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ABOUT EXPANSION OF NUMBER OF MODELS WHICH HAVE PAIRS OF LAX'S

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Abstract: Let's consider the generalized KdV equations, when the pairs of Lax's is not under construction directly. In this problem we want to find such variables in which it is possible to construct the exact solutions and pairs of Lax's the equation connected with the initial. Possibility of research asymptotic properties of initial model opens using the theory of inverse scattering problem. The constructed new nonlinear equations don't enter into hierarchy of the equations of type of KdV and earlier other authors weren't considered.

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1. Introduction

In works [1]-[4] methods of the solutions of the Kortweg-de Vris (KdV) equation with the help of pairs of P.D. Lax have been offered. Perturbation of the KdV soliton have been considered in [5-7].

To consider equation KdV V.I. Arnold at "International conference on differential equations and dynamical systems" in town Suzdal at 2008 year, advice to us. We have constructed an *accompany* matrix and exact solutions. We have solutions the new equation which is a condition of resolvability of linear algebraic equationsystem. Have shown what solutions it has.

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Let's consider the generalized KdV equations, when the pairs of Lax's is not under construction directly. In this problem we want to find such variables in which it is possible to construct the exact solutions and pairs of Lax's the equation connected with the initial. Possibility of research asymptotic properties of initial model opens using the theory of inverse scattering problem. The constructed new nonlinear equations don't enter into hierarchy of the equations of type of KdV and earlier other authors weren't considered [8].

In works [14]-[16] we managed to prove that exists (to within elementary transformations) a unique *accompanying* matrix of the functional linear algebraic equationsystem for the KdV equation, and in the given work for the generalized KdV equation. Our results supplement and specify the theory, we hope.

A new method of construction of exact solutions for generalized KdV equations is proposed in this article and for partial differential equations (PDE) in the [10-13] too. The authors in mathematics used change of variables. However, they did not notice an important property of broad class of PDE, which was discovered in articles [10-13]. This property gives the possibility of expressing PDE as $\mathbf{A_1 X = b}$. This is a linear algebraic equations system with regards derivatives to old variables on new variables.

2. The Generalized Kortweg-de Vris Equations is Reduce to Linear Algebraic Equationsystem

Let us consider summarize KdV equation:

$$Z^{'}{}_{t} + R(t,\ Z,\ Z^{'}{}_{x}) + Z^{'''}{}_{xxx} = F(t,\ Z). \tag{1}$$

The proposed algorithm works providing that all functions are continuously differentiable functions. Let's make an arbitrary replacement of variables:

$$Z(x,t)|_{x=x(\xi,\ \delta),t=t(\xi,\delta)} = U(\xi,\delta). \tag{2}$$

The inverse replace of variables defines the function $Z(x,t)$ of (1) from the function:

$$Z(x,t) = U(\xi,\delta)|_{\xi=\xi(x,t),\ \delta=\delta(x,t)}.$$

We note that

$$\det J = x^{'}{}_{\xi} t^{'}{}_{\delta} - t^{'}{}_{\xi} x^{'}{}_{\delta} \neq 0$$

is nonezero. An inverse transformation exist, at least locally:

$$\xi = \xi(x,t), \delta = \delta(x,t).$$

The derivatives of the old independent variables on the new variables are determind as follows:

$$\frac{\partial x}{\partial \xi} = \det J \frac{\partial \delta}{\partial t},\ \frac{\partial t}{\partial \xi} = -\det J \frac{\partial \delta}{\partial x}, \frac{\partial x}{\partial \delta} = -\det J \frac{\partial \xi}{\partial t},\ \frac{\partial t}{\partial \delta} = \det J \frac{\partial \xi}{\partial x}. \tag{3}$$

Let us introduce the following relation, like [9]:

$$\begin{aligned} &\frac{\partial Z}{\partial x}|_{x=x(\xi,\ \delta),\ t=t(\xi,\ \delta)} = Y(\xi,\delta), \\ &\frac{\partial Z}{\partial t}|_{x=x(\xi,\ \delta),\ t=t(\xi,\ \delta)} = T(\xi,\delta) \\ &\frac{\partial Y(\xi(x,t),\delta(x,t))}{\partial x}|_{x=x(\xi,\ \delta),\ t=t(\xi,\ \delta)} = M(\xi,\delta). \end{aligned} \tag{4}$$

The functions $Y(\xi,\delta), T(\xi,\delta), M(\xi,\delta)$ are unknown. Using (2) and (3), (4) we obtain the formulas:

$$\left(\frac{\partial U}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial U}{\partial \delta}\frac{\partial t}{\partial \xi}\right) = Y(\xi,\delta)[x'_{\xi}t'_{\delta} - t'_{\xi}x'_{\delta}], \tag{5}$$

$$\left(-\frac{\partial U}{\partial \xi}\frac{\partial x}{\partial \delta} + \frac{\partial U}{\partial \delta}\frac{\partial x}{\partial \xi}\right) = T(\xi,\delta)[x'_{\xi}t'_{\delta} - t'_{\xi}x'_{\delta}]. \tag{6}$$

Using (2) and (3), (4) we obtain the formulas:

$$\left(\frac{\partial Y}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial Y}{\partial \delta}\frac{\partial t}{\partial \xi}\right) = [x'_{\xi}t'_{\delta} - t'_{\xi}x'_{\delta}]\ M(\xi,\delta). \tag{7}$$

Equation (1) takes the form

$$\det J + \left(\frac{\partial M}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial M}{\partial \delta}\frac{\partial t}{\partial \xi}\right)/[T + R(t(\xi,\delta),U,\ Y) - F(t(\xi,\delta),U)] = 0. \tag{8}$$

As Z is continuously differentiable function, with the necessary of

$$\frac{\partial}{\partial t}Z'_{x} = \frac{\partial}{\partial x}Z'_{t} \tag{9}$$

in the variables ξ, δ. Taking into consideration on (2), (4), we can write this equality in the form

$$\frac{\partial x}{\partial \delta}\frac{\partial Y}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial Y}{\partial \delta} + \frac{\partial t}{\partial \delta}\frac{\partial T}{\partial \xi} - \frac{\partial t}{\partial \xi}\frac{\partial T}{\partial \delta} = 0. \tag{10}$$

In this problem we have five equations (5)-(8), (10) for four variables:

$$x^{'}{}_{\xi},\ x^{'}{}_{\delta},\ \ t^{'}{}_{\xi},\ t^{'}{}_{\delta}.$$

In this problem we have five variants. We consider all five variants.

We consider system (5), (6), (7), (10) as linear algebraic equationsystem regarding the derivatives

$$x^{'}{}_{\xi},\ x^{'}{}_{\delta},\ \ t^{'}{}_{\xi},\ t^{'}{}_{\delta},$$

for example.

The equation (8) remains in a stock.

Theorem 1. *The implicit linear algebraic equationsystem (5), (6), (7), (10), with regarding the derivatives* $x^{'}{}_{\xi},\ x^{'}{}_{\delta},\ t^{'}{}_{\xi},\ t^{'}{}_{\delta}$, *has the unique solution:*

$$\frac{\partial x}{\partial \xi} = \Psi_1(\xi,\delta), \quad \frac{\partial x}{\partial \delta} = \Psi_2(\xi,\delta), \tag{11}$$

$$\frac{\partial t}{\partial \xi} = \Psi_3(\xi,\delta), \quad \frac{\partial t}{\partial \delta} = \Psi_4(\xi,\delta). \tag{12}$$

The functions $\Psi_j(\xi,\delta),\ j = 1 \div\ 4$ *obvious are calculable, like [10-16]. It is new equationsystem, where*

$$\Psi_1(\xi,\delta) \stackrel{\text{def}}{=} [T\ Y^{'}{}_{\xi}(Y^{'}{}_{\xi}U^{'}{}_{\delta} - U^{'}{}_{\xi}Y^{'}{}_{\delta}) + U^{'}{}_{\xi}[Y\ (Y^{'}{}_{\delta}T^{'}{}_{\xi} - Y^{'}{}_{\xi}T^{'}) + M\ (T^{'}{}_{\delta}U^{'}{}_{\xi} - T^{'}{}_{\xi}U^{'}{}_{\delta})]]/P_1(\xi,\delta), \tag{13}$$

$$\Psi_2(\xi,\delta) \stackrel{\text{def}}{=} [Y\ (Y^{'}{}_{\delta}T^{'}{}_{\xi} - T^{'}{}_{\delta}Y^{'}{}_{\xi})U^{'}{}_{\delta} + M\ U^{'}{}_{\delta}(T^{'}{}_{\delta}U^{'}{}_{\xi} - U^{'}{}_{\delta}T^{'}{}_{\xi}) + T\ Y^{'}{}_{\delta}(Y^{'}{}_{\xi}U^{'}{}_{\delta} - Y^{'}{}_{\delta}U^{'}{}_{\xi})]/P_1(\xi,\delta), \tag{14}$$

$$\Psi_3(\xi,\delta) \stackrel{\text{def}}{=} (-Y\ \ Y^{'}{}_{\xi} + M\ \ U^{'}{}_{\xi})(Y^{'}{}_{\xi}U^{'}{}_{\delta} - Y^{'}{}_{\delta}U^{'}{}_{\xi})/P_1(\xi,\delta), \tag{15}$$

$$\Psi_4(\xi,\delta) \stackrel{\text{def}}{=} (Y\ Y^{'}{}_{\delta} - M\ U^{'}{}_{\delta})\ (Y^{'}{}_{\delta}U^{'}{}_{\xi} - U^{'}{}_{\delta}Y^{'}{}_{\xi})/P_1(\xi,\delta), \tag{16}$$

$$P_1(\xi,\delta)) = Y^2(T^{'}{}_{\xi}Y^{'}{}_{\delta} - T^{'}{}_{\delta}Y^{'}{}_{\xi}) + M\ Y\ (T^{'}{}_{\delta}U^{'}{}_{\xi} - U^{'}{}_{\delta}T^{'}{}_{\xi}) M\ T\ (U^{'}{}_{\delta}Y^{'}{}_{\xi} - U^{'}{}_{\xi}Y^{'}{}_{\delta}), \tag{17}$$

and moreover

$$\det J = (U^{'}{}_{\xi}Y^{'}{}_{\delta} - U^{'}{}_{\delta}Y^{'}{}_{\xi})^2/P_1(\xi,\delta), \tag{18}$$

Proof. Equation (10) is linear. We can express any three derivatives $x^{'}{}_{\xi},\ x^{'}{}_{\delta},\ t^{'}{}_{\xi}$, by means of one derivative. For example, we can express them by means of $t^{'}{}_{\delta}$. We can substitute it to last equation and obtain a linear algebraic equation for $t^{'}{}_{\delta}$ in all variants **(!)**.

Substituting (11)-(12) in (8).

From relation (8) takes the form

$$\begin{aligned} T(\xi,\delta) = [M\ (M^{'}{}_{\xi}U^{'}{}_{\delta} - M^{'}{}_{\delta}U^{'}{}_{\xi}) + Y\ (M^{'}{}_{\delta}Y^{'}{}_{\xi} - M^{'}{}_{\xi}Y^{'}{}_{\delta}) \\ + (F(t(\xi,\delta),U) - R(t(\xi,\delta),U,Y))\,(Y^{'}{}_{\delta}U^{'}{}_{\xi} - Y^{'}{}_{\xi}U^{'}{}_{\delta})]/P_2, \end{aligned} \tag{19}$$

where $P_2 = Y^{'}{}_{\delta}U^{'}{}_{\xi} - Y^{'}{}_{\xi}U^{'}{}_{\delta}$.

Remark 1. In the articles [13]-[16] we used linear algebraic equationsystem (6), (7), (8), (10), on the first step of proof. We can substitute it to last equation (5) and obtain (19).

We consider all five variants. In the equationsystem the function $T(\xi,\delta)$ takes the form (19). It is property of this problem.

Let's go to the second stage of proof. We can substitute (19) in the (11), (12) and in $\det J$ (18). We obtain the formulas $\tilde{\det} J$.

We obtained $\mathbf{A_1 X} = \mathbf{b}$. It is shown that generalized KdV equation is equivalent to the system of functional liner algebraic equationsystem $\mathbf{A_1 X} = \mathbf{b}$. The new mathematic object - is *accompany* matrix of the generalized KdV equation.

Theorem 2. *Let's implicit liner algebraic equationsystem (6), (10), (7), (5), and (19) is equal equation (1).*

The liner algebraic equationsystem (6), (10), (7), (5) with (19) has the form:

$$\begin{pmatrix} U^{'}{}_{\delta} & -U^{'}{}_{\xi} & 0 & 0 \\ -Y^{'}{}_{\delta} & Y^{'}{}_{\xi} & -T^{'}{}_{\delta} & T^{'}{}_{\xi} \\ 0 & 0 & -Y^{'}{}_{\delta} & Y^{'}{}_{\xi} \\ 0 & 0 & -U^{'}{}_{\delta} & U^{'}{}_{\xi} \end{pmatrix} \begin{pmatrix} x'_{\xi} \\ x'_{\delta} \\ t'_{\xi} \\ t'_{\delta} \end{pmatrix} = \begin{pmatrix} b_1 \\ 0 \\ b_3 \\ b_4 \end{pmatrix}.$$

Vectors $\mathbf{X},\ \mathbf{b}$ *have the form* $\mathbf{X} = (\mathbf{x}^{'}{}_{\xi},\ \mathbf{x}^{'}{}_{\delta},\ \mathbf{t}^{'}{}_{\xi},\ \mathbf{t}^{'}{}_{\delta})^{\tau}$, $\mathbf{b} = (\mathbf{b_1}, \mathbf{0}, \mathbf{b_3}, \mathbf{b_4})^{\tau}$, where $b_1 = T\ \tilde{\det} J,\quad b_3 = M\ \tilde{\det} J,\quad b_4 = Y\ \tilde{\det} J.$

Vector symbol τ *means conjuation. Eigen values of matrix* $\mathbf{A_1}$ *have the form*

$$\lambda_{1,2} = [-Y^{'}{}_{\delta} + U^{'}{}_{\xi} \pm \sqrt{D_1}]/2,$$

$$\lambda_{3,4} = [Y^{'}{}_{\xi} + U^{'}{}_{\delta} \pm \sqrt{D_2}]/2,$$
$$D_1 = (Y^{'}{}_{\delta})^2 + 2Y^{'}{}_{\delta}\, U^{'}{}_{\xi} + (U^{'}{}_{\xi})^2 - 4\, Y^{'}{}_{\xi} U^{'}{}_{\delta},$$
$$D_2 = (Y^{'}{}_{\xi})^2 - 2Y^{'}{}_{\xi} U^{'}{}_{\delta} + (U^{'}{}_{\delta})^2 + 4\, U^{'}{}_{\xi} Y^{'}{}_{\delta}.$$

Proof. At the third stage, consider the new first- order system (11), (12) with the functions $x = x(\xi, \delta)$, $t = t(\xi, \delta)$.

It is well known that the solvability of a system of this type is verified by calculating the second mixed derivatives of the functions $x = x(\xi, \delta)$, and $t = t(\xi, \delta)$ on the arguments ξ and δ:

$$x^{''}{}_{\xi\,\delta} = x^{''}{}_{\delta\,\xi}, \quad t^{''}{}_{\xi\,\delta} = t^{''}{}_{\delta\,\xi}. \tag{20}$$

This result look in the [10]-[13].

Remark 2. If some free functions U, Y, M satisfy the condition (20), then systems (11), (12) and (5)-(10) are solvable.

This property (Theorems 1 and 2) of generalized KdV equation was not know before.

Let's choose in this article the elementary form $t(\xi, \delta) = \xi$, $t^{'}{}_{\xi} = 1$, $t^{'}{}_{\delta} = 0$.

From $t^{'}{}_{\delta} = 0$, $\Psi_4 = 0$ (16) we obtain:

$$M(\xi, \delta) = Y\; Y^{'}{}_{\delta}/U^{'}{}_{\delta}. \tag{21}$$

This assumption leads an accompanying matrix $\mathbf{A_1}$ to the diagonal kind.

Suppose that

$$Y(\xi, \delta) = G(\xi, U(\xi, \delta)). \tag{22}$$

We obtain the following result.

Theorem 3. *Let's we have equation (1) and let one (2)-(21) and* $t(\xi, \delta) = \xi$. *Exact solution of the generalized KdV equation has the form (2) and equation-system (11), (12) has the form*

$$x^{'}{}_{\xi} = R(\xi\; U,\; G(\xi, U))/G + (G^{'}{}_{U}(\xi, U))^2 + G\; G^{''}{}_{UU}$$
$$+U^{'}{}_{\xi}/G(\xi, U) - F(\xi, U)/G(\xi, U),$$
$$x^{'}{}_{\delta} = U^{'}{}_{\delta}/G(\xi, U), \qquad t^{'}{}_{\xi} = 1, \qquad t^{'}{}_{\delta} = 0, \tag{23}$$

$\tilde{\det} J = -U^{'}{}_{\delta}/R(\xi, U)$. *Exact solution* $Z(x, t)$ *of the generalized KdV equation (1) in parametric, non obvious form*

$$x(\xi, \delta) = s(\xi) + \int (1/G(\xi, U)) d\; U, \tag{24}$$

where $s(\xi)$-free are continuously differentiable function.

The solvability conditions $x''_{\xi\delta} = x''_{\delta\xi}$ of equationsystem (23) are solvable

$$G'_{\xi} + 3\ G^2\ G'_U\ G''_{UU} + G^3\ G'''_{UUU} - G\ F'_U + F\ G'_U + G\ G'_U\ R'_G + GR'_U - R(\xi, U, G)G'_U = 0. \quad (25)$$

Proof. We have counted all five variants of introduction of differential communications without preservation laws.

We have counted all five variants of introduction of differential communications with the account of laws of preservation. One such variant is described in work [13].

Results of research of variants the following: intermediate calculations various, but as a result all of them lead to the differential communications (19) and the theorem 2. The equation has exact solutions which are reduced by standard methods to the ODE, but we are not going to discuss here their [8], [13], [17].

3. Construction of Pairs Lax's for Equation (25)

Example of construction of pairs Lax's:

Theorem 4. *Let's in the generalized KdV equation (1) functions $R(t, Z, Z'_x)$, $F(t, Z)$ have the form*

$$\begin{aligned} &R(t, Z, Z'_x) = b(t)\ Z(x,t)Z'_x - \alpha(t)\ (Z'_x)^2 - \sigma(t)\ (Z'_x)^3, \\ &b(\xi) = [2\ \sigma(t)\ c'(t) - c(t)\ \sigma'(t)]/[2\ c^2(t)], \quad \alpha(t) = 3\ c(t), \\ &F(t, Z) = -\phi(t) - Z(x,t)\ c'(t)/c(t). \end{aligned} \quad (26)$$

where $c(t)$, $\phi(t)$-free are continuously differentiable function. Let's Theorem 3 hold.

The equation (25) have the form:

$$\begin{aligned} &G'_{\xi} + 3\ G^2\ G'_U\ G''_{UU} + G^3\ G'''_{UUU} - U\ G'_U c'(t)/c(\xi) \\ &-2\ G^3\ \sigma(t)\ G'_U - \phi(\xi)\ G'_U\ - 3\ c(t)\ G^2\ G'_U + G\ c'(\xi)/c(\xi) \\ &+G^2\ [2\ \sigma(\xi)\ c'(\xi) - c(\xi)\ \sigma'(\xi)]/[2\ c^2(\xi)] = 0 \end{aligned} \quad (27)$$

and have pairs Lax's

$$\Psi''_{UU} = (c(\xi)/G(\xi, U) + \sigma(\xi)/2)\,\Psi(\xi, U),$$

$$\Psi'_{\xi} = \Psi(\xi,U)\left(k(\xi) - c'(\xi)/(2\ c(\xi)) + c(\xi)\ G(\xi,U)G'_{U}(\xi,U)\right) +$$
$$+\Psi'_{U}\left(\phi(\xi) - c(\xi)G^2(\xi,U) + U\ c'(\xi)/c(\xi)\right), \qquad (28)$$

where $k(\xi)$-free are continuously differentiable function.

Proof. The direct way to construct of pairs Lax's with the function (26) for the equation (1) doesn't possible [1]-[8].

The proof is spent by check of a condition of compatibility

$$\left(\Psi''_{UU}\right)'_{\xi} = \left(\Psi'_{\xi}\right)''_{UU}.$$

Potential in the of Schrodinger equation on the solutions like solitons is the hole with infinitely high walls.

Example. We have many possibility.

Let's one $c(t) = C_1 \exp(a\ t)$, $\sigma(t) = C_2 \exp(a\ t)$. It is possible to investigate the asymptotic of the Cauchy problem solutions using the theory of inverse scattering problem [5]-[7].

Let's one $c(t) = C_1$, $\sigma(t) = C_2$, and $b = 0$ and transformation of variables $Z'_x = V(x,t)$. Possibility of research asymptotic properties of the equation KdV

$$V'_t - 2\ \alpha\ V\ V'_x - 3\ \sigma\ V^2\ V'_x + V'''_{xxx} = 0. \qquad (29)$$

The nonlinear equations (29) don't enter into hierarchy of the equations of type of KdV, see [8].

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