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CONVERGENCE OF THE SBA ALGORITHM APPLYING TO SOLVE VOLTERRA NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND KIND

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Abstract: In this paper we shwo the convergence of the SBA method on the Volterra nonlinear integro-differential equations of second kind.

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1. Introduction

In this paper, we show the convergence of the SBA method (combination of the principle of Picard, Adomian method and the successive approximations) on the nonlinear integro-differential equations of Volterra second kind under the

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form:

$$\begin{cases} \varphi'(x) = f(x) + \lambda \int\limits_a^x K(x,t) g(\varphi(t))\, dt \\ \varphi(a) = c \end{cases}$$

where g is such as $g(\varphi(t)) = l(\varphi(t)) + N(\varphi(t))$, $\lambda > 0$, $l\ (\varphi(t)) = \varphi(t)$, $a \leq t \leq x \leq T < +\infty$ and N is nonlinear.

2. Convergence of the SBA Method

Theorem 1. *Let us consider the following Volterra integro-differential second kind*

$$(p) : \begin{cases} \varphi'(x) = f(x) + \lambda \int\limits_a^x K(x,t)\, g(\varphi(t))\, dt \\ \varphi(a) = c \end{cases} \tag{1}$$

where $g(\varphi(t)) = R(t) + N(\varphi(t)), \lambda > 0, \varphi \in C^1(\Sigma), f \in \Sigma$ and $K \in C(\Sigma \times \Omega)$

(p) is equivalent to

$$\varphi(x) = c + \int\limits_a^x f(z)\, dz + \lambda \int\limits_a^x \int\limits_a^z K(z,t)\, R(t)\, dtdz + \lambda \int\limits_a^x \int\limits_a^z K(z,t)\, N(\varphi(t))\, dtdz \tag{2}$$

Appliying the SBA method to (E), we get:

$$(p_{app}) \begin{cases} \varphi_0^k(x) = c + \int\limits_a^x f(z)\, dz + \lambda \int\limits_a^x \int\limits_a^z K(z,t)\, N\left(\varphi^{k-1}(t)\right) dtdz, k \geq 1 \\ \varphi_{n+1}^k(x) = \lambda \int\limits_a^x \int\limits_a^z K(z,t)\, \varphi_n^k(t)\, dtdz; n \geq 0 \end{cases} \tag{3}$$

We suppose the following hypotheses:

H_1: $\forall x \in \Sigma, |f(x)| \leq \beta, |c| = \mu$

H_2 : $\forall (x,t) \in \Sigma \times \Omega, |K(x,t)| \leq M$

H_3 : $\exists \varphi^0 \in H$ sach as $N\left(\varphi^0(t)\right) = 0$ and for $k = p, N(\varphi^p(t)) = 0$.

If the SBA algorithm associated to (E_{app}) converges to the step $k = 1$, then the solution $\varphi(x)$ of the equation (E) is unique, such as $\varphi(x) = \lim\limits_{k \to +\infty} \varphi^k(x)$.

Proof. At the step $k = 1$, we have: □

$$(p_{app}) : \begin{cases} \varphi_0^1(x) = c + \int_a^x f(z)\,dz \\ \varphi_{n+1}^1(x) = \lambda \int_a^x \int_a^z K(z,t)\,\varphi_n^1(t)\,dtdz; n \geq 0 \end{cases} \tag{4}$$

Appliying H_1 and H_2
$\Rightarrow$

$$(p_{app}) : \begin{cases} \left|\varphi_0^1(x)\right| \leq |c| + \int_a^x |f(z)|\,dz \leq \mu + \beta(x-a) \\ \left|\varphi_{n+1}^1(x)\right| \leq \lambda \int_a^x \int_a^z |K(z,t)| \left|\varphi_n^1(t)\right| dtdz; n \geq 0 \end{cases} \tag{5}$$

then

$$\begin{cases} \left|\varphi_0^1(x)\right| \leq \mu + \beta(x-a) \\ \left|\varphi_1^1(x)\right| \leq \lambda M \left[\mu \frac{(x-a)^2}{2!} + \beta \frac{(x-a)^3}{3!}\right] \\ \left|\varphi_2^1(x)\right| \leq \lambda^2 M^2 \left[\mu \frac{(x-a)^4}{4!} + \beta \frac{(x-a)^5}{5!}\right] \\ \dots \\ \left|\varphi_n^1(x)\right| \leq \lambda^n M^n \left[\mu \frac{(x-a)^{2n}}{(2n)!} + \beta \frac{(x-a)^{2n+1}}{(2n+1)!}\right] \end{cases} \tag{6}$$

$$\psi_1^p(x) = \mu \sum_{n=0}^{p} \frac{\left[\sqrt{\lambda M}(x-a)\right]^{2n}}{(2n)!} + \frac{\beta}{\sqrt{\lambda M}} \sum_{n=0}^{p} \frac{\left[\sqrt{\lambda M}(x-a)\right]^{2n+1}}{(2n+1)!}$$

so we obtain

$$\lim_{p \to +\infty} \psi_1^p(x) = \mu ch\left(\sqrt{\lambda M}(x-a)\right) + \gamma sh\left(\sqrt{\lambda M}(x-a)\right) \tag{7}$$

where $\gamma = \dfrac{\beta}{\sqrt{\lambda M}}$

which prouved that $\left(\sum_{n=0}^{+\infty} \varphi_n^1(x)\right)$ is absolutly congergent.

We supposed at the step $k = p \geq 1$,we have $N(\varphi^p(x)) = 0$ and we get at the step $k = p + 1$:

$$\begin{cases} \left|\varphi_0^{p+1}(x)\right| \leq \mu + \beta(x-a) \\ \left|\varphi_1^{p+1}(x)\right| \leq \lambda M\left[\mu\frac{(x-a)^2}{2!} + \beta\frac{(x-a)^3}{3!}\right] \\ \left|\varphi_2^{p+1}(x)\right| \leq \lambda^2 M^2\left[\mu\frac{(x-a)^4}{4!} + \beta\frac{(x-a)^5}{5!}\right] \\ \dots \\ \left|\varphi_n^{p+1}(x)\right| \leq \lambda^n M^n\left[\mu\frac{(x-a)^{2n}}{(2n)!} + \beta\frac{(x-a)^{2n+1}}{(2n+1)!}\right] \end{cases} \quad (8)$$

$\Rightarrow$

$$\sum_{n=0}^{+\infty}\left|\varphi_n^{p+1}(x)\right| \leq \mu\sum_{n=0}^{+\infty}\frac{\left[\sqrt{\lambda M}(x-a)\right]^{2n}}{(2n)!} + \gamma\sum_{n=0}^{+\infty}\frac{\left[\sqrt{\lambda M}(x-a)\right]^{2n+1}}{(2n+1)!}$$

$\Rightarrow$

$$\sum_{n=0}^{+\infty}\left|\varphi_n^{p+1}(x)\right| \leq \mu ch\left(\sqrt{\lambda M}(x-a)\right) + \gamma sh\left(\sqrt{\lambda M}(x-a)\right). \quad (9)$$

which prouved that $\left(\sum_{n=0}^{+\infty}\varphi_n^{p+1}(x)\right)$ is absolutly congergent, then we get $\varphi(x) = \lim_{k\to+\infty}\varphi^k(x)$.

Now, let us suppose that the equation (E) admits two distinct solutions $\varphi(x)$ and $\phi(x)$.

Taking $\delta(x) = \varphi(x) - \phi(x)$, appliying the SBA algorithm with the preceding hypotheses, we have:

$$\begin{cases} \delta_0^k(x) = 0 \ ; \ k = 1, 2, \cdots \\ \delta_n^k(x) = \lambda\int_a^x K(x,t)\delta_{n-1}^k(t)\,dt \ ; \ n = 1, 2, \cdots \end{cases} \quad (10)$$

which solution at each step k is $\delta^k(x) = 0$. Then we get:

$$\delta(x) = \lim_{k\to+\infty}\delta^k(x) = 0.$$

Hence

$$\forall t \in [a, x], \delta(t) = \varphi(t) - \phi(t) = 0 \Rightarrow \varphi(t) = \phi(t) \quad (11)$$

which is opposed to our hypothesis, so the solution of the equation (E) is unique.

3. Illustrative Examples

In this section, we solve some examples of linear and nonlinear integral differential equation Volterra second kind.

3.1. Example 1

Let us consider the following nonlinear integral differential equation second kind of Volterra, which is the canonical form of Adomian:

$$(E):\begin{cases} \dfrac{d\varphi(x)}{dx} = 1 + x(\ln(2)-1) + \int\limits_0^x \dfrac{\varphi(t)}{x+t}dt + \int\limits_0^x \dfrac{\varphi^5(t) - t^2\varphi^3(t)}{x+t}dt \\ \varphi(0) = 0 \end{cases} \tag{12}$$

where $\varphi \in C^1([0;T])$ and $0 \leq t < x \leq T < +\infty$.

We obtain:

$$\varphi(x) = x + (\ln(2)-1)\frac{x^2}{2!} + \int\limits_0^x\int\limits_0^z \frac{\varphi(t)}{z+t}dtdz + \int\limits_0^x\int\limits_0^z \frac{\varphi^5(t) - t^2\varphi^3(t)}{z+t}dtdz$$

The modified SBA algorithm for this equation is the following:

$$\begin{cases} \varphi_0^k(x) = x + \int\limits_0^x\int\limits_0^z \dfrac{N\left(\varphi^{k-1}(t)\right)}{z+t}dtdz\ ;\ k \geq 1 \\ \varphi_1^k(x) = (\ln(2)-1)\dfrac{x^2}{2!} + \int\limits_0^x\int\limits_0^z \dfrac{\varphi_0^k(t)}{z+t}dtdz \\ \varphi_n^k(x) = \int\limits_0^x\int\limits_0^z \dfrac{\varphi_{n-1}^k(t)}{z+t}dtdz\ ;\ n \geq 2 \end{cases}$$

where $N(\varphi(t)) = \varphi^5(t) - t^2\varphi^3(t)$.

At the step $k = 1$,appliying the principle of Picard, for $\varphi^0(x) = 0$, we have $N\left(\varphi^0(x)\right) = 0$ and we calculate: $\varphi_0^1(x), \varphi_1^1(x), ..., \varphi_n^1(x)$.

$$\begin{cases} \varphi_0^1(x) = x \\ \varphi_1^1(x) = (\ln(2)-1)\dfrac{x^2}{2!} + \int\limits_0^x\int\limits_0^z \dfrac{\varphi_0^k(t)}{z+t}dtdz = 0 \\ ... \\ \varphi_n^1(x) = \int\limits_0^x\int\limits_0^z \dfrac{\varphi_{n-1}^k(t)}{z+t}dtdz = 0; n \geq 2 \end{cases}$$

converges to $\varphi^1(x) = \sum_{n=0}^{+\infty} \varphi_n^1(x) \Rightarrow \varphi^1(x) = x.$

At the step $k = 2$, we have $N\left(\varphi^1(x)\right) = 0$ and we calculate: $\varphi_0^2(x), \varphi_1^2(x)$, ..., $\varphi_n^2(x)$.

$$\begin{cases} \varphi_0^2(x) = x \\ \varphi_1^2(x) = 0 \\ \ldots \\ \varphi_n^2(x) = 0; n \geq 1 \end{cases}$$

converges to $\varphi^2(x) = \sum_{n=0}^{+\infty} \varphi_n^2(x) \Rightarrow \varphi^2(x) = x.$

In the recurcive way, we obtain: $\varphi^1(x) = \varphi^2(x) = ... = \varphi^k(x) = x$.Therefore, we obtain the exact solution of equation (E) :

$$\varphi(x) = \lim_{k \to +\infty} \varphi^k(x) = x. \tag{13}$$

3.2. Example 2

Let us consider the following nonlinear integral equation second kind of Volterra, which is the canonical form of Adomian:

$$(F): \begin{cases} \dfrac{d\varphi(x)}{dx} &= e^x(1 - xe^x) + \int_0^x e^{2x-t}\varphi(t)\,dt + \\ & \int_0^x e^{2x-t}\left(\sqrt{e^{-2t}}\varphi^3(t) - \varphi^2(t)\right)dt \\ \varphi(0) &= 1 \end{cases} \tag{14}$$

where $\varphi \in C^1([0;T])$ and $0 \leq t < x \leq T < +\infty$.

We obtain:

$$\varphi(x) = e^x - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} - \frac{1}{4} + \int_0^x\int_0^z e^{2z-t}\varphi(t)\,dtdz + \int_0^x\int_0^z e^{2z-t}N(\varphi(t))\,dtdz$$

Where $N(\varphi(t)) = \sqrt{e^{-2t}}\varphi^3(t) - \varphi^2(t)$.

The modified SBA algorithm for this equation is the following:

$$\begin{cases} \varphi_0^k(x) = e^x + \int_0^x\int_0^z e^{2z-t}N\left(\varphi^{k-1}(t)\right)dtdz \ ; \ k \geq 1 \\ \varphi_1^k(x) = -\dfrac{1}{2}xe^{2x} + \dfrac{1}{4}e^{2x} - \dfrac{1}{4} + \displaystyle\int_0^x\int_0^z e^{2z-t}\varphi_0^k(t)\,dtdz \\ \varphi_n^k(x) = \int_0^x \dfrac{\varphi_{n-1}^k(t)}{x+t}dt \ ; \ n \geq 2 \end{cases}$$

At the step $k=1$,appliying the principle of Picard, for $\varphi^0(x)=0$, we have $N\left(\varphi^0(x)\right)=0$ and calculate $\varphi_0^1(x), \varphi_1^1(x), ..., \varphi_n^1(x)$.

$$\begin{cases} \varphi_0^1(x)=e^x \\ \varphi_1^1(x)=0 \\ ... \\ \varphi_n^1(x)=0; n\geq 1 \end{cases}$$

converges to $\varphi^1(x)=\sum\limits_{n=0}^{+\infty}\varphi_n^1(x)\Rightarrow\varphi^1(x)=e^x$.

At the step $k=2$, we have $N\left(\varphi^1(x)\right)=0$ we calculate: $\varphi_0^2(x), \varphi_1^2(x), ...,$ $\varphi_n^2(x)$.

$$\begin{cases} \varphi_0^2(x)=e^x \\ \varphi_1^2(x)=0 \\ \varphi_2^2(x)=0 \\ ... \\ \varphi_n^2(x)=0; n\geq 1 \end{cases}$$

converges to $\varphi^2(x)=\sum\limits_{n=0}^{+\infty}\varphi_n^2(x)\Rightarrow\varphi^2(x)=e^x$.

In the recurcive way, we get: $\varphi^1(x)=\varphi^2(x)=...=\varphi^k(x)=e^x$.Therefore, we obtain the exact solution of equation (F)

$$\varphi(x)=\lim_{k\to+\infty}\varphi^k(x)=e^x. \tag{15}$$

3.3. Example 3

Let us consider the following nonlinear integral equation second kind of Volterra, which is the canonical form of Adomian:

$$(H):\begin{cases} \dfrac{d\varphi(x)}{dx} & = & 1-2\sqrt{x}-\dfrac{2}{3}x\sqrt{x}+\displaystyle\int_0^x\frac{\varphi(t)}{\sqrt{x-t}}dt+ \\ & & \displaystyle\int_0^x\frac{(t\varphi(t))^2-\varphi^4(t)}{\sqrt{x-t}}dt \\ \varphi(0) & = & 0 \end{cases} \tag{16}$$

where $\varphi\in C^1([0;T])$ and $0\leq t<x\leq T<+\infty$.

We obtain:

$$\varphi(x)=x-\frac{8}{15}x^2\sqrt{x}+\int_0^x\int_0^z\frac{\varphi(t)}{\sqrt{z-t}}dtdz+\int_0^x\int_0^z\frac{\left(t^2\varphi^2(t)-\varphi^4(t)\right)}{\sqrt{z-t}}dtdz,$$

Here $N(\varphi(t)) = (t\varphi(t))^2 - \varphi^4(t)$.

The modified SBA algorithm for this equation is the following:

$$\begin{cases} \varphi_0^k(x) = x + \int\limits_0^x \int\limits_0^z \frac{N\left(\varphi^{k-1}(t)\right)}{\sqrt{z-t}} dtdz \; ; \; k \geq 1 \\ \varphi_1^k(x) = -\frac{8}{15}x^2\sqrt{x} + \int\limits_0^x \int\limits_0^z \frac{\varphi_0^k(t)}{\sqrt{z-t}} dtdz \\ \varphi_n^k(x) = \int\limits_0^x \frac{\varphi_{n-1}^k(t)}{\sqrt{x-t}} dt \; ; \; n \geq 2 \end{cases}$$

At the step $k = 1$,appliying the principle of Picard, for $\varphi^0(x) = 0$, we have $N\left(\varphi^0(x)\right) = 0$ and calculate $\varphi_0^1(x), \varphi_1^1(x), ..., \varphi_n^1(x)$.

$$\begin{cases} \varphi_0^1(x) = x \\ \varphi_1^1(x) = 0 \\ ... \\ \varphi_n^1(x) = 0; n \geq 1 \end{cases}$$

converges to $\varphi^1(x) = \sum\limits_{n=0}^{+\infty} \varphi_n^1(x) \Rightarrow \varphi^1(x) = x$.

At the step $k = 2$, we have $N\left(\varphi^1(x)\right) = 0$ we calculate: $\varphi_0^2(x), \varphi_1^2(x), ...,$ $\varphi_n^2(x)$.

$$\begin{cases} \varphi_0^2(x) = x \\ \varphi_1^2(x) = 0 \\ \varphi_2^2(x) = 0 \\ ... \\ \varphi_n^2(x) = 0; n \geq 1 \end{cases}$$

converges to $\varphi^2(x) = \sum\limits_{n=0}^{+\infty} \varphi_n^2(x) \Rightarrow \varphi^2(x) = x$.

In the recurcive way, we get: $\varphi^1(x) = \varphi^2(x) = ... = \varphi^k(x) = x$.Therefore, we obtain the exact solution of equation (H)

$$\varphi(x) = \lim_{k \to +\infty} \varphi^k(x) = x. \tag{17}$$

4. Conclusion

In this paper, we prouved the convergence of the algorithm SBA for the non-linear integral equation Volterra of second kind and applied this convergence to solve successfuly this kind of integral equations.

References

[1] A.M. Wazwaz, A reliable treatment of singular Emden-Flower initial value problems and boundary value problems, *Applied Mathematics and Computaion*, **217** (2011), 19387-10395.

[2] A.M. Wazwaz, A new algorithm for solving differential equations of Lane-Emden type, *Applied Mathematics and Computation*, **118** (2001), 287-310.

[3] Bakari Abbo, Abba Danna, Pare Youssouf and Blaise Some, Extension of the new iterative approach of Adomian algorithm to partial equations (PDEs) strongly nonlinear with initial and boundary conditions, *Far East J. Math. Sci. (FJMS)*, **75**, No. 2 (2013), 245-255.

[4] Bakari Abbo, N. N'garasta, B. Mampassi, B. Some and Longin Some, A new approach of the Adomian algorithm for solving nonlinear ordinary or partial differential equations, *Far East J. Appl. Math.*, **23**, No. 3 (2006), 299-312.

[5] Bakari Abbo, Nouvel algorithme numérique de résolution des équations différentielles ordinaires (EDO) et des équations aux dérivées partielles non linéaires" Thèse de Doctorat unique, Université de Ouagadougou, Janvier 2007.

[6] K. Abbaoui and Y. Cherruault, The decomposition method applied to the Cauchy problem, *Kybernetes*, **28**, No. 1 (1999), 68-74.

[7] S. Kheilifa and Y. Cherruault, The decomposition method for solving first order partial differntial equations, *Kybernetes*, **31**, No. 6 (2002), 844-871.

[8] Pare Youssouf, Saley Bisso, So Ousseni and Blaise Some, A numerical method for solving Cauchy evolution problem of partial differential equations (PDEs) for several variables, *Inter. J. Numer. Method. Appl.*, **1**, No. 1 (2009), 87-100.

[9] Pare Youssouf, Yaro Rasmane, Elysee Gouba and Blaise Some, Solving a system of nonlinear equations second kind of Volterra by the SBA method, *Far. East. J. Appl. Math.*, **71**, No. 1 (2012), 43-54.

[10] Youssouf Pare, Françis Bassono and Blaise Some, A new technique for numerical resolution of new nonlinear integral equations of Fredholm by SBA method, *Far East J. Appl. Math.*, **70**, No. 1 (2012), 21-33.

[11] Youssouf Pare, Abbo Bakari, Rasmane Yaro and Blaise Some, Solving first king Abel integral equations using the SBA numerical method, *Nonl. Analysis and Differential Equations*, **1**, No. 3 (2013), 115-128.