |p(t)I_+^\alpha(D_+^\alpha u(t))| \\
&\leq \frac{\|u\|_\alpha}{\sqrt{2\alpha-1}.\Gamma(\alpha)} \cdot \sup_{t>0} p(t)t^{\alpha-\frac{1}{2}},
\end{aligned}$$

and so,

$$\|u\|_{\infty,p} \leq M\|u\|_\alpha.$$

□

From the definition of the norm in $E_0^\alpha(0, +\infty)$, it is easy to see that

Proposition 2.6. *$E_0^\alpha(0, +\infty)$ embeds continuously in $L^2(0, +\infty)$.*

To prove the compactness embedding of $E_0^\alpha(0, +\infty)$ in $C_p([0, +\infty))$ we follow the ideas in [3].

Lemma 2.7. *[3] Let $D \subset C_p([0, +\infty))$ be a bounded set. Then D is relatively compact if the following conditions hold:*

(a) D is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, i.e.,

$$\forall J \subset [0,+\infty) \text{ compact subinterval }, \forall \varepsilon > 0, \exists \delta > 0, \forall t_1, t_2 \in J : \\ |t_1 - t_2| < \delta \Longrightarrow |p(t_1)u(t_1) - p(t_2)u(t_2)| \leq \varepsilon, \ \forall u \in D,$$

(b) D is equiconvergent at $+\infty$ i.e.,

$$\forall \varepsilon > 0, \exists T = T(\varepsilon) > 0 \text{ such that} \\ \forall t_1, \ t_2 : t_1, t_2 \geq T(\varepsilon) \Longrightarrow |p(t_1)u(t_1) - p(t_2)u(t_2)| \leq \varepsilon, \ \ \forall u \in D.$$

Lemma 2.8. *The embedding*

$$E_0^\alpha(0,+\infty) \hookrightarrow C_p([0,+\infty))$$

is compact.

Proof. Let $D \subset E_0^\alpha(0,+\infty)$ be a bounded set. Then it is bounded in $C_p([0,+\infty))$ by Lemma 2.5. Let $R > 0$ be such that for all $u \in D$, $\|u\|_\alpha \leq R$. We will apply Lemma 2.7.

(a) D is equicontinuous on every compact interval of $[0,+\infty)$.

Let $u \in D$ and $t_1, t_2 \in J \subset [0,+\infty)$, where J is a compact sub-interval and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|p(t_1)I_+^\alpha u(t_1) \ &- \ p(t_2)I_+^\alpha u(t_2)| = \frac{1}{\Gamma(\alpha)}\Big|p(t_1)\int_0^{t_1}(t_1-s)^{\alpha-1}u(s)ds \\
&- \ p(t_2)\int_0^{t_2}(t_2-s)^{\alpha-1}u(s)ds\Big| \\
&\leq \ \frac{1}{\Gamma(\alpha)}\Big|p(t_1)\int_0^{t_1}(t_1-s)^{\alpha-1}u(s)ds \\
&- \ p(t_2)\int_0^{t_1}(t_2-s)^{\alpha-1}u(s)ds\Big| \\
&+ \ \frac{p(t_2)}{\Gamma(\alpha)}\Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}u(s)ds\Big| \\
&\leq \ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|p(t_1)(t_1-s)^{\alpha-1} - p(t_2)(t_2-s)^{\alpha-1})||u(s)|ds \\
&+ \ \frac{p(t_2)}{\Gamma(\alpha)}\Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}u(s)ds\Big| \\
&\leq \ \frac{\|u\|_{L^2}}{\Gamma(\alpha)}\Big[\Big(\int_0^{t_1}\Big(p(t_1)(t_1-s)^{\alpha-1} - p(t_2)(t_2-s)^{\alpha-1}\Big)^2 ds\Big)^{1/2}
\end{aligned}$$

$$+ \quad p(t_2)\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2}ds\Big] \tag{2.1}$$

so, we have

$$\begin{aligned}
|p(t_1)u(t_1) &- p(t_2)u(t_2)| = |p(t_1)I_+^\alpha D_+^\alpha u(t_1) - p(t_2)I_{0^+}^\alpha D_+^\alpha u(t_2)| \\
&\le \frac{\|D_+^\alpha u\|_{L^2}}{\Gamma(\alpha)}\Big(\int_0^{t_1}\Big(p(t_1)(t_1-s)^{\alpha-1} - p(t_2)(t_2-s)^{\alpha-1}\Big)^2 ds\Big)^{1/2} \\
&+ \frac{\|D_+^\alpha u\|_{L^2}}{\Gamma(\alpha)}p(t_2)\Big(\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2}ds\Big)^{1/2} \\
&\le \frac{R}{\Gamma(\alpha)}\Big(\int_0^{t_1}\Big(p(t_1)(t_1-s)^{\alpha-1} - p(t_2)(t_2-s)^{\alpha-1}\Big)^2 ds\Big)^{1/2} \\
&+ \frac{R}{\Gamma(\alpha)}p(t_2)\Big(\int_{t_1}^{t_2}(t_2-s)^{2(\alpha-1)}ds\Big)^{1/2} \longrightarrow 0,
\end{aligned}$$

as $|t_1 - t_2| \longrightarrow 0$.

(b) D is equiconvergent at $+\infty$.

For $t \in [0, +\infty)$ and $u \in D$, using the fact that $pu(+\infty) = 0$ and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|p(t)u(t) - p(+\infty)u(+\infty)| &= |p(t)u(t)| = |p(t)I_+^\alpha D_{0^+}^\alpha u(t)| \\
&= \frac{p(t)}{\Gamma(\alpha)}\Big|\int_0^t (t-s)^{\alpha-1}\, D_{0^+}^\alpha u(s)ds\Big| \\
&\le \frac{p(t)\|u\|_\alpha}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{2(\alpha-1)}ds\Big)^{\frac{1}{2}} \\
&= \frac{p(t)\|u\|_\alpha}{\sqrt{2\alpha-1}\Gamma(\alpha)}\cdot t^{\alpha-\frac{1}{2}} \\
&\le \frac{Rp(t)}{\sqrt{2\alpha-1}\Gamma(\alpha)}\cdot t^{\alpha-\frac{1}{2}} \longrightarrow 0
\end{aligned}$$

as $t \longrightarrow +\infty$. □

The space $E_0^\alpha(0, +\infty)$ is endowed with the structure of a Hilbert space together with the inner product

$$\begin{aligned}
(.,.)_{E_0^\alpha(0,+\infty)} : E_0^\alpha(0,+\infty) \times E_0^\alpha(0,+\infty) &\longrightarrow \mathbb{R} \\
(u,v) &\longrightarrow (u,v)_{E_0^\alpha(0,+\infty)} = \int_0^{+\infty} u(t)v(t)dt \\
&+ \int_0^{+\infty} D_+^\alpha u(t) D_+^\alpha v(t)dt.
\end{aligned}$$

Now we are in a position to give our main existence result. Assume p satisfies the conditions in Section 1 and also suppose the following conditions hold:

(H1) There exist functions a, b with $\frac{a}{p} \in L^2(0,+\infty)$, $b \in L^2(0,+\infty)$ such that

$$|f(t,x)| \leq a(t)|x| + b(t) \; \forall\, t \in [0,+\infty), \forall\, x \in \mathbb{R}$$

with

$$M \|\frac{a}{p}\|_{L^2} < 1.$$

(H2) $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is decreasing with respect to the second variable, i.e.

$$f(t,x_1) \leq f(t,x_2) \quad \text{for a.e. } \; t \in [0,+\infty) \quad \text{and } \; x_1, x_2 \in \mathbb{R}, x_1 \geq x_2.$$

We define the operator $A : E_0^\alpha(0,+\infty) \longrightarrow (E_0^\alpha(0,+\infty))^*$ by $A = I - F$, where the operator I and F are defined from $E_0^\alpha(0,+\infty)$ into $(E_0^\alpha(0,+\infty))^*$ as

$$\langle I(u), v\rangle = \int_0^{+\infty} D_+^\alpha u(t) D_+^\alpha v(t)dt + \int_0^{+\infty} u(t)v(t)dt$$

and

$$\langle F(u), v\rangle = \int_0^{+\infty} f(t,u(t))v(t)dt.$$

We search for a weak solution of problem (1.1) which is a function $u \in E_0^\alpha(0,+\infty)$ which satisfies the operator equation $Au = 0$.

Theorem 2.9. *Assume that f satisfies the hypotheses $(H1)$ and $(H2)$. Then problem (1.1) has a weak solution.*

Proof. We will use Theorem 1.3 and we divide our proof into five steps.

Step 1. A is bounded.

Note, the functional

$$\psi(u) = \frac{1}{2}\Big(\int_0^{+\infty} |D_+^\alpha u(t)|^2 dt + \int_0^{+\infty} |u(t)|^2 dt\Big) = \|u\|_\alpha^2$$

is of class C^1 and I is the derivative operator of ψ in the weak sense, so I is continuous.

Let $u \in E_0^\alpha(0,+\infty)$, be such that $\|u\|_\alpha < R$. Using the Cauchy-Schwartz inequality, we obtain

$$\|I(u)\|_{(E_0^\alpha(0,+\infty))^*} = \sup_{\|v\|_\alpha \leq 1} \Big|\langle I(u), v\rangle\Big|$$

$$
\begin{aligned}
&= \sup_{\|v\|_\alpha\le 1}\Big|\Big(\int_0^{+\infty} D_+^\alpha u(t) D_+^\alpha v(t)dt + \int_0^{+\infty} u(t)v(t)dt\Big)\Big| \\
&\le \sup_{\|v\|_\alpha\le 1}\Big[\Big(\int_0^{+\infty} |D_+^\alpha u(t)|^2\Big)^{\frac12}\Big(\int_0^{+\infty} |D_+^\alpha v(t)|^2\Big)^{\frac12} \\
&\quad + \Big(\int_0^{+\infty} u^2(t)\Big)^{\frac12}\Big(\int_0^{+\infty} v^2(t)\Big)^{\frac12}\Big] \\
&\le \sup_{\|v\|_\alpha\le 1}\big(\|D_+^\alpha u\|_{L^2}\|D_+^\alpha v\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2}\big) \\
&\le \sup_{\|v\|_\alpha\le 1}\big(\|v\|_\alpha\|u\|_\alpha\big) \\
&\le \|u\|_\alpha \le R
\end{aligned}
$$

and

$$
\begin{aligned}
\|F(u)\|_{(E_0^\alpha(0,+\infty))^*} &= \sup_{\|v\|_\alpha\le 1}\Big|\langle F(u), v\rangle\Big| = \sup_{\|v\|_\alpha\le 1}\Big|\int_0^{+\infty} f(t,u(t))v(t)dt\Big| \\
&\le \sup_{\|v\|_\alpha\le 1}\Big(\int_0^{+\infty} a(t)|u(t)||v(t)|dt + \int_0^{+\infty} b(t)|v(t)|dt\Big) \\
&= \sup_{\|v\|_\alpha\le 1}\Big(\int_0^{+\infty} \frac{a(t)}{p(t)}p(t)|u(t)||v(t)|dt + \int_0^{+\infty} b(t)|v(t)|dt\Big) \\
&\le \sup_{\|v\|_\alpha\le 1}\Big[\|u\|_{p,\infty}\Big(\int_0^{+\infty} \frac{a(t)}{p(t)}|v(t)|dt\Big) + \Big(\int_0^{+\infty} b(t)|v(t)|dt\Big)\Big] \\
&\le \sup_{\|v\|_\alpha\le 1}\Big[M\|u\|_\alpha\Big(\int_0^{+\infty} \frac{a^2(t)}{p^2(t)}\Big)^{\frac12}\Big(\int_0^{+\infty} |v(t)|^2\Big)^{\frac12} \\
&\quad + \Big(\int_0^{+\infty} b^2(t)dt)\Big)^{\frac12}\Big(\int_0^{+\infty} |v(t)|^2\Big)^{\frac12}\Big] \\
&\le \sup_{\|v\|_\alpha\le 1}\|v\|_\alpha\Big[M\|u\|_\alpha\Big(\int_0^{+\infty} \frac{a^2(t)}{p^2(t)}\Big)^{\frac12} + \Big(\int_0^{+\infty} b^2(t)dt)\Big)^{\frac12}\Big] \\
&\le M\|u\|_\alpha\|\frac{a}{p}\|_{L^2} + \|b\|_{L^2}.
\end{aligned}
$$

Hence A is bounded.

Step 2. A is demicontinuous.

We prove that F is strongly continuous that is, if $u_n \rightharpoonup u$ then $F(u_n) \longrightarrow F(u)$. Let (u_n) be such that $u_n \rightharpoonup u$ in $E_0^\alpha(0,+\infty)$. Now (u_n) is bounded in $E_0^\alpha(0,+\infty)$ and by Lemma 2.5, we have that (u_n) is bounded in $C_p([0,+\infty))$, and by Lemma 2.8, we have $u_n \longrightarrow u$ in $C_p([0,+\infty))$. Then

$$\begin{aligned}
\|F(u)-F(u_n)\|_{(E_0^\alpha(0,+\infty))^*} &= \sup_{\|v\|_\alpha\le 1}\left|\left\langle F(u)-F(u_n),v\right\rangle\right| \\
&= \sup_{\|v\|_\alpha\le 1}\left|\int_0^{+\infty}\big(f(t,u(t))-f(t,u_n(t))\big)v(t)dt\right| \\
&\le \sup_{\|v\|_\alpha\le 1}\left(\int_0^{+\infty}\Big(f(t,u(t))v(t)dt\Big)\right. \\
&\quad \left.+\sup_{\|v\|_\alpha\le 1}\left(\int_0^{+\infty}\Big(f(t,u_n(t))v(t)dt\Big)\right.\right. \\
&\le \sup_{\|v\|_\alpha\le 1}\left(\int_0^{+\infty}a(t)|u(t)||v(t)|dt\right) \\
&\quad +\sup_{\|v\|_\alpha\le 1}\left(\int_0^{+\infty}a(t)|u_n(t)||v(t)|dt\right) \\
&\le M\|\frac{a}{p}\|_{L^2}\big(\|u\|_\alpha\|+\|u_n\|_\alpha\big)+2\|b\|_{L^2} \\
&\le M\|\frac{a}{p}\|_{L^2}C+2\|b\|_{L^2},
\end{aligned}$$

for some constant $C>0$. Since $u_n \longrightarrow u$ in $C_p([0,+\infty))$, we obtain

$$\int_0^{+\infty}(f(t,u(t))-f(t,u_n(t)))v(t)dt \longrightarrow 0 \text{ as } n\longrightarrow +\infty.$$

Thus F is strongly continuous and therefore it is continuous. From the fact that I is continuous, we deduce that the operator A is continuous. Thus A is demicontinuous.

Step 3. A is monotone.
Note

$$\begin{aligned}
\langle I(u)-I(v),u-v\rangle &= \int_0^{+\infty}\big(D_+^\alpha u(t)-D_+^\alpha v(t)\big)^2dt \\
&\quad +\int_0^{+\infty}(u(t)-v(t))^2dt \\
&=\|u-v\|_\alpha^2, \qquad (2.2)
\end{aligned}$$

so, I is strongly monotone.

Also, since f is decreasing with respect to the second variable,

$$\langle F(u) - F(v), u - v \rangle = \int_0^{+\infty} (f(t, u(t)) - f(t, v(t)))(u(t) - v(t))dt \leq 0.$$

Thus A is strongly monotone.

Step 4. A is a coercive operator.

Now

$$\begin{aligned}
\frac{1}{\|u\|_\alpha}\langle A(u), u\rangle =& \frac{1}{\|u\|_\alpha}\Big[\int_0^{+\infty}(|D_+^\alpha u(t)|^2 + |u(t)|^2)dt - \int_0^{+\infty} f(t, u(t))u(t)dt\Big] \\
\geq& \frac{1}{\|u\|_\alpha}\Big[\|u\|_\alpha^2 - \int_0^{+\infty}(a(t)|u(t)||u(t)| + b(t)|u(t)|)dt\Big] \\
\geq& \frac{1}{\|u\|_\alpha}\Big[\|u\|_\alpha^2 - \Big(\int_0^{+\infty} a^2(t)|u(t)|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty}|u(t)|^2 dt\Big) \\
& - \int_0^{+\infty} b(t)|u(t)|dt\Big] \\
\geq& \frac{1}{\|u\|_\alpha}\Big[\|u\|_\alpha^2 - \Big(\int_0^{+\infty}\frac{a^2(t)}{p^2(t)}\cdot p^2(t)|u(t)|^2 dt\Big)^{\frac{1}{2}}\|u\|_{L^2} \\
& - \|b\|_{L^2}\|u\|_{L^2}\Big] \\
\geq& \frac{1}{\|u\|_\alpha}\Big[\|u\|_\alpha^2 - \|\frac{a}{p}\|_{L^2}\|u\|_{\infty,p}\|u\|_\alpha - \|b\|_{L^2}\|u\|_\alpha\Big] \\
\geq& \frac{1}{\|u\|_\alpha}\Big[\|u\|_\alpha^2 - M\|\frac{a}{p}\|_{L^2}\|u\|_\alpha^2 - \|b\|_{L^2}\|u\|_\alpha\Big] \\
=& \Big(1 - M\|\frac{a}{p}\|_{L^2}\Big)\|u\|_\alpha - \|b\|_{L^2},
\end{aligned}$$

so A is coercive.

The assumptions in the statement of Theorem 1.3 are fulfilled, so problem (1.1) has a weak solution.

Step 5. Uniqueness.

Let u and v be weak solutions of problem (1.1) with $u \neq v$. From (2.2) and (2.3) it follows that

$$\langle A(u) - A(v), u - v\rangle \geq \|u - v\|_\alpha^2 \geq 0.$$

Then $u = v$ and the proof is complete. □

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