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ALGEBRAIZATION OF BRIOSCHI-HALPHEN EQUATION

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Abstract: The Brioschi-Halphen Equation (BHE) is a second order differential equation in the complex domain obtained by a two-step variable transformation of the Lamé equation in the Weierstrass form. In this paper, we present a new algebraization of the BHE by writing its operator as an element of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. We also obtain a Lie algebraic potential from Lie algebraic equation so formed by using a suitable guage transformation. A polynomial solution of the Lie algebraic equation obtained is presented.

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1. Introduction

The Brioschi-Halphen equation (BHE) is a second order differential equation in the complex domain of the form

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$$\Big(4(w-e_1)(w-e_2)(w-e_3)D^2+(1-2n)(6w^2-\frac{1}{2}g_2)D + 4n(2n-1)w+B\Big)\Psi=0, \quad (1.1)$$

where, $D=\frac{\mathrm{d}}{\mathrm{d}w}$, B is a constant and n is a positive integer (ref. [2]). In the equation (1.1), we may put $V_0(w)=4n(2n-1)w$. Then, the equation (1.1) can be re-written in the form

$$\left\{(4w^3-g_2w-g_3)D^2+(1-2n)(6w^2-\frac{1}{2}g_2)D+V_0(w)+B\right\}\Psi=0, \quad (1.2)$$

where the invariant constants g_2 and g_3 are given by

$$g_2=4(e_1e_2+e_2e_3+e_1e_3) \quad \text{and} \quad g_3=4e_1e_2e_3 \quad (1.3)$$

where $e_j=\wp(\omega_j), j=1,2,3$ and each ω_j is the half-period of the Weierstrass elliptic $\wp$-function.

In this work, we present a new algebraization of equation (1.1) by writing it in terms of the generators of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. We also obtain a Lie algebraic potential, gauge potential and exactly solvable potential from the Lie algebraic equation, whose solution is obtained explicitly in closed form in terms of Jacobi polynomial. In Section **2**, we set down the necessary mathematical formalism while our main results are shown in Section **3**.

2. Mathematical Formalism

Following Shifman ([5], p. 2912), in order to transform equation (1.1) into a Lie algebraic equation, we consider a second order differential operator

$$\mathcal{H}:=-\frac{1}{2}P_4(w)\frac{\mathrm{d}^2}{\mathrm{d}w^2}+P_3(w)\frac{\mathrm{d}}{\mathrm{d}w}+P_2(w), \quad (2.1)$$

where each $P_j(w)$ $(j=2,3,4)$ is a polynomial of degree j given by

$$\begin{aligned}
P_4(w) &= -2[c_{++}w^4-2c_{+0}w^3+(c_{00}-2c_{+-})w^2+2c_{-0}w+c_{--}],\\
P_3(w) &= -2kc_{++}w^3+(3kc_{+0}-c_+)w^2+(-k(c_{00}-2c_{+-})+c_0)w\\
&\qquad +(-kc_{-0}+c_-),\\
P_2(w) &= 2jkc_{++}w^2+(-2jkc_{+0}+2jc_+)w.
\end{aligned}$$

Here also, following Shifman ([5], p. 2913), for the operator $\mathcal{H}$ given in (2.1) to be exactly solvable, we assume that $c_{+-}=0$. Let $\mathfrak{g}$ be the Lie algebra

of a certain Lie group G. For $1 \leq a, b \leq \nu$ where $\nu = \dim(\mathfrak{g})$, let $\{\mathcal{J}^\alpha, \alpha = a, b\}$ denote the generators of the Lie algebra $\mathfrak{g}$. By rewriting the differential operator in (2.1) as the quadratic combination of the generators $\mathcal{J}^\alpha$, we get a Lie algebraic operator (cf: Ganguli [4], p.4)

$$\mathcal{H} = \sum_{a,b=0,\pm} c_{a\,b}\, \mathcal{J}^a \mathcal{J}^b + \sum_{a=0,\pm} c_a\, \mathcal{J}^a. \tag{2.2}$$

Here, $c_{a\,b}$ and c_a are complex (or real) numbers and $c_{a\,b}$ form the entries of the metric $\{g^{ab}\}$ of dimension N . The generators $\mathcal{J}^\alpha(\alpha = a, b)$ of $\mathfrak{g}$ leave invariant the space, say $\mathscr{P}_\nu$, of polynomials of degree ν in the variable w. By construction, the differential operator given in (2.2) also preserves $\mathscr{P}_\nu$. The resulting representation in (2.2) is known in the mathematics literature to be isomorphic to the matrix representation of the operator (2.1) in finite dimension [see Brihaye and Godart ([3], pp. 5284-5285) and Shifman ([5], Appendix A, p. 2946)]. It is called an algebraization of the operator (2.1) and it is an element of the universal enveloping algebra of the Lie algebra of the Lie group under consideration.

3. $\mathfrak{sl}(2, \mathbb{C})$-Algebraization of BHE

We are now ready to apply the formalism given in Section **2** above to obtain a new algebraization of the BHE (2.1). Thereafter, we deduce Lie algebraic potential, gauge potential and exactly solvable potential generated by $\mathfrak{sl}(2, \mathbb{C})$ algebra. A solution of the Lie algebraic equation obtained is presented in terms of Jacobi polynomial.

We begin by multiplying equation (1.2) by w to obtain

$$\Big\{(4w^4 - g_2w^2 - g_3w)D^2 + (1-2n)(6w^3 - \frac{1}{2}g_2w)D + 4n(2n-1)w^2 + Bw\Big\}\Psi = 0. \tag{3.1}$$

By comparing the coefficients of the operator of equation (3.1) and the coefficients of the differential operator (2.1) we obtain

$$\begin{array}{llll} c_{++} = 4, & c_{+-} = 0, & c_{-+} = 0, & c_{--} = 0 \\ c_{00} = -g_2, & c_{0-} = 0, & c_{-0} = -\frac{g_3}{4}, & c_{+0} = 0 \\ c_{0+} = 0, & c_0 = -\frac{(3j-1)}{4}g_2, & c_- = -\frac{3}{16}(3j-1)^2 g_2, & c_+ = 0 \\ j = \frac{2}{3}n, & k = \frac{3(3j-1)}{4}, & & B = 0 \end{array} \tag{3.2}$$

The structure metric of the BHE is therefore given by

$$\{g^{ab}\} = \begin{pmatrix} c_{++} & c_{+0} & c_{+-} \\ c_{0+} & c_{00} & c_{0-} \\ c_{-+} & c_{-0} & c_{--} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -g_2 & 0 \\ 0 & -\frac{g_3}{2} & 0 \end{pmatrix}. \tag{3.3}$$

The norm of $\{g^{ab}\}$ is given as

$$\|g\| = \det\{g^{ab}\} = 0 \tag{3.4}$$

and the modulus of the metric vector $g^a = (c_+, c_0, c_-)$ is expressed as

$$|g^a| = \frac{(3j-1)g_2}{4}\sqrt{1 + \frac{9(3j-1)^2}{16}}.$$

Equation (3.4) means that the elements of the inverse of the metric $\{g^{ab}\}$, which does not exist, are not involved in the algebraization process of equation (3.1) presented in what follows. We consider the finite dimensional $\mathfrak{sl}(2,\mathbb{C})$ representation for BHE in bilinear form with spin j in a space of polynomials of the basis functions

$$\mathscr{B} := \{v_{m,j} = w^{j+m} | -j \leq m \leq j\}.$$

The generators of the algebra $\mathfrak{sl}(2,\mathbb{C})$ are

$$\mathcal{J}_+ := w^2\frac{\mathrm{d}}{\mathrm{d}w} - 2jw, \quad \mathcal{J}_0 := w\frac{\mathrm{d}}{\mathrm{d}w} - j, \quad \mathcal{J}_- := \frac{\mathrm{d}}{\mathrm{d}w} \tag{3.5}$$

which obey the commutation relations

$$[\mathcal{J}_0, \mathcal{J}_+] = \mathcal{J}_+, \quad [\mathcal{J}_-, \mathcal{J}_+] = 2\mathcal{J}_0, \quad [\mathcal{J}_-, \mathcal{J}_0] = \mathcal{J}_- \ .$$

(cf: Turbiner [6], p.469). The operator of equation (3.1) is now written as an element of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ as

$$\mathcal{H} := 4\mathcal{J}_+^2 - g_2\mathcal{J}_0^2 - \frac{g_3}{2}\mathcal{J}_-\mathcal{J}_0 - \frac{3j-1}{4}g_2\mathcal{J}_0 - \frac{3}{16}(3j-1)^2 g_2\mathcal{J}_-. \tag{3.6}$$

The eigenvalue equation resulting from the differential expression (3.6) above is therefore obtained as

$$\mathcal{H}\Psi = \frac{3j(3j-1)}{4}\Psi. \tag{3.7}$$

In this case, $\frac{3j(3j-1)}{4}$ is the eigenvalue of the Casimir operator (3.6). After necessary re-arrangements, equation (3.7) becomes

$$\left(H_1 - \frac{j(j+1)g_2}{4}\right)\Psi = \frac{3j(3j-1)}{4}\Psi, \tag{3.8}$$

where H_1 is again a Casimir operator with eigenvalues E_j given as

$$E_j = \frac{j(j+1)g_2}{4} + \frac{3j(3j-1)}{4}.$$

Explicitly H_1 may be expressed as

$$\begin{aligned} H_1 =&(4w^4 - g_2w^2 - \frac{g_3}{2}w)\frac{\mathrm{d}^2}{\mathrm{d}w^2} \\ &+ \left(8(1-2j)w^3 - \frac{1}{4}(2g_3 + (3-5j)g_2)w - \frac{3(3j-1)^2g_2 - 8g_3(j-1)}{16}\right)\frac{\mathrm{d}}{\mathrm{d}w} \\ &+ 8j(2j-1)w^2. \end{aligned} \tag{3.9}$$

Now, a deformation of the expression in (3.9) yields

$$H_1{}' = -\frac{1}{2}D^2 + \mathscr{A}(w)D + \Delta V(w), \quad D := \frac{\mathrm{d}}{\mathrm{d}w}, \tag{3.10}$$

where $\Delta V(w)$, called the Lie algebraic potential, is given by

$$\Delta V(w) = -\frac{8j(2j-1)w}{8w^3 - 2g_2w - g_3}.$$

The gauge potential $\mathscr{A}(w)$ for the Lie algebraic operator $H_1{}'$ in (3.10) is

$$\begin{aligned} &\mathscr{A}(w) = \\ &\quad - \frac{128(1-2j)w^3 - 4(2g_3 + (3-5j)g_2)w - (3(3j-1)^2g_2 - 8g_3(j-1))}{128w^4 - 32g_2w^2 - 16g_3w}, \end{aligned} \tag{3.11}$$

and the corresponding phase function is obtained as

$$\begin{aligned} &\Phi(w) = \int^w \mathscr{A}(u)\mathrm{d}u \\ &= -\int^w \frac{128(1-2j)u^3 - 4(2g_3 + (3-5j)g_2)u - (3(3j-1)^2g_2 - 8g_3(j-1))}{128u^4 - 32g_2u^2 - 16g_3u}\mathrm{d}u \end{aligned} \tag{3.12}$$

(cf: Shifman [5], p.2907). Exactly solvable potential is obtained from the Lie algebraic potential and the gauge potential by the relation

$$V(w) \equiv \Delta V(w) - \frac{1}{2}\mathscr{A}'(w) - \mathscr{A}(w)^2, \quad ' := \frac{\mathrm{d}}{\mathrm{d}w} \tag{3.13}$$

and which, in the case of this work, is obtained as

$$\begin{aligned} V(w) = {} & \frac{8j(1-2j)w}{8w^3 - 2g_2w - g_3} + \frac{1}{2(128w^4 - 32g_2w^2 - 16g_3w)^2} \\ & \times \big[(128w^4 - 32g_2w^2 - 16g_3w)(384(1-2j)w^2 - 8g_3 - 4(3-5j)g_2) \\ & - (128(1-2j)w^3 - 4(2g_3 + (3-5j)g_2)w - (3(3j-1)^2g_2 - 8g_3(j-1))) \\ & \times (512w^3 - 64g_2w - 16g_3)\big] \\ & + \frac{[128(1-2j)w^3 - 4(g_3 + (3-5j)g_2)w - 3(3j-1)^2g_2 + 8g_3(j-1)]^2}{256(8w^4 - 2g_2w^2 - g_3w)^2}. \end{aligned}$$

Using the basis functions defined above we get

$$\mathcal{J}_- v_{m,j} = (m+j)v_{j,m-1}, \quad \mathcal{J}_+ = w^2\mathcal{J}_- - 2jw, \quad \mathcal{J}_0 = w\mathcal{J}_- - j. \tag{3.14}$$

Equation (3.8), which may be re-written (by substituting the expressions in (3.14) into (3.6)) as

$$\begin{aligned} \Big\{4(w^2\mathcal{J}_- - 2jw)^2 - g_2(w\mathcal{J}_- - j)^2 - \frac{g_3}{2}\mathcal{J}_-(w\mathcal{J}_- - j) - \frac{(3j-1)g_2}{4}(w\mathcal{J}_- - j) \\ - \frac{3}{16}(3j-1)^2 g_2\mathcal{J}_-\Big\}\Psi(w) = -\frac{3j(3j-1)}{4}\Psi(w) \end{aligned} \tag{3.15}$$

is solved, using method of point canonical transformation, by

$$\begin{aligned} \Psi(w) &= \exp(\Phi(w))\mathscr{P}_m(z(w)) \\ &= \exp\Big(-\int^w \frac{128(1-2j)u^3 - 4(2g_3 + (3-5j)g_2)u - (3(3j-1)^2g_2 - 8g_3(j-1))}{128u^4 - 32g_2u^2 - 16g_3u}\mathrm{d}u\Big) \\ &\quad \times \mathscr{P}_m(z(w)), \end{aligned} \tag{3.16}$$

where $\mathscr{P}_m(z(w))$ is a polynomial in $z \equiv z(w)$ of degree m given as

$$\begin{aligned} \mathscr{P}_m(z(w)) = \left(\frac{(-1)^m\, 2^{\omega-\frac{1}{2}}}{\sqrt{c}}\right) \frac{\Gamma(m+\omega-\gamma+1)\Gamma(2m+\omega)}{(2m+\omega)\Gamma(m+\omega)} \\ \times z(w)^{-\frac{(1+\gamma)}{2}}(z(w)-1)^{-\frac{\alpha+\beta-\gamma+1}{2}} \end{aligned}$$

$$\times P_m^{(\omega-\gamma,\gamma-1)}(2z(w)-1), \quad (3.17)$$

where

$$c = \frac{j(j+1)g_2 + 3j(3j-1) + 4m(m+\omega)}{4[3(3j-1)^2 g_2 - 8(j-1)g_3]} \;\; \forall\, j \in [\frac{1}{2}, K] \text{ with } \; K < \infty \, ;$$

$$z(w) = \frac{1}{c}\exp(cw)$$

and γ, ω are the constraint constants of the Gauss Hypergeometric Equation (GHE)

$$z(z-1)\ddot{F}(z) + [\gamma(z-1) - (\alpha+\beta-\gamma+1)z]\dot{F}(z) - m(m+\omega)F(z) = 0,$$
$$\dot{} = \frac{\mathrm{d}}{\mathrm{d}z}. \quad (3.18)$$

Here, $m = -\alpha$ and $m + \omega = \beta$. The polynomial $P_m^{(\omega-\gamma,\gamma-1)}(\cdot)$ is the Jacobi polynomial of order m [cf: Abramowitz and Stegun ([1],§22.5-8, pp. 782-9)] given by

$$\begin{aligned} P_m^{(\omega-\gamma,\gamma-1)}(2z-1) &= \left(\begin{array}{c} m+\omega-\gamma \\ m \end{array} \right) F\left(-m, m+\omega, \omega-\gamma+1 \middle| 1-z \right) \\ &= \frac{\Gamma(\omega-\gamma+m+1)}{m!\Gamma(\omega+m)} \sum_{n=0}^{m} \left(\begin{array}{c} m \\ n \end{array} \right) \frac{\Gamma(\omega+n+m)}{\Gamma(\omega-\gamma+n+1)} (1-z)^n \, . \end{aligned}$$

4. Conclusion

In this work, the BHE has been re-written via $\mathfrak{sl}(2,\mathbb{C})$ generators. The algebraic solution of the Lie algebraic equation obtained has been presented in terms of Jacobi polynomial and exactly solvable potential generated by $\mathfrak{sl}(2,\mathbb{C})$ has been displayed.

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