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LIAPUNOV-TYPE INTEGRAL INEQUALITIES FOR TWO-DIMENSIONAL PARTIAL DYNAMIC SYSTEMS ON TIME SCALES

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Abstract: In this paper, Liapunov-type integral inequalities has been obtained for two-dimensional partial dynamic systems on time scales. As an applications, we prove that every weakly oscillatory proper solution of the two-dimensional Emden-Fowler-type equation is bounded on $[s_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}$, where $\mathbb{T}$ is a time scale.

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1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [15] in his Ph. D. thesis in 1988 in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory; see the survey paper of Agarwal et. al. [1] and references cited therein and a book on the subject of time scales by Bohner and Peterson [3]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represents the classical theories of differential equations and difference equations respectively.

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In [17], Russian mathematician Liapunov proved that if $y(t)$ is a non-trivial solution of

$$y^{\Delta\Delta} + p(t)y = 0 \tag{1.1}$$

with $y(a) = 0 = y(b)$, where $a, b \in \mathbb{R}$ with $a < b$ and $y(t) \neq 0$ for $t \in (a, b)$, then

$$\int_a^b |p(t)|dt > \frac{4}{b-a} \tag{1.2}$$

holds, where $p \in L^1_{loc}$.

This result has found applications in differential and difference equations in the study of various properties of solutions of (1.1) and it is useful tools in oscillation theory, disconjugacy and eigen value problems (see [6] -[12], [14], [19]).

Bohner et al. [3] extended the Liapunov inequality (1.2) on time scale $\mathbb{T}$ for the dynamic equation

$$y^{\Delta\Delta}(t) + p(t)y^{\sigma}(t) = 0, \tag{1.3}$$

where $p(t)$ is a positive rd-continious function defined on $\mathbb{T}$. They proved, by using the quadratic functional equation

$$F(y) = \int_a^b [(y^{\Delta}(t))^2 - p(t)(y^{\sigma})^2]\Delta t = 0,$$

that if $y(t)$ is a nontrivial solution of (1.3) with $y(a) = 0 = y(b)(a < b)$, then

$$\int_a^b p(t)\Delta t > \frac{(b-a)}{f(d)},$$

where $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = (t-a)(t-b)$ and $d \in \mathbb{T}$ such that $f(d) = \max\{f(t) : t \in [a, b]\}$. In particular, using the fact that, $a < c < b$ and

$$\frac{1}{c-a} + \frac{1}{b-c} = \frac{(a+b-2c)^2}{(b-a)(c-a)(b-c)} + \frac{4}{b-a} > \frac{4}{b-a},$$

they obtained

$$\int_a^b p(t)\Delta t > \frac{4}{b-a}.$$

In [20], Aydin Tiryaki et. al. established a Liapunov-type inequalities for the nonlinear systems of differential equations of the form

$$\begin{aligned} x'(t) &= \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t), \\ u'(t) &= -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t), \end{aligned} \tag{1.4}$$

where $\gamma > 1, \beta > 1$ are real constants. $\beta_1, \beta_2 : [0, \infty) \subset \mathbb{R} \to \mathbb{R}$. are continuous functions such that $\beta_1(t) > 0$ for $t \in [0, \infty)$ and $\alpha_1 : [0, \infty) \to \mathbb{R}$ is a continuous function.

In [21], Ünal and Cakmak derived the liapunov-type inequalities for the nonlinear system

$$\begin{aligned} x^{\Delta}(t) &= \alpha_1(t)x(\sigma(t)) + \beta_1(t)|u(t)|^{\gamma-2}u(t), \\ u^{\Delta}(t) &= -\beta_2(t)|x(\sigma(t))|^{\alpha-2}x(\sigma(t)) - \alpha_1(t)u(t), \end{aligned} \tag{1.5}$$

on a time scale $\mathbb{T}$, where α_1, β_1 and β_2 are real rd-continuous functions on $\mathbb{T}$ with $1 - \mu(t)\alpha_1(t) \neq 0$ and $\beta_1(t) > 0, \alpha > 1$ is a constant and α is a conjugate to γ, that is, $\dfrac{1}{\alpha} + \dfrac{1}{\gamma} = 1$. It is interesting to observe that the second order half-linear dynamic equation

$$[r(t)|x^{\Delta}(t)|^{\alpha-2}x^{\Delta}(t)]^{\Delta} + q(t)|x(\sigma(t))|^{\alpha-2}x(\sigma(t)) = 0, \tag{1.6}$$

where r and q are real rd-continuous functions with $r(t) > 0$ for all $t \in \mathbb{T}$ and $\alpha > 1$, can be written as equivalent nonlinear system (1.5) on $\mathbb{T}$. Indeed, $x(t)$ be a solution of (1.5) and set $u(t) = r(t)|x^{\Delta}(t)|^{\alpha-2}x^{\Delta}(t)$. Then we have

$$x^{\Delta}(t) = r^{1-\gamma}(t)|u(t)|^{\gamma-2}u(t), \quad u^{\Delta}(t) = -q(t)|x(\sigma(t))|^{\alpha-2}x(\sigma(t)). \tag{1.7}$$

Hence (1.6) is equivalent to (1.5) with

$$\alpha_1(t) = 0, \quad \beta_1(t) = r^{1-\gamma}(t), \quad \beta_2(t) = q(t).$$

We also note here that the nonlinear system (1.5) with $\alpha = \gamma = 2$ cover not only the recent paper by Jiang and Zhou [16]

$$\begin{aligned} x^{\Delta}(t) &= \alpha_1(t)x(\sigma(t)) + \beta_1(t)u(t), \\ u^{\Delta}(t) &= -\beta_2(t)x(\sigma(t)) - \alpha_1(t)u(t), \end{aligned}$$

but also the linear Hamiltonian system (when $\mathbb{T} = \mathbb{R}$, see [13, 18])

$$x'(t) = \alpha_1(t)x(\sigma(t)) + \beta_1(t)u(t),$$

$$u'(t) = -\beta_2(t)x(\sigma(t)) - \alpha_1(t)u(t),$$

and discrete Hamiltonian system (when $\mathbb{T} = \mathbb{Z}$, see [2, 13])

$$\begin{aligned}\Delta x(t) &= \alpha_1(t)x(t+1) + \beta_1(t)u(t),\\ \Delta u(t) &= -\beta_2(t)x(t+1) - \alpha_1(t)u(t).\end{aligned}$$

Consider the two-dimensional nonlinear partial dynamic systems of the form

$$\begin{aligned}\frac{\partial^2 x(s,t)}{\Delta_1 s \Delta_2 t} &= \alpha_1(s,t)x(\sigma(s),\sigma(t)) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),\\ \frac{\partial^2 u(s,t)}{\Delta_1 s \Delta_2 t} &= -\beta_2(s,t)|x(\sigma(s),\sigma(t))|^{\beta-2}x(\sigma(s),\sigma(t)) - \alpha_1(s,t)u(s,t).\end{aligned} \tag{1.8}$$

We shall assume the existence of nontrivial system $x(s,t)$ and $u(s,t)$ of the system (1.8), and furthermore, (1.8) satisfying the following assumptions
(A_1) $\gamma > 1, \beta > 1$ are real constants;
(A_2) $\beta_1(s,t), \beta_2(s,t) : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ are rd-continious functions such that $\beta_1(s,t) > 0$ for $(s,t) \in \mathbb{T}_1 \times \mathbb{T}_2$; where $\mathbb{T}_1 = [s_0,\infty)_{\mathbb{T}} = [s_0,\infty) \cap \mathbb{T}$ and $\mathbb{T}_2 = [t_0,\infty)_{\mathbb{T}} = [t_0,\infty) \cap \mathbb{T}$; and
(A_3) $\alpha_1(s,t) : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is a rd-continuous function,
where $\sigma(t)$ is the forward jump operator defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$.

Note that (1.8) in its general form involves some different types of differential and difference systems depending on the choice of time scales $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, (1.8) becomes a even order differential system. When $\mathbb{T} = \mathbb{Z}$, (1.8) is an even order difference system. When $\mathbb{T} = h\mathbb{Z}$, then (1.8) becomes a generalized difference system and when $\mathbb{T} = q^{\mathbb{N}}$, then (1.8) becomes a quantum difference system. Note also that results in this paper can be applied on the time scales $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, $\mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, $\mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$ and when $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$, where $\{t_n\}$ is a set of harmonic numbers.

The organizations of the paper is as follows. Section 2 will give some preliminaries on time scales. In Section 3, Liapunov- type integral inequality has been derived for the two-dimensional partial dynamic systems. As an application, we prove that every weakly oscillatory proper solution of two-dimensional Emden-Flower-type equation is bounded on $[s_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$, where $\mathbb{T}$ is a time scale.

2. Preliminaries on Time Scales

2.1. Function of Single Variable

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. On any time scale we define the "forward and backward jump operators" by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

We make the convention:

$$\inf \phi = sup\mathbb{T}, \quad \sup \phi = \inf \mathbb{T}.$$

A point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) = t$, right dense if $\sigma(t) = t$, left scattered if $\rho(t) < t$, right scattered if $\sigma(t) > t$. The points that are simultaneously right-dense and left-dense are called dense.

The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

and

$$\nu(t) = t - \rho(t)$$

are called, respectively, the forward and backward graininess functions.

If $\mathbb{T}$ has a right- scattered minimum m, then define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has left-scattered maximum M, then define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Finally, put $\mathbb{T}_k^k = \mathbb{T}_k \cap \mathbb{T}^k$. For a function $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^k$ the delta derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is right-dense, then derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point and if there exists a finite left limit at all left dense points.

The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The derivative and the shift operator σ are related by the formula

$$f^\sigma = f + \mu f^\Delta, \quad where \quad f^\sigma = f \circ \sigma.$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nonincreasing, and nondecreasing on $[a, b]$ if $t_1, t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \leq f(t_1)$, $f(t_2) \geq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nonincreasing, and nondecreasing on $[a, b]$ if $f^\Delta(t) > 0$, $f^\Delta(t) < 0$, $f^\Delta(t) \leq 0$, $f^\Delta(t) \geq 0$, for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g:

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)),$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

The integration by parts formula read as

$$\int_a^b f^\Delta(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) + \int_a^b f^\sigma(t)g^\Delta(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s.$$

Theorem 2.1. *(Holder's Inequality) Let $a, b \in \mathbb{T}$. For rd- continuous $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^p \Delta t \right\}^{\frac{1}{p}} \left\{ \int_a^b |g(t)|^q \Delta t \right\}^{\frac{1}{q}},$$

where $p > 1$ and $q = p/(p - 1)$.

The special case $p = q = 2$ reduces to the Cauchy-Schwarz Inequality.

Theorem 2.2. *Let $a, b \in \mathbb{T}$. For rd- continuous $f, g : [a, b] \to \mathbb{R}$, we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^2 \Delta t \right\}^{\frac{1}{2}} \left\{ \int_a^b |g(t)|^2 \Delta t \right\}^{\frac{1}{2}}.$$

2.2. Partial Derivatives

Let $n \in \mathbb{N}$ be fixed. Further, for each $i \in \{1, 2, ..., n\}$ let $\mathbb{T}_i$ denote a time scale, that is, $\mathbb{T}_i$ is a nonempty closed subset of the real numbers $\mathbb{R}$. Let us set

$$\Gamma^n = \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n = \{t = (t_1, t_2, ..., t_n) : t_i \in \mathbb{T}_i \quad for \quad all \quad i \in \{1, 2, ..., n\}\}.$$

We call Γ^n an *n-dimensional time scale.* The set Γ^n is a complete metric space with the metric defined by

$$d(t, s) = \left(\sum_{i=1}^{n} |t_i - s_i|^2 \right)^{1/2} \quad for \quad t, s \in \Gamma^n.$$

Consequently, according to the well-known theory of general metric spaces, we have for Γ^n the fundamental concepts such as open balls, neighbourhoods of points, open sets, closed sets, compact sets, and so on. In particular, for a given number $\delta > 0$, the δ- neighbourhood $U_\delta(t^0)$ of a given point $t^0 \in \Gamma^n$ is the set of all points $t \in \Gamma^n$ such that $d(t^0, t) < \delta$. By a neighbourhood of a point $t^0 \in \Gamma^n$ is meant an arbitrary set in Γ^n containing a $\delta-$ neighbourhood of the point t^0. Also we have for functions $f : \Gamma^n \to \mathbb{R}$ the concept of limit, continuity, and properties of continuous functions on general complete metric spaces.

In the following we introduce the partial derivatives for functions $f : \Gamma^n \to \mathbb{R}$. This is possible due to the special structure of the metric space Γ^n.

Let σ_i and ρ_i denote, respectively, the *forward and backward jump operators* in $\mathbb{T}_i$. For $u \in \mathbb{T}_i$ the *forward jump operator* $\sigma_i : \mathbb{T}_i \to \mathbb{T}_i$ is defined by

$$\sigma_i(t) = \inf\{v \in \mathbb{T}_i : v > u\}$$

and the *backward jump operator* $\rho : \mathbb{T}_i \to \mathbb{T}_i$ is defined

$$\rho_i(t) = \sup\{v \in \mathbb{T}_i : v < u\}.$$

In this defination we put $\sigma_i(\max \mathbb{T}_i) = \max \mathbb{T}_i$ if $\mathbb{T}_i$ has finite maximum, and $\rho_i(\min \mathbb{T}_i) = \min \mathbb{T}_i$ if $\mathbb{T}_i$ has a finite minimum. If $\sigma_i(u) > u$, then we say that u is *right-scattered* (in $\mathbb{T}_i$) while any u with $\rho_i(u) < u$ is *left-scattered* (in $\mathbb{T}_i$).

Also, if $u < \max \mathbb{T}_i$ and $\sigma_i(u) = u$, then u is called *right dense* (in $\mathbb{T}_i$), and if $u > \min \mathbb{T}_i$ and $\rho_i(u) = u$, then u is called *left dense* (in $\mathbb{T}_i$). If $\mathbb{T}_i$ has left-scattered maximum M, then we define $\mathbb{T}_i^k = \mathbb{T}_i \setminus \{M\}$, otherwise $\mathbb{T}_i^k = \mathbb{T}_i$. If $\mathbb{T}_i$ is aright-scattered minimum m, then we define $(\mathbb{T}_i)_k = \mathbb{T}_i \setminus \{m\}$, otherwise $(\mathbb{T})_k = \mathbb{T}_i$.

Let $f : \Gamma^n \to \mathbb{R}$ be a function. The *partial delta derivative* of f with respect to $t_i \in \mathbb{T}_i^k$ defined as the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1, t_2, ..., t_{i-1}, \sigma_i(t_i), t_{i+1}, ..., t_n) - f(t_1, t_2, ..., t_{i-1}, s_i, t_{i+1}, ..., t_n)}{\sigma_i(t_i) - s_i}$$

provided that this limit exists as a finite number, and it is denoted by any of the following symbols :

$$\frac{\partial f(t_1, ..., t_n)}{\Delta_i t_i}, \quad \frac{\partial f(t)}{\Delta_i t_i}, \quad \frac{\partial f}{\Delta_i t_i}(t), \quad f_{t_i}^{\Delta_i}(t).$$

If f has partial derivatives $\frac{\partial f(t)}{\Delta_1 t_1}, ..., \frac{\partial f(t)}{\Delta_n t_n}$, then we can also consider their partial delta derivatives. These are called *second order* partial delta derivatives. We write

$$\frac{\partial^2 f(t)}{\Delta_i t_i^2} \ \ and \ \ \frac{\partial^2 f(t)}{\Delta_j t_j \Delta_i t_i} \ \ or \ \ \{f_{t_i t_i}^{\Delta_i \Delta_i}(t) \ \ and \ \ f_{t_i t_j}^{\Delta_i \Delta_j}(t)\}$$

partial delta derivatives of $\frac{\partial f(t)}{\Delta_i t_i}$ with respect to t_i and with respect to t_j, respectively. Thus

$$\frac{\partial^2 f(t)}{\Delta_i t_i^2} = \frac{\partial}{\Delta_i t_i}\left(\frac{\partial f(t)}{\Delta_i t_i}\right) \ \ and \ \ \frac{\partial^2 f(t)}{\Delta_j t_j \Delta_i t_i} = \frac{\partial}{\Delta_j t_j}\left(\frac{\partial f(t)}{\Delta_i t_i}\right).$$

Higher order partial delta derivatives are similarly defined. The *partial nabla derivatives* of f with respect to $t_i \in (\mathbb{T}_i)_k$ is defined as the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \rho_i(t_i)}} \frac{f(t_1, t_2, ..., t_{i-1}, \rho_i(t_i), t_{i+1}, ..., t_n) - f(t_1, t_2, ..., t_{i-1}, s_i, t_{i+1}, ..., t_n)}{\rho_i(t_i) - s_i}$$

and denoted by $\frac{\partial f(t)}{\nabla_i t_i}$, provided that this limit exists as a finite number. In an obvious way we can define higher order partial nabla derivatives and also mixed derivatives obtained by combining both delta and nabla differentiations such as, for instance,

$$\frac{\partial^2 f(t)}{\Delta_i t_i \Delta_j t_j} \quad or \frac{\partial^3 f(t)}{\Delta_i t_i^2 \Delta_j t_j}.$$

3. Main Results

In this section, we establish the Liapunov-type inequality for two-dimensional nonlinear partial dynamic systems (1.8).

Theorem 3.1. *Let the hypothesis $(A_1)-(A_3)$ hold. If the nonlinear partial dynamic system (1.8) has a real solution $(x(s,t), u(s,t))$ such that $x(\sigma(a),t) = x(\sigma(b),t) = x(s,\sigma(c)) = x(s,\sigma(d)) = 0$ for $(s,t) \in [\sigma(a),\sigma(b)]_{\mathbb{T}} \times [\sigma(c),\sigma(d)]_{\mathbb{T}}$ and*

$$(\partial u(s,t)/\Delta_1 s)(\partial x(\sigma(s),t)/\Delta_2 t) + (\partial u(s,t)/\Delta_2 t)(\partial x(s,\sigma(t))/\Delta_1 s) = 0$$

and $x(s,t)$ is not identically zero on $[\sigma(a),\sigma(b)]_{\mathbb{T}} \times [\sigma(c),\sigma(d)]_{\mathbb{T}}$, with $\sigma(a) < \sigma(b), \sigma(c) < \sigma(d)$, then

$$2 \le \int_{\sigma(a)}^{\sigma(b)} \int_{\sigma(c)}^{\sigma(d)} |\alpha_1(s,t)|\Delta_2 t \Delta_1 s + M^{\beta/\alpha - 1} \left(\int_{\sigma(a)}^{\sigma(b)} \int_{\sigma(c)}^{\sigma(d)} |\beta_1(s,t)|\Delta_2 t \Delta_1 s \right)^{1/\gamma} \times \left(\int_{\sigma(a)}^{\sigma(b)} \int_{\sigma(c)}^{\sigma(d)} \beta_2^+(s,t)\Delta_2 t \Delta_1 s \right)^{1/\alpha}, \quad (3.1)$$

where $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$, $M = \max_{\substack{\sigma(a) \le s \le \sigma(b) \\ \sigma(c) \le t \le \sigma(d)}} |x(s,t)| > 0$, $\beta_2^+(s,t) = \max_{\sigma(a) < s < \sigma(b)} \{\beta_2(s,t), 0\}$ *is the non negative part of* $\beta_2(s,t)$.

Proof. Since $x(\sigma(a),t) = x(\sigma(b),t) = x(s,\sigma(c)) = x(s,\sigma(d)) = 0$ and $x(s,t)$ is not identically zero on for $(\sigma(a),\sigma(b))_{\mathbb{T}} \times (\sigma(c),\sigma(d))_{\mathbb{T}}$, we choose $(\tau_1,\tau_2) \in (\sigma(a),\sigma(b))_{\mathbb{T}} \times (\sigma(c),\sigma(d))_{\mathbb{T}}$ such that

$$|x(\tau_1,\tau_2)| = \max_{\substack{\sigma(a) < s < \sigma(b) \\ \sigma(c) < t < \sigma(d)}} |x(s,t)| > 0.$$

Let $M = |x(\tau_1,\tau_2)| > 0$. Integrating the first equation of the system (1.8) over t from $\sigma(c)$ to τ_2 and over s from $\sigma(a)$ to τ_1, respectively, we obtain

$$\int_{\sigma(a)}^{\tau_1} \int_{\sigma(c)}^{\tau_2} \frac{\partial^2 x(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s = \int_{\sigma(a)}^{\tau_1} \int_{\sigma(c)}^{\tau_2} \Big(\alpha_1(s,t) x(\sigma(s),\sigma(t))$$

$$+\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\Big)\Delta_2 t\Delta_1 s. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}\int_{\sigma(a)}^{\tau_1}\int_{\sigma(c)}^{\tau_2}\frac{\partial^2 x(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s &= \int_{\sigma(a)}^{\tau_1}\int_{\sigma(c)}^{\tau_2}\frac{\partial}{\Delta_2 t}\left(\frac{\partial x(s,t)}{\Delta_1 s}\right)\Delta_2 t\Delta_1 s\\ &= \int_{\sigma(a)}^{\tau_1}\frac{\partial x(s,\tau_2)}{\Delta_1 s}\Delta_1 s - \int_{\sigma(a)}^{\tau_1}\frac{\partial x(s,\sigma(c))}{\Delta_1 s}\Delta_1 s\\ &= x(\tau_1,\tau_2) - x(\sigma(a),\tau_2) - x(\tau_1,\sigma(c)) - x(\sigma(a),\sigma(c))\\ &= x(\tau_1,\tau_2). \end{aligned} \tag{3.3}$$

Hence,

$$\begin{aligned}&x(\tau_1,\tau_2)\\ &= \int_{\sigma(a)}^{\tau_1}\int_{\sigma(c)}^{\tau_2}\Big(\alpha_1(s,t)x(\sigma(s),\sigma(t)) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\Big)\Delta_2 t\Delta_1 s.\end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned}&x(\tau_1,\tau_2)\\ &= \int_{\tau_1}^{\sigma(b)}\int_{\tau_2}^{\sigma(d)}\Big(\alpha_1(s,t)x(\sigma(s),\sigma(t)) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\Big)\Delta_2 t\Delta_1 s.\end{aligned} \tag{3.5}$$

Use of the triangle inequality in (3.4) and (3.5), we obtain

$$\begin{aligned}&|x(\tau_1,\tau_2)|\\ &= \int_{\sigma(a)}^{\tau_1}\int_{\sigma(c)}^{\tau_2}|\alpha_1(s,t)||x(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s + \beta_1(s,t)|u(s,t)|^{\gamma-1}\Delta_2 t\Delta_1 s.\end{aligned} \tag{3.6}$$

$$\begin{aligned}&|x(\tau_1,\tau_2)|\\ &= \int_{\tau_1}^{\sigma(b)}\int_{\tau_2}^{\sigma(d)}|\alpha_1(s,t)||x(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s + \beta_1(s,t)|u(s,t)|^{\gamma-1}\Delta_2 t\Delta_1 s.\end{aligned} \tag{3.7}$$

Summing (3.6) and (3.7), we obtain

$$\begin{aligned} 2|x(\tau_1,\tau_2)| &= \int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\alpha_1(s,t)||x(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s \\ &+ \int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)|u(s,t)|^{\gamma-1}\Delta_2 t\Delta_1 s. \end{aligned} \tag{3.8}$$

By using Hölder's inequality on the second integral of (3.8) with indices α and γ, we obtain

$$\begin{aligned} &\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)|u(s,t)|^{\gamma-1}\Delta_2 t\Delta_1 s \\ &= \int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)^{1/\gamma}\beta_1(s,t)^{1/\alpha}|u(s,t)|^{\gamma-1}\Delta_2 t\Delta_1 s \\ &= \left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)\Delta_2 t\Delta_1 s\right)^{1/\gamma} \\ &\times \left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)|u(s,t)|^{\alpha(\gamma-1)}\Delta_2 t\Delta_1 s\right)^{1/\alpha} \\ &= \left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)\Delta_2 t\Delta_1 s\right)^{1/\gamma}\left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)|u(s,t)|^{\gamma}\Delta_2 t\Delta_1 s\right)^{1/\alpha}, \end{aligned}$$

where $\dfrac{1}{\alpha}+\dfrac{1}{\gamma}=1$.

Consequently, (3.8) yields

$$\begin{aligned} 2|x(\tau_1,\tau_2)| = &\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\alpha_1(s,t)||x(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s \\ &+ \left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)\Delta_2 t\Delta_1 s\right)^{1/\gamma} \\ &\times \left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)|u(s,t)|^{\gamma}\Delta_2 t\Delta_1 s\right)^{1/\alpha}. \end{aligned} \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} \frac{\partial^2(x(s,t)u(s,t))}{\Delta_1 s\Delta_2 t} &= \frac{\partial}{\Delta_1 s}\left(\frac{\partial x(s,t)}{\Delta_2 t}u(s,t)+x(s,\sigma(t))\frac{\partial u(s,t)}{\Delta_2 t}\right) \\ &= \frac{\partial^2 x(s,t)}{\Delta_1 s\Delta_2 t}u(s,t)+\frac{\partial x(\sigma(s),t)}{\Delta_2 t}\frac{\partial u(s,t)}{\Delta_1 s} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\partial x(s,\sigma(t))}{\Delta_1 s}\frac{\partial u(s,t)}{\Delta_2 t} \\
&+ \quad x(\sigma(s),\sigma(t))\frac{\partial^2 u(s,t)}{\Delta_1 s\Delta_2 t} \\
&= \quad u(s,t)\frac{\partial^2 x(s,t)}{\Delta_1 s\Delta_2 t} + x(\sigma(s),\sigma(t))\frac{\partial^2 u(s,t)}{\Delta_1 s\Delta_2 t}. \qquad (3.10)
\end{aligned}
$$

Multiplying first equation of (1.8) by $u(s,t)$ and second equation of (1.8) by $x(\sigma(s),\sigma(t))$, we obtain

$$
\begin{aligned}
u(s,t)\frac{\partial^2 x(s,t)}{\Delta_1 s\Delta_2 t} &= \alpha_1(s,t)x(\sigma(s),\sigma(t))u(s,t) + \beta_1(s,t)|u(s,t)|^\gamma, \\
x(\sigma(s),\sigma(t))\frac{\partial^2 u(s,t)}{\Delta_1 s\Delta_2 t} &= -\beta_2(s,t)|x(\sigma(s),\sigma(t))|^\beta - \alpha_1(s,t)x(\sigma(s),\sigma(t))u(s,t).
\end{aligned}
$$

Adding the preceding two inequalities, yields

$$
\begin{aligned}
u(s,t)\frac{\partial^2 x(s,t)}{\Delta_1 s\Delta_2 t} + x(\sigma(s),\sigma(t))\frac{\partial^2 u(s,t)}{\Delta_1 s\Delta_2 t} &= \beta_1(s,t)|u(s,t)|^\gamma \\
&\quad - \beta_2(s,t)|x(\sigma(s),\sigma(t))|^\beta. \qquad (3.11)
\end{aligned}
$$

From (3.10) and (3.11), we get

$$
\frac{\partial^2}{\Delta_1 s\Delta_2 t}[x(s,t)u(s,t)] = \beta_1(s,t)|u(s,t)|^\gamma - \beta_2(s,t)|x(\sigma(s),\sigma(t))|^\beta \qquad (3.12)
$$

Integrating the left side of the (3.12) over t from $\sigma(c)$ to $\sigma(d)$ and over s from $\sigma(a)$ to $\sigma(b)$, respectively, we get

$$
\begin{aligned}
&\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \frac{\partial^2}{\Delta_1 s\Delta_2 t}[x(s,t)u(s,t)]\Delta_2 t\Delta_1 s \\
&= \int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \frac{\partial}{\Delta_2 t}\left[\frac{\partial x(s,t)u(s,t)}{\Delta_1 s}\right]\Delta_2 t\Delta_1 s \\
&= \int_{\sigma(a)}^{\sigma(b)} \left[\frac{\partial}{\Delta_1 s}x(s,t)u(s,t)\Big|_{\sigma(c)}^{\sigma(d)}\right]\Delta_1 s \\
&= \int_{\sigma(a)}^{\sigma(b)} \left[\frac{\partial}{\Delta_1 s}x(s,\sigma(d))u(s,\sigma(d)) - \frac{\partial}{\Delta_1 s}x(s,\sigma(c))u(s,\sigma(c)\right]\Delta_1 s \\
&= x(\sigma(b),\sigma(d))u(\sigma(b),\sigma(d)) - x(\sigma(a),\sigma(d))u(\sigma(a),\sigma(d)) \\
&\quad - x(\sigma(b),\sigma(c))u(\sigma(b),\sigma(c)) + x(\sigma(a),\sigma(c))u(\sigma(a),\sigma(c)) = 0. \qquad (3.13)
\end{aligned}
$$

Integrating both the sides of (3.12) over t from $\sigma(c)$ to $\sigma(d)$ and over s from $\sigma(a)$ to $\sigma(b)$, respectively, and noting that $x(\sigma(a),t) = 0 = x(\sigma(b),t)$, we get

$$\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)|u(s,t)|^{\gamma}\Delta_2 t\Delta_1 s = \int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_2(s,t)|x(\sigma(s),\sigma(t))|^{\beta}\Delta_2 t\Delta_1 s. \quad (3.14)$$

Substituting (3.14) and (3.9), we get

$$\begin{aligned} 2|x(\tau_1,\tau_2)| &= \int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\alpha_1(s,t)||x(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s \\ &+ \left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_1(s,t)\Delta_2 t\Delta_1 s\right)^{1/\gamma} \\ &\times\left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_2(s,t)|x(\sigma(s),\sigma(t))|^{\beta}\Delta_2 t\Delta_1 s\right)^{1/\alpha}. \end{aligned}$$

Since $M = \max_{\substack{\sigma(a)\le s\le\sigma(b)\\ \sigma(c)\le t\le\sigma(d)}} |x(s,t)| > 0$, $\beta_2^{+}(s,t) = \max_{\sigma(a)<s<\sigma(b)} \{\beta_2(s,t),0\}$, we get

$$\begin{aligned} 2M &\le M\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\alpha_1(s,t)|\Delta_2 t\Delta_1 s \\ &+ M^{\beta/\alpha}\left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\beta_1(s,t)|\Delta_2 t\Delta_1 s\right)^{1/\gamma} \\ &\times\left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_2^{+}(s,t)\Delta_2 t\Delta_1 s\right)^{1/\alpha}, \end{aligned}$$

That is,

$$\begin{aligned} 2 &\le \int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\alpha_1(s,t)|\Delta_2 t\Delta_1 s \\ &+ M^{\beta/\alpha-1}\left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\beta_1(s,t)|\Delta_2 t\Delta_1 s\right)^{1/\gamma} \\ &\times\left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} \beta_2^{+}(s,t)\Delta_2 t\Delta_1 s\right)^{1/\alpha}, \end{aligned}$$

Hence proof of the theorem is complete. □

Remark 3.2. Let $x(s,t), u(s,t), \alpha_1(s,t)$ and $\beta_1(s,t)$ changes to $x(t)$, $u(t)$, $\alpha_1(t)$ and $\beta_1(t)$ in (1.8), then the two dimensional-partial dynamic system (1.8) reduces to the nonlinear dynamic system

$$\begin{aligned} x^{\Delta}(t) &= \alpha_1(t)x(\sigma(t)) + \beta_1(t)|u(t)|^{\gamma-2}u(t) \\ u^{\Delta}(t) &= -\beta_2(t)|x(\sigma(t))|^{\beta-2}x(\sigma(t)) - \alpha_1(t)u(t) \end{aligned} \tag{3.15}$$

and with suitable changes, the inequality (3.1) becomes

$$2 \le \int_{\sigma(a)}^{\sigma(b)} |\alpha_1(t)|\Delta t + M^{\beta/\alpha-1}\left(\int_{\sigma(a)}^{\sigma(b)} |\beta_1(t)|\Delta t\right)^{1/\gamma} \times \left(\int_{\sigma(a)}^{\sigma(b)} \beta_2^+(t)\Delta t\right)^{1/\alpha}. \tag{3.16}$$

The similar-type of inequality as obtained by Ünal et.al[21].

Remark 3.3. If $\mathbb{T} = \mathbb{R}$, and let $x(s,t), u(s,t), \alpha_1(s,t)$ and $\beta_1(s,t)$ changes to $x(t), u(t), \alpha_1(t)$ and $\beta_1(t)$ in (1.8), then the two dimensional-partial differential system (1.8) reduces to the nonlinear differential system

$$\begin{aligned} x'(t) &= \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t) \\ u'(t) &= -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t) \end{aligned}$$

and with suitable changes, the inequality (3.1) becomes

$$2 \le \int_a^b |\alpha_1(t)|dt + M^{\beta/\alpha-1}\left(\int_a^b |\beta_1(t)|dt\right)^{1/\gamma}\left(\int_a^b \beta_2^+(t)dt\right)^{1/\alpha}. \tag{3.17}$$

This is same as obtained by Tiryaki et. al [20].

Remark 3.4. Consider the two-dimensional Emden-Fowler-type dynamic equation

$$\frac{\partial}{\Delta_1 s\Delta_2 t}\left(r(s,t)\left|\frac{\partial x(s,t)}{\Delta_1 s\Delta_2 t}\right|^{\alpha-2}\frac{\partial x(s,t)}{\Delta_1 s\Delta_2 t}\right) + q(s,t)|x(\sigma(s),\sigma(t))|^{\beta-2}x(\sigma(s),\sigma(t)) = 0, \tag{3.18}$$

where, $\alpha > 1$ is a constant, $r(s,t)$ and $q(s,t)$ are real rd-continuous functions, and $r(s,t) > 0$ for all $(s,t) \in \mathbb{T}_1 \times \mathbb{T}_2$. The equivalent dynamic system for (3.18) is given by

$$\frac{\partial^2 x(s,t)}{\Delta_1 s\Delta_2 t} = \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),$$

$$\frac{\partial^2 u(s,t)}{\Delta_1 s \Delta_2 t} = -\beta_2(s,t)|x(\sigma(s),\sigma(t))|^{\beta-2}x(\sigma(s),\sigma(t)), \tag{3.19}$$

where, $\beta_1(s,t) = r(s,t)^{1-\gamma}$ and $\beta_2(s,t) = q(s,t)$. Clearly, Theorem 3.1 for two dimensional nonlinear dynamic system (1.8) with $\alpha_1(s,t) = 0$ is satisfied for the system (3.19). Therefore, Eq. (3.1) becomes

$$2 \le M^{\beta/\alpha-1}\left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\beta_1(s,t)|\Delta_2 t\Delta_1 s\right)^{1/\gamma} \times \left(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\beta_2^+(s,t)|\Delta_2 t\Delta_1 s\right)^{1/\alpha}. \tag{3.20}$$

Definition 3.5. A nontrivial solution $(x(s,t),u(s,t))$ of the system (3.19) defined on $[s_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$ is said to be proper if and only if

$$\sup\{|x(s,t)| + |u(s,t)| : \sigma(a) \le s < \infty, \sigma(c) \le t < \infty\},$$

for any $\sigma(a) \ge s_0, \sigma(c) \ge t_0$. A proper solution $(x(s,t),u(s,t))$ of the system (3.19) is called weakly oscillatory if and only if at least one component has sequence of zeros (Generalized Zeros) tending to $+\infty$.

Theorem 3.6. *If $|x(\tau_1,\tau_2)| = \max\{|x(s,t)| : \sigma(a) < s < \sigma(b), \sigma(c) < t < \sigma(d)\}$, where $\sigma(a) > s_0, \sigma(c) > t_0$ and $s_0, \sigma(a)$, and $\sigma(b) \in \mathbb{T}_1$ and $t_0, \sigma(c)$, and $\sigma(d) \in \mathbb{T}_2, u(\tau_1,t)$ is bounded on $[t_0,\infty)_{\mathbb{T}}$ and $u(s,\tau_2)$ is bounded $[s_0,\infty)_{\mathbb{T}}$,*

$$\int^{\infty}\int^{\infty} \beta_1(s,t)\Delta_2 t\Delta_1 s < \infty, \qquad \int^{\infty}\int^{\infty} |\beta_2(s,t)|\Delta_2 t\Delta_1 s < \infty, \tag{3.21}$$

then every weakly oscillatory proper solution of nonlinear dynamic system (3.19) is bounded on $I = [s_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$.

Proof. Let $x(s,t), u(s,t)$ be any nontrivial weakly oscillatory proper solution of nonlinear system (3.19) on I such that $x(s,t)$ has a sequence of zeros tending to $+\infty$. Suppose to the contrary that $\limsup_{t\to\infty} |x(s,t)| = \infty$, then given any positive M_0, we can find positive numbers S_0 and T_0 such that $|x(s,t)| > M_0$ for all $s > S_0, t > T_0$. Since $x(s,t)$ is an oscillatory solution, there exists $(\sigma(a),\sigma(b))_{\mathbb{T}} \times (\sigma(c),\sigma(d))_{\mathbb{T}} \in \mathbb{T}_1 \times \mathbb{T}_2$, with $\sigma(a) > S_0, \sigma(c) > T_0$, such that $x(\sigma(a),t) = x(\sigma(b),t) = x(s,\sigma(c)) = x(s,\sigma(d)) = 0$ and $x(s,t) > 0$ for $(s,t) \in (\sigma(a),\sigma(b))_{\mathbb{T}} \times (\sigma(c),\sigma(d))_{\mathbb{T}}$.

Choose (τ_1,τ_2) in $(\sigma(a),\sigma(b))_{\mathbb{T}} \times (\sigma(c),\sigma(d))_{\mathbb{T}}$ such that $M = |x(\tau_1,\tau_2)| = \max\{|x(s,t)| : \sigma(a) < s < \sigma(b), \sigma(c) < t < \sigma(d)\} > M_0$.

In view of (3.21), we can choose S_0 and T_0 large enough such that for every $\sigma(a) > S_0, \sigma(c) > T_0$,

$$\begin{aligned}\int_{\sigma(a)}^{\infty}\int_{\sigma(c)}^{\infty} \beta_1(s,t)\Delta_2 t\Delta_1 s &< M^{-(\beta-\alpha)/(\alpha-1)}, \\ \int_{\sigma(a)}^{\infty}\int_{\sigma(c)}^{\infty} |\beta_2(s,t)|\Delta_2 t\Delta_1 s &< 1. \end{aligned} \tag{3.22}$$

Taking α^{th} power of both sides of (3.20), we obtain

$$\begin{aligned} 2^{\alpha} &\leq M^{\beta-\alpha}\Big(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\beta_1(s,t)|\Delta_2 t\Delta_1 s\Big)^{\alpha-1} \\ &\quad\times\Big(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\beta_2^{+}(s,t)|\Delta_2 t\Delta_1 s\Big) \\ &\leq M^{\beta-\alpha}\Big(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\beta_1(s,t)|\Delta_2 t\Delta_1 s\Big)^{\alpha-1} \\ &\quad\times\Big(\int_{\sigma(a)}^{\sigma(b)}\int_{\sigma(c)}^{\sigma(d)} |\beta_2(s,t)|\Delta_2 t\Delta_1 s\Big) \\ &< M^{\beta-\alpha}M^{-(\beta-\alpha)} = 1, \end{aligned} \tag{3.23}$$

where, $\alpha > 1, \beta^{+}(s,t) \leq |\beta_2(s,t)|$, is a contradiction. Hence $|x(s,t)|$ is bounded on I. Therefore, there exists a positive constant K such that

$$|x(s,t)| \leq K, \quad \text{for all} \quad (s,t) \in I.$$

On the other hand, integrating the second equation of the system (3.19) over t from τ_2 to t and over s from τ_1 to s, we obtain

$$\begin{aligned} &u(s,t) - u(\tau_1,t) - u(s,\tau_2) + u(\tau_1,\tau_2) \\ &\qquad = \int_{\tau_1}^{s}\int_{\tau_2}^{t} -\beta_2(s,t)|x(\sigma(s),\sigma(t))|^{\beta-2}x(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s. \end{aligned}$$

Since $u(\tau_1,t)$ is bounded on $[t_0,\infty)_{\mathbb{T}}$ and $u(s,\tau_2)$ is bounded on $[s_0,\infty)_{\mathbb{T}}$, and in view of triangle inequality, we have

$$\begin{aligned} |u(s,t)| \leq& |u(\tau_1,t) + u(s,\tau_2) - u(\tau_1,\tau_2)| \\ &+ \int_{\tau_1}^{s}\int_{\tau_2}^{t} |\beta_2(s,t)||x(\sigma(s),\sigma(t))|^{\beta-2}x(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s, \end{aligned}$$

$$\leq |u(\tau_1,t)+u(s,\tau_2)-C| + K^{\beta-1}\int_{\tau_1}^{s}\int_{\tau_2}^{t}|\beta_2(s,t)|\Delta_2 t\Delta_1 s, \qquad (3.24)$$

where $u(\tau_1,\tau_2) = C$ is a constant. Since $\int_{\tau_1}^{s}\int_{\tau_2}^{t}|\beta_2(s,t)|\Delta_2 t\Delta_1 s < \infty$, then from (3.24), we obtain $|u(s,t)|$ is bounded on I. Consequently,

$$\limsup_{t\to\infty}\{|x(s,t)|+|u(s,t)|\} \leq \limsup_{t\to\infty}|x(s,t)| + \limsup_{t\to\infty}|u(s,t)|,$$

That is, $\limsup_{t\to\infty}\{|x(s,t)|+|u(s,t)|\}$ is bounded on I. Hence proof of the theorem is complete. □

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