

ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE WAVE EQUATION OF HARTREE TYPE

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Abstract: We study the scattering problem for the wave equation of Hartree type with small initial data of slow decay. We show the existence of the scattering operator under some conditions. We also show that those conditions are sharp by presenting a non scattering result. In the present paper, we introduce a modified weighted norm as a new tool.

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1. Introduction

This is the third of the four papers including [5], [12] and [13] concerned with the Cauchy and scattering problems for the wave equation of a Hartree type with small initial data in three dimensions.

We consider the scattering problem for the wave equation

$$\partial_t^2 u - \Delta u = V_1(x)u + (V_2 * |u|^{p-1})u, \quad t \in \mathbf{R}, x \in \mathbf{R}^3, \quad (1.1)$$

with initial data $u(0, x)$, $u_t(0, x) = O(|x|^{-1-k})$ with $k > 0$ as $|x| \rightarrow \infty$, where

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$p \geq 3$, $V_1(x) = O(|x|^{-\gamma_1})$ with $\gamma_1 > 0$ as $|x| \rightarrow \infty$, $V_2(x) = \pm|x|^{-\gamma_2}$ with $\gamma_2 > 0$, and $*$ denotes spatial convolution.

For the usual Hartree type, say, $p = 3$, it is proved in [10] that the associated Cauchy problem has global solutions for small initial data under the conditions $\gamma_1 > 2$, $2 < \gamma_2 < 3$ and $k > (5 - \gamma_2)/2$. On the other hand, it is also shown that if any of these three conditions are relaxed, then there exist arbitrarily small initial data such that the corresponding solutions blow up in a finite time. The existence of scattering operators is shown in [11], where the proof contains only key lemmas. For the case $V_1(x) \equiv 0$, see for example [2], [6] and the references cited therein.

We here remark that in the usual Hartree type, the effect of the power ($p = 3$) is hidden in the conditions above. It seems unclear how the power affects the existence of the global solutions and scattering operators. This study has started to reveal the effect.

Consider the case $p \geq 3$ and assume $\gamma_2 \geq 1$. The second author [13] has shown conditions for the existence of scattering operators for the fast decay case $k > 1$. The conditions are

$$\gamma_1 > 2, \quad p + \gamma_2 > 5, \quad k(p - 1) > 5 - \gamma_2, \quad \text{and} \quad 1 < \gamma_2 < 3,$$

and the cases $p + \gamma_2 = 5$, $k(p - 1) = 5 - \gamma_2$ and $\gamma_2 = 1$ are admissible in special cases.

On the other hand, if $\gamma_1 < 2$, $p + \gamma_2 < 5$ or $k(p - 1) < 5 - \gamma_2$, then blow up in a finite time occurs. This blow up result will be proved in the sequel [5].

It follows from these results that the critical case is when $\gamma_1 = 2$, $p + \gamma_2 = 5$ and $k(p - 1) = 5 - \gamma_2$. Note that the scaling argument also provides $k(p - 1) = 5 - \gamma_2$ as the critical one if $V_1(x)$ vanishes.

In the present paper, we focus on the case $k \leq 1$ and $\gamma_2 \geq 1$. We refer the reader to [5] for the case $\gamma_2 < 1$ together with $\gamma_2 = 1$ and $k(p - 1) = 5 - \gamma_2$ (see Remark after Theorem 2.2 below), and to [13] for the case $k > 1$.

We here mention known results about the asymptotic behavior of solutions to the wave equation of power nonlinearity:

$$\partial_t^2 u - \Delta u = V_1(x)u \pm |u|^{p-1}u, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^3 \quad (1.2)$$

with the initial data which behave like $|x|^{-k-1}$. It is known that there exist global solutions of (1.2) for small initial data if $\gamma_1 > 2$, $p > 1 + \sqrt{2}$ and $k(p - 1) \geq 2$. We have proved in [4] that if

$$k \leq 1, \quad V_1(x) \not\equiv 0,$$

or

$$kp \leq 5/2, \quad V_1(x) \equiv 0,$$

then the global solutions do not necessarily approach the free solutions asymptotically in the sense of the energy norm as $t \rightarrow \pm\infty$.

In view of these results, going back to (1.1), one might expect that the potentials $V_1(x)$ and $V_2(x)$ may affect the asymptotic behavior of solutions. In this paper we show that there hold similar results to the case of power nonlinearity. Our main goal of the paper is to find conditions for the existence of the scattering operator for (1.1). We also show that those conditions are sharp by presenting a non scattering result. To prove these results, we need the estimates above and below of the convolution $V_2 * |u|^{p-1}$. We here emphasize that the usual weighted norm used so far is not useful to show the scattering state for some cases. In this paper we introduce a modified weighted norm as a new one.

The paper is organized as follows. In Section 2 we give conditions, and then state our main results. The first part of Section 3 is devoted to establish the weighted estimates of the convolution $V_2 * |u|^{p-1}$. Then we show the basic estimate for the existence proof. Finally, our main results are proved in Section 4.

2. Main results

Before stating our main theorems, we make some hypotheses.

Let $u_0^-(t, x)$ be the C^2 -solution of the Cauchy problem

$$u_{tt} - \Delta u = 0, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^3 \quad (2.1)$$

with the initial data $u(0, x) = \varphi(x)$, $u_t(0, x) = \psi(x)$ satisfying

(H1) $\varphi(x) \in C^3(\mathbf{R}^3)$, $\psi(x) \in C^2(\mathbf{R}^3)$,

$$\sum_{|\alpha| \leq 2} \{|\nabla \partial_x^\alpha \varphi(x)| + |\partial_x^\alpha \psi(x)|\} \leq \frac{\varepsilon_0}{\langle x \rangle^{1+k}}$$

with $0 < k \leq 1$, where $\langle x \rangle = 1 + |x|$ and $\varepsilon_0 > 0$ is a small parameter.

We solve the integral equation

$$u(t, x) = u_0^-(t, x) + \frac{1}{4\pi} \int_{-\infty}^t \frac{1}{t-s} \int_{|y-x|=t-s} F(u(s, y)) dS_y ds \quad (2.2)$$

with $F(u) = V_1(x)u + (V_2 * G(u))u$, where V_1 , V_2 and $G(u)$ satisfy the following conditions:

(H2) The potential $V_1(x)$ satisfies

$$\sum_{|\alpha| \leq 2} |\partial_x^\alpha V_1(x)| \leq \frac{\nu_1}{\langle x \rangle^{\gamma_1}}$$

with $\gamma_1 > 2$, where $\nu_1 > 0$ is a small parameter.

(H3) The potential $V_2 = V_2(x)$ satisfies $|V_2(x)| \leq \nu_2 |x|^{-\gamma_2}$ with $1 \leq \gamma_2 < 3$ and some constant $\nu_2 > 0$.

(H4) $G(u) \in C^2(\mathbf{R})$ and there exist $p > 3$, $A_1 > 0$ and $A_2 > 0$ such that

$$\begin{aligned} |G^{(j)}(u)| &\leq A_1 |u|^{p-1-j} \quad \text{for } |u| \leq 1, \quad j = 0, 1, 2, \\ |G''(u) - G''(v)| &\leq \begin{cases} A_2 |u - v|^{p-3} & \text{if } 3 < p < 4, \\ A_2 (|u| + |v|)^{p-4} |u - v| & \text{if } p \geq 4 \end{cases} \end{aligned}$$

for $|u|, |v| \leq 1$.

(H5) Assume

$$k(p-1) \geq 5 - \gamma_2 \quad \text{if } 1 < \gamma_2 < 3,$$

$$k(p-1) > 5 - \gamma_2 \quad \text{if } \gamma_2 = 1.$$

We define the energy norm by

$$\|u(t)\|_e = \left\{ \|\nabla u(t)\|_{L^2(\mathbf{R}^3)}^2 + \|\partial_t u(t)\|_{L^2(\mathbf{R}^3)}^2 \right\}^{\frac{1}{2}}.$$

Our goal is to prove the following three theorems.

Theorem 2.1 (Existence). *Assume (H1) through (H5). If ε_0 and ν_1 are sufficiently small, depending on p , k , γ_1 , γ_2 , ν_2 , A_1 and A_2 , then there exists a unique global C^2 -solution of (2.2).*

Theorem 2.2 (Scattering). *Let the assumptions of Theorem 2.1 hold and let $V_1(x) \equiv 0$. If $pk > (11 - 2\gamma_2)/2$, then the solution u provided by Theorem 2.1 satisfies*

$$\|u(t) - u_0^-(t)\|_e \longrightarrow 0 \quad (t \rightarrow -\infty). \quad (2.3)$$

Furthermore, there exists a unique C^2 -solution $u_0^+(t, x)$ of (2.1) such that for the solution $u(t, x)$ provided by Theorem 2.1,

$$\|u(t) - u_0^+(t)\|_e \longrightarrow 0 \quad (t \rightarrow +\infty). \quad (2.4)$$

Thus, one can define the scattering operator

$$S : (u_0^-(0), \partial_t u_0^-(0)) \mapsto (u_0^+(0), \partial_t u_0^+(0))$$

for $u_{tt} - \Delta u = F(u)$.

Remark. (i) For the case $V_1(x) \not\equiv 0$, it is proved in [13] that there exist scattering operators for $u_{tt} - \Delta u = F(u)$ if $k > 1$.

(ii) In (H5) the case $\gamma_2 = 1$ and $k(p-1) = 5 - \gamma_2$ is excluded. We treat this critical case in [5].

We next consider the Cauchy problem

$$\begin{cases} \square u = V_1(x)u + (V_2 * G(u))u, & (t, x) \in (0, \infty) \times \mathbf{R}^3 \\ u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x), & x \in \mathbf{R}^3. \end{cases} \quad (2.5)$$

Corollary 1 (Global existence). *Assume (H1) through (H5). If ε_0 and ν_1 are sufficiently small, depending on $p, k, \gamma_1, \gamma_2, \nu_2, A_1$ and A_2 , then there exists a unique global C^2 -solution of (2.5).*

Theorem 2.3 (No scattering). *Let the assumptions of Corollary 1 hold and let $G(u) = |u|^{p-1}$ or $|u|^{p-2}u$. Moreover, assume*

$$V_1(x) \geq 0, \quad V_2(x) \geq 0 \quad \text{for the case } V_1(x) \not\equiv 0, \quad (2.6)$$

or

$$V_2(x) = |x|^{-\gamma_2}, \quad pk \leq (11 - 2\gamma_2)/2 \quad \text{for the case } V_1(x) \equiv 0. \quad (2.7)$$

If the initial data are chosen so that

$$\varphi(x) \equiv 0, \quad \psi(x) \geq \varepsilon_1 \langle x \rangle^{-1-k}$$

for some small $\varepsilon_1 > 0$, then the solution provided by Corollary 1 does not satisfy

$$\|u(t) - u_0(t)\|_e \rightarrow 0 \quad (t \rightarrow \infty), \quad (2.8)$$

where u_0 is the solution of $\square u = 0$ with the initial data $(0, \psi(x))$ satisfying the assumption above.

Remark. If $V_1(x) \equiv 0$ and $1 < \gamma_2 < 3$, from the assumptions of Corollary 1 and Theorem 2.3, we see that the case of no scattering arises when $k \leq 1/2$ and $(11 - 2\gamma_2)/2 \geq pk \geq 5 - \gamma_2 + k$.

3. Key estimates

In this section we collect the estimates needed to prove the main result. The following lemma is proved in Asakura [1].

Lemma 3.1. *Let $\varphi(x)$ and $\psi(x)$ satisfy (H1). Then*

(i) *The solution u^0 of (2.1) with the data $\varphi(x)$ and $\psi(x)$ satisfies*

$$\sum_{|\alpha| \leq 2} |\partial_x^\alpha u^0(t, x)| \leq \frac{C_k \varepsilon_0}{\langle |t| + |x| \rangle^k} \left(1 + \log \frac{\langle |t| + |x| \rangle}{\langle |t| - |x| \rangle} \right)^{\delta_{1k}},$$

where $\langle x \rangle = 1 + |x|$, δ_{1k} is the usual Kronecker delta and the constant C_k depends only on k .

(ii) *Moreover, if $0 < k < 1$, then*

$$\sum_{|\alpha| \leq 2} |\partial_x^\alpha u^0(t, x)| \leq \frac{C_k \varepsilon_0 \langle t \rangle^{\delta-k}}{\langle |t| + |x| \rangle^\delta}$$

holds for $0 < k \leq \delta \leq 1 + k$.

Proof. Part (i) is proved in [1]. Part (ii) is shown by Lemma 3.4 (2) below. Note that it is enough to consider the integral $\frac{1}{|x|} \int_{||x|-t|}^{|x|+t} \frac{1}{\langle \lambda \rangle^k} d\lambda$ to show the estimate. $\square$

Based on the estimates above, one might be led to introduce a weighted norm for a function $u(t, x)$ by setting

$$\|u\| = \sup_{\substack{t \in \mathbf{R} \\ x \in \mathbf{R}^3}} \langle |t| + |x| \rangle^k \left(1 + \log \frac{\langle |t| + |x| \rangle}{\langle |t| - |x| \rangle} \right)^{-\delta_{1k}} |u(t, x)|. \quad (3.1)$$

When $k(p-1) > 4$ and $\gamma_2 + k \leq 3/2$, however, this norm does not suffice for the proof of Theorem 2.2, so we have to replace it by the norm

$$\|u\| = \sup_{\substack{t \in \mathbf{R} \\ x \in \mathbf{R}^3}} \langle |t| + |x| \rangle^\delta \langle t \rangle^{-\delta+k} |u(t, x)| \quad (3.2)$$

for that particular case. Here, δ is defined by the formula

$$\delta = \begin{cases} \min \left\{ k + \gamma_1 - 2, 2k + \gamma_2 - \frac{1}{2}, pk - 5 + \gamma_2 \right\} & (V_1(x) \not\equiv 0), \\ \min \left\{ 2k + \gamma_2 - \frac{1}{2}, pk - 5 + \gamma_2 \right\} & (V_1(x) \equiv 0) \end{cases}$$

and one easily checks that $k \leq \delta \leq 1 + k$ when $\gamma_1 > 2$, $\gamma_2 \geq 1$, $k(p-1) > 4$ and $\gamma_2 + k \leq 3/2$.

We also define the function $\text{Log}(q : s)$ by

$$\text{Log}(q : s) = \begin{cases} 1 & (q \neq 0), \\ 1 + \log \langle s \rangle & (q = 0). \end{cases}$$

We denote by $X \lesssim Y$ the statement $X \leq CY$ for some constant $C > 0$ depending on $p, k, \gamma_1, \gamma_2, \nu_2, A_1$ and A_2 .

Proposition 3.2. *Let $|V(x)| \leq \nu|x|^{-\gamma}$ with $\nu > 0$ for $x \in \mathbf{R}^3$. Assume $k(p-1) > 3 - \gamma$ and $\gamma < 3$.*

If u satisfies $\|u\| < \infty$, then there holds

$$|V * |u|^{p-1}(s, y)| \lesssim \begin{cases} \frac{\|u\|^{p-1}}{\langle |y| + |s| \rangle^{\gamma \langle s \rangle^{k(p-1)-3}}} \left(1 + \log \frac{\langle |y| + |s| \rangle}{\langle |y| - |s| \rangle} \right)^{\delta_{1k}(p-1)} & (k(p-1) > 3), \\ \frac{\|u\|^{p-1}}{\langle |y| + |s| \rangle^{\gamma}} \left(1 + \log \frac{\langle |y| + |s| \rangle}{\langle s \rangle} \right) \left(1 + \log \frac{\langle |y| + |s| \rangle}{\langle |y| - |s| \rangle} \right)^{\delta_{1k}(p-1)} & (k(p-1) = 3), \\ \frac{\|u\|^{p-1}}{\langle |y| + |s| \rangle^{k(p-1)+\gamma-3}} \left(1 + \log \frac{\langle |y| + |s| \rangle}{\langle |y| - |s| \rangle} \right)^{\delta_{1k}(p-1)} & (3 - \gamma < k(p-1) < 3) \end{cases} \quad (3.3)$$

for all $s \in \mathbf{R}$ and $y \in \mathbf{R}^3$.

Remark. See [12] for the weighted estimates of the convolution for the case $k > 1$.

Proof. First of all, we note that both of the norms (3.1) and (3.2) are such that

$$|u(t, x)| \leq \frac{\|u\|}{\langle |t| + |x| \rangle^k} \left(1 + \log \frac{\langle |t| + |x| \rangle}{\langle |t| - |x| \rangle} \right)^{\delta_{1k}}$$

for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}^3$. Using our assumption on V , we then get

$$\begin{aligned} & |V * |u|^{p-1}(s, y)| \\ & \leq |\nu| \int |y - z|^{-\gamma} |u|^{p-1}(s, z) dz \end{aligned}$$

$$\lesssim \|u\|^{p-1} \int |y-z|^{-\gamma} \langle |s| + |z| \rangle^{-k(p-1)} \left(1 + \log \frac{\langle |s| + |z| \rangle}{\langle |s| - |z| \rangle} \right)^{\delta_{1k}(p-1)} dz.$$

We note that the last integral with the region $|y-z| \leq 1/2$ is easy to treat since $1 + |s| + |z|$ and $1 + ||s| - |z||$ are equivalent to $1 + |s| + |y|$ and $1 + ||s| - |y||$, respectively if $|y-z| \leq 1/2$. Hence, we have only to treat the integral region $|y-z| \geq 1/2$. Then since $|y-z| \geq (1 + |y-z|)/3$, it suffices to estimate

$$\int \frac{1}{\langle y-z \rangle^\gamma \langle |s| + |z| \rangle^{k(p-1)}} \left(1 + \log \frac{\langle |s| + |z| \rangle}{\langle |s| - |z| \rangle} \right)^{\delta_{1k}(p-1)} dz. \quad (3.4)$$

We write the integral (3.4) as the sum of seven integrals I_j , $j = 1, \dots, 7$ whose region of integration is described below.

Region 1. $|z| \geq 2|s|$ and $|z| \leq |y|/2$. In this region, $|z| + |s|$ and $|z| - |s|$ are equivalent, while $|y-z|$ is equivalent to $|y| + |z|$. Hence, we have

$$I_1 \lesssim \langle |y| + |s| \rangle^{-\gamma} \int_{2|s| \leq |z| \leq |y|} \langle |s| + |z| \rangle^{-k(p-1)} dz.$$

Switching to polar coordinates, we obtain

$$\begin{aligned} I_1 &\lesssim \langle |y| + |s| \rangle^{-\gamma} \int_{2|s|}^{|y|} \langle |s| + \eta \rangle^{2-k(p-1)} d\eta \\ &\lesssim \begin{cases} \frac{1}{\langle |y| + |s| \rangle^\gamma \langle s \rangle^{k(p-1)-3}} & (k(p-1) > 3), \\ \frac{1}{\langle |y| + |s| \rangle^\gamma} \left(1 + \log \frac{\langle |y| + |s| \rangle}{\langle s \rangle} \right) & (k(p-1) = 3), \\ \frac{1}{\langle |y| + |s| \rangle^{k(p-1)+\gamma-3}} & (3 - \gamma < k(p-1) < 3). \end{cases} \end{aligned} \quad (3.5)$$

Region 2. $|z| \geq 2|s|$ and $|z| \geq 2|y|$. As above, $|z| + |s|$ and $|z| - |s|$ are equivalent, while $|z-y|$ is equivalent to $|z|$. Since $3 - \gamma < k(p-1)$, we obtain

$$I_2 \lesssim \int_{|z| \geq |y| + |s|} \langle z \rangle^{-\gamma-k(p-1)} dz \lesssim \langle |y| + |s| \rangle^{3-\gamma-k(p-1)}. \quad (3.6)$$

Region 3. $|z| \geq 2|s|$ and $|y|/2 \leq |z| \leq 2|y|$. As before,

$$I_3 \lesssim \langle |y| + |s| \rangle^{-k(p-1)} \int_{|z| \leq 2|y|} \langle y-z \rangle^{-\gamma} dz$$

$$\begin{aligned}
&\lesssim \langle |y| + |s| \rangle^{-k(p-1)} \int_{|y-z| \leq 3|y|} \langle y-z \rangle^{-\gamma} dz \\
&\lesssim \langle |y| + |s| \rangle^{3-\gamma-k(p-1)}
\end{aligned} \tag{3.7}$$

since $\gamma < 3$.

Region 4. $|z| \leq 2|s|$ and $|z| \leq |y|/2$. In this region $|y-z|$ is equivalent to $|y|$, hence

$$I_4 \lesssim \langle s \rangle^{-k(p-1)} \int_{|z| \leq \min\{2|s|, |y|\}} \langle y \rangle^{-\gamma} \left(1 + \log \frac{|s| + |z|}{||s| - |z||} \right)^{\delta_{1k}(p-1)} dz.$$

If $|y| \geq |s|$, then switching to polar coordinates, we have

$$\begin{aligned}
I_4 &\lesssim \langle |y| + |s| \rangle^{-\gamma} \langle s \rangle^{-k(p-1)} \int_0^{2|s|} \eta^2 \left(1 + \log \frac{|s| + \eta}{||s| - \eta|} \right)^{\delta_{1k}(p-1)} d\eta \\
&\lesssim \langle |y| + |s| \rangle^{-\gamma} \langle s \rangle^{3-k(p-1)} \int_0^2 \theta^2 \left(1 + \log \frac{1 + \theta}{|1 - \theta|} \right)^{\delta_{1k}(p-1)} d\theta,
\end{aligned}$$

where $\eta = |s|\theta$. We can easily see that the last integral is bounded, thus we obtain

$$I_4 \lesssim \begin{cases} \langle |y| + |s| \rangle^{-\gamma} \langle s \rangle^{3-k(p-1)} & (k(p-1) \geq 3), \\ \langle |y| + |s| \rangle^{3-\gamma-k(p-1)} & (3-\gamma < k(p-1) < 3). \end{cases} \tag{3.8}$$

If $|y| \leq |s|$, then we obtain in the same way as before,

$$\begin{aligned}
I_4 &\lesssim \langle s \rangle^{-k(p-1)} \int_0^{2|s|} \eta^{2-\gamma} \left(1 + \log \frac{|s| + \eta}{||s| - \eta|} \right)^{\delta_{1k}(p-1)} d\eta \\
&\lesssim \langle s \rangle^{3-\gamma-k(p-1)} \int_0^2 \theta^{2-\gamma} \left(1 + \log \frac{1 + \theta}{|1 - \theta|} \right)^{\delta_{1k}(p-1)} d\theta \\
&\lesssim \langle |y| + |s| \rangle^{-\gamma} \langle s \rangle^{3-k(p-1)}.
\end{aligned} \tag{3.9}$$

Region 5. $|z| \leq 2|s|$ and $|z| \geq 2|y|$. In this region $|z-y|$ is equivalent to $|z|$, hence

$$\begin{aligned}
I_5 &\lesssim \langle s \rangle^{-k(p-1)} \int_{|z| \leq 2|s|} |z|^{-\gamma} \left(1 + \log \frac{|s| + |z|}{||s| - |z||} \right)^{\delta_{1k}(p-1)} dz \\
&\lesssim \langle s \rangle^{-k(p-1)} \int_0^{2|s|} \eta^{2-\gamma} \left(1 + \log \frac{|s| + \eta}{||s| - \eta|} \right)^{\delta_{1k}(p-1)} d\eta
\end{aligned}$$

with $\eta = |z|$, which is exactly the same estimate as the first one for I_4 in the case $|y| \leq |s|$. Thus we obtain (3.9).

Region 6. $|z| \leq 2|s|$, $|y|/2 \leq |z| \leq 2|y|$ and $||s| - |z|| \geq |y - z|$. For this region, we have

$$I_6 \lesssim \langle |y| + |s| \rangle^{-k(p-1)} \int_{|z| \leq 2|s|} \langle y - z \rangle^{-\gamma} \left(1 + \log \frac{\langle |s| + |z| \rangle}{\langle y - z \rangle} \right)^{\delta_{1k}(p-1)} dz$$

as well as $|y - z| \leq 6|s|$. Switching to polar coordinates yields

$$I_6 \lesssim \langle |y| + |s| \rangle^{-k(p-1)} \int_0^{6|s|} \langle \rho \rangle^{2-\gamma} \left(1 + \log \frac{\langle s \rangle}{\langle \rho \rangle} \right)^{\delta_{1k}(p-1)} d\rho,$$

where $\rho = |y - z|$. Using [4, Lemma 3] we conclude that

$$I_6 \lesssim \langle |y| + |s| \rangle^{-k(p-1)} \langle s \rangle^{3-\gamma} \lesssim \langle |y| + |s| \rangle^{3-\gamma-k(p-1)}. \quad (3.10)$$

Region 7. $|z| \leq 2|s|$, $|y|/2 \leq |z| \leq 2|y|$ and $||s| - |z|| \leq |y - z|$. For this region, we have

$$I_7 \lesssim \langle |y| + |s| \rangle^{-k(p-1)} \int_{|z| \leq 2|s|} \langle y - z \rangle^{-\gamma} \left(1 + \log \frac{|s| + |z|}{||s| - |z||} \right)^{\delta_{1k}(p-1)} dz$$

and we may assume that $||s| - |z|| \leq |y - z|$ within the region of integration. In order to estimate the last integral, we apply the formula:

$$\frac{1}{4\pi t} \int_{|y-x|=t} g(|y|) dS_y = \frac{1}{2|x|} \int_{|t-|x||}^{t+|x|} \lambda g(\lambda) d\lambda, \quad (3.11)$$

for $t > 0$ and a continuous function $g(|x|)$ on $[0, \infty)$ for $x \in \mathbf{R}^3$. See [7] for its proof.

Using (3.11) implies

$$I_7 \lesssim \langle |y| + |s| \rangle^{-k(p-1)} \int_0^{2|s|} \left(1 + \log \frac{|s| + \eta}{||s| - \eta|} \right)^{\delta_{1k}(p-1)} \int_{||s| - \eta|}^{|y| + \eta} \langle \rho \rangle^{1-\gamma} d\rho d\eta$$

with $\rho = |y - z|$ and $\eta = |z|$. If $\gamma > 2$, then the last equation gives

$$\begin{aligned} I_7 &\lesssim \langle |y| + |s| \rangle^{-k(p-1)} \int_0^{2|s|} \left(1 + \log \frac{|s| + \eta}{||s| - \eta|} \right)^{\delta_{1k}(p-1)} ||s| - \eta|^{2-\gamma} d\eta \\ &= |s|^{3-\gamma} \langle |y| + |s| \rangle^{-k(p-1)} \int_0^2 \left(1 + \log \frac{1 + \theta}{|1 - \theta|} \right)^{\delta_{1k}(p-1)} |1 - \theta|^{2-\gamma} d\theta \end{aligned} \quad (3.12)$$

with $\theta = \eta/|s|$. If $\gamma < 2$, then we similarly have

$$\begin{aligned} I_7 &\lesssim \langle |y| + |s| \rangle^{2-\gamma-k(p-1)} \int_0^{2|s|} \left(1 + \log \frac{|s| + \eta}{||s| - \eta|} \right)^{\delta_{1k}(p-1)} d\eta \\ &= |s| \langle |y| + |s| \rangle^{2-\gamma-k(p-1)} \int_0^2 \left(1 + \log \frac{1 + \theta}{|1 - \theta|} \right)^{\delta_{1k}(p-1)} d\theta. \end{aligned} \quad (3.13)$$

If $\gamma = 2$, then we have

$$\begin{aligned} I_7 &\lesssim \langle |y| + |s| \rangle^{-k(p-1)} \\ &\quad \int_0^{2|s|} \left(1 + \log \frac{|s| + \eta}{||s| - \eta|} \right)^{\delta_{1k}(p-1)} \left(1 + \log \frac{|y| + \eta}{||s| - \eta|} \right) d\eta \\ &\lesssim \langle |y| + |s| \rangle^{-k(p-1)} \int_0^{2|s|} \left(1 + \log \frac{|s| + \eta}{||s| - \eta|} \right)^{1+\delta_{1k}(p-1)} d\eta \end{aligned} \quad (3.14)$$

since $|y| \leq 4|s|$ for Region 7. Thus as before, from (3.12), (3.13) and (3.14), we obtain

$$I_7 \lesssim \langle |y| + |s| \rangle^{3-\gamma-k(p-1)} \quad (3.15)$$

in any case whatsoever. Combining (3.5)-(3.10) and (3.15) together, we obtain the desired estimates, which completes the proof. $\square$

Remark. If we consider only the case $k < 1$, we do not need to divide the integral region into seven parts, and the proof becomes simpler.

We next show a basic estimate for the proof of Theorem 2.1. We define

$$L[F(u)](t, x) = \frac{1}{4\pi} \int_{-\infty}^t \frac{1}{t-s} \int_{|y-x|=t-s} F(u(s, y)) dS_y ds. \quad (3.16)$$

Lemma 3.3. *Let p, k, γ_1 and γ_2 satisfy the assumptions of Theorem 2.1. Then*

$$\|L[F(u)]\| \lesssim \nu_1 \|u\| + \|u\|^p,$$

where L is defined in (4.3).

We use the following two lemmas repeatedly to prove Lemma 3.3.

Lemma 3.4.

(1) *Let $0 < k \leq 1$. Then*

$$\frac{1}{r} \int_{|r-t|}^{r+t} \frac{1}{\langle \lambda \rangle^k} d\lambda \leq \frac{C_k}{\langle t+r \rangle^k} \left(1 + \log \frac{\langle t+r \rangle}{\langle t-r \rangle} \right)^{\delta_{1k}}$$

for all $t, r \geq 0$, where the constant C_k depends only on k .

(2) Let $0 < k < 1$ and $k \leq \delta \leq k + 1$. Then

$$\frac{1}{r} \int_{|r-t|}^{r+t} \frac{1}{\langle \lambda \rangle^k} d\lambda \leq \frac{C_k \langle t \rangle^{\delta-k}}{\langle t+r \rangle^\delta}$$

for all $t, r \geq 0$.

Proof. (1) See [1] for a proof. (2) From the proof of Proposition 1.1 in [1], we see that

$$\frac{1}{r} \int_{|r-t|}^{r+t} \frac{1}{\langle \lambda \rangle^k} d\lambda \leq \frac{C_k \min\{\langle t \rangle, \langle r \rangle\}}{\langle r \rangle \langle t+r \rangle^k} \quad (3.17)$$

holds if $k < 1$. Let $k \leq \delta \leq k + 1$.

If $r \geq t$, then we have

$$\frac{1}{r} \int_{|r-t|}^{r+t} \frac{1}{\langle \lambda \rangle^k} d\lambda \lesssim \frac{1}{\langle t+r \rangle^k} \left(\frac{\langle t \rangle}{\langle r \rangle} \right)^{\delta-k} \lesssim \frac{\langle t \rangle^{\delta-k}}{\langle t+r \rangle^\delta}.$$

If $t \geq r$, then from (3.17)

$$\frac{1}{r} \int_{|r-t|}^{r+t} \frac{1}{\langle \lambda \rangle^k} d\lambda \lesssim \frac{\langle t \rangle}{\langle t+r \rangle^{k+1}} \lesssim \frac{1}{\langle t+r \rangle^k} \left(\frac{\langle t \rangle}{\langle t+r \rangle} \right)^{\delta-k} = \frac{\langle t \rangle^{\delta-k}}{\langle t+r \rangle^\delta}.$$

This completes the proof of Lemma 3.4. □

Lemma 3.5. (1) Let $\alpha \geq 0$ and let $\mu \geq 0, a \geq 0$. Then

$$\int_0^\alpha \frac{1}{\langle \alpha - \beta \rangle^\mu} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right)^a d\beta \lesssim \frac{\text{Log}(\mu - 1 : \alpha)}{\langle \alpha \rangle^{[\mu-1]_-}},$$

where $[m]_- = \min\{0, m\}$.

(2) Let $\sigma \geq \alpha \geq 0$ and let $\mu > 0, \nu \geq 0, \mu + \nu > 1, a \geq 0$. Then

$$\int_\sigma^\infty \frac{1}{\langle \beta \rangle^\mu \langle \beta - \alpha \rangle^\nu} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^a d\beta \lesssim \frac{\text{Log}(\nu - 1 : \sigma)}{\langle \sigma \rangle^{\mu + [\nu-1]_-}} \left(1 + \log \frac{\langle \sigma \rangle}{\langle \alpha \rangle} \right)^a.$$

Proof. (1) Split the integral on the left-hand side into two parts. Then using Proposition 4.3 of [9] yields

$$\begin{aligned} & \left(\int_0^{\alpha/2} + \int_{\alpha/2}^\alpha \right) \frac{1}{\langle \alpha - \beta \rangle^\mu} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right)^a d\beta \\ & \lesssim \frac{1}{\langle \alpha \rangle^\mu} \int_0^{\alpha/2} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right)^a d\beta + \int_{\alpha/2}^\alpha \frac{1}{\langle \alpha - \beta \rangle^\mu} d\beta \end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{1}{\langle \alpha \rangle^{\mu-1}} + \frac{\text{Log}(\mu-1 : \alpha)}{\langle \alpha \rangle^{[\mu-1]_-}} \\
&\lesssim \frac{\text{Log}(\mu-1 : \alpha)}{\langle \alpha \rangle^{[\mu-1]_-}}.
\end{aligned}$$

(2) As above,

$$\begin{aligned}
&\left(\int_{\sigma}^{2\sigma} + \int_{2\sigma}^{\infty} \right) \frac{1}{\langle \beta \rangle^{\mu} \langle \beta - \alpha \rangle^{\nu}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^a d\beta \\
&\lesssim \frac{1}{\langle \sigma \rangle^{\mu}} \left(1 + \log \frac{\langle \sigma \rangle}{\langle \alpha \rangle} \right)^a \int_{\sigma}^{2\sigma} \frac{1}{\langle \beta - \alpha \rangle^{\nu}} d\beta \\
&\quad + \int_{2\sigma}^{\infty} \frac{1}{\langle \beta \rangle^{\mu+\nu}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^a d\beta.
\end{aligned}$$

Applying Lemma 4 of [4] for the last integral yields the desired estimate. $\square$

Proof of Lemma 3.3 for the norm (3.1). We see from [4] and [8] that

$$\|L[V_1 u]\| \lesssim \nu_1 \|u\|$$

holds. Hence it suffices to prove

$$\|L[(V_2 * G(u))u]\| \lesssim \|u\|^p. \quad (3.18)$$

For simplicity, throughout the proof, we denote γ_2 by γ . To show (3.18), according to Proposition 3.2, we have to distinguish three cases $k(p-1) > 3$, $k(p-1) = 3$ and $3 - \gamma < k(p-1) < 3$.

Case 1. $k(p-1) > 3$. @ By (3.1), (3.3) and (3.11), $L[(V_2 * G(u))u](t, x)$ is estimated by the following type of integral:

$$\begin{aligned}
I \equiv \frac{\|u\|^p}{r} \int_{-\infty}^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + |s| \rangle^{\gamma+k} \langle s \rangle^{k(p-1)-3}} \\
\left(1 + \log \frac{\langle \lambda + |s| \rangle}{\langle \lambda - |s| \rangle} \right)^{\delta_{1k} p} d\lambda ds.
\end{aligned}$$

To prove (3.18), we have only to show that

$$I \lesssim \frac{\|u\|^p}{\langle |t| + r \rangle^k} \left(1 + \log \frac{\langle |t| + r \rangle}{\langle |t| - r \rangle} \right)^{\delta_{1k}}. \quad (3.19)$$

We first estimate the integral for the region $0 \leq s \leq t$ to show

$$I_+ \equiv \frac{\|u\|^p}{r} \int_0^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + s \rangle^{\gamma+k} \langle s \rangle^{k(p-1)-3}}$$

$$\left(1 + \log \frac{\langle \lambda + s \rangle}{\langle \lambda - s \rangle}\right)^{\delta_{1k}p} d\lambda ds \quad (3.20)$$

$$\lesssim \frac{\|u\|^p}{\langle t+r \rangle^k} \left(1 + \log \frac{\langle t+r \rangle}{\langle t-r \rangle}\right)^{\delta_{1k}}. \quad (3.21)$$

Changing variables by

$$\alpha = \lambda + s, \quad \beta = \lambda - s, \quad (3.22)$$

in (3.20), we have

$$\begin{aligned} I_+ &\lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^{\gamma+k-1}} \int_{-\alpha}^{\alpha} \frac{1}{\langle \alpha - \beta \rangle^{k(p-1)-3}} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle}\right)^{\delta_{1k}p} d\beta d\alpha \\ &\lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^{\gamma+k-1}} \int_0^{\alpha} \frac{1}{\langle \alpha - \beta \rangle^{k(p-1)-3}} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle}\right)^{\delta_{1k}p} d\beta d\alpha. \end{aligned}$$

Lemma 3.5 (1) yields

$$I_+ \lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{\text{Log}(k(p-1) - 4 : \alpha)}{\langle \alpha \rangle^{\gamma+k-1+[k(p-1)-4]_-}} d\alpha. \quad (3.23)$$

Note that

$$\gamma - 1 + [k(p-1) - 4]_- \begin{cases} \geq 0 & (k(p-1) \neq 4), \\ > 0 & (k(p-1) = 4) \end{cases} \quad (3.24)$$

by assumption. Hence, using Lemma 3.4, we obtain

$$I_+ \lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^k} d\alpha \lesssim \frac{\|u\|^p}{\langle t+r \rangle^k} \left(1 + \log \frac{\langle t+r \rangle}{\langle t-r \rangle}\right)^{\delta_{1k}}.$$

Hence (3.21) is proved.

We next treat the integral region $s \leq 0$ when $t \geq 0$ and also the case $t < 0$ to show

$$\begin{aligned} I_- &\equiv \frac{\|u\|^p}{r} \int_{-\infty}^{\min\{0,t\}} \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + |s| \rangle^{\gamma+k} \langle s \rangle^{k(p-1)-3}} \\ &\quad \left(1 + \log \frac{\langle \lambda + |s| \rangle}{\langle \lambda - |s| \rangle}\right)^{\delta_{1k}p} d\lambda ds \\ &\lesssim \frac{\|u\|^p}{\langle |t| + r \rangle^k} \left(1 + \log \frac{\langle |t| + r \rangle}{\langle |t| - r \rangle}\right)^{\delta_{1k}}. \end{aligned} \quad (3.25)$$

Let $|t| \geq r$. Then changing variables by (3.22), we have

$$I_- \lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \int_{\alpha}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma+k-1} \langle \beta - \alpha \rangle^{k(p-1)-3}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k} p} d\beta d\alpha$$

and by Lemma 3.5 (2),

$$I_- \lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \frac{\text{Log}(k(p-1) - 4 : \alpha)}{\langle \alpha \rangle^{\gamma+k-1+[k(p-1)-4]_-}} d\alpha.$$

This is exactly the same estimate as (3.23). Hence, proceeding as before, we see that (3.25) holds for $|t| \geq r$.

We next assume $r \geq |t|$. If $r \geq t \geq 0$, then by (3.22),

$$\begin{aligned} I_- &\lesssim \frac{\|u\|^p}{r} \int_{t-r}^{r-t} \int_{r-t}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma+k-1} \langle \beta - \alpha \rangle^{k(p-1)-3}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k} p} d\beta d\alpha \\ &\quad + \frac{\|u\|^p}{r} \int_{r-t}^{t+r} \int_{\alpha}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma+k-1} \langle \beta - \alpha \rangle^{k(p-1)-3}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k} p} d\beta d\alpha. \end{aligned}$$

The last term is estimated in the same way as before. We denote by I'_- the first term on the right-hand side above.

On the other hand, if $r \geq -t \geq 0$, then since $t + r \leq r - t$,

$$\begin{aligned} I_- &\lesssim \frac{\|u\|^p}{r} \int_{t-r}^{r-t} \int_{r-t}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma+k-1} \langle \beta - \alpha \rangle^{k(p-1)-3}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k} p} d\beta d\alpha \\ &= I'_-. \end{aligned}$$

Hence, we have only to estimate I'_- for case $r \geq |t|$. We may here assume $r \geq 1$ since otherwise it is easy to show the desired estimate. Note that

$$I'_- \lesssim \frac{\|u\|^p}{\langle r \rangle} \int_0^{r-t} \int_{r-t}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma+k-1} \langle \beta - \alpha \rangle^{k(p-1)-3}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k} p} d\beta d\alpha.$$

Then using Lemma 3.5 (2) and (3.24), we obtain

$$\begin{aligned} I'_- &\lesssim \frac{\|u\|^p \text{Log}(k(p-1) - 4 : r-t)}{\langle r + |t| \rangle \langle r-t \rangle^{\gamma+k-1+[k(p-1)-4]_-}} \int_0^{r-t} \left(1 + \log \frac{\langle r-t \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k} p} d\alpha \\ &\lesssim \frac{\|u\|^p}{\langle r + |t| \rangle^k}. \end{aligned}$$

Here we have used Lemma 3.5 (1) with $\mu = 0$ for the case $k = 1$. Hence (3.25) holds for $r \geq |t|$. Therefore, it follows from (3.21) and (3.25) that (3.19) holds for $k(p-1) > 3$.

Case 2. $k(p-1) = 3$.@ By (3.1), (3.3) and (3.11), $L[(V_2 * G(u))u](t, x)$ is estimated by the following type of integral:

$$II \equiv \frac{\|u\|^p}{r} \int_{-\infty}^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + |s| \rangle^{\gamma+k}} \left(1 + \log \frac{\langle \lambda + |s| \rangle}{\langle \lambda - |s| \rangle} \right)^{\delta_{1k}p} \left(1 + \log \frac{\langle \lambda + |s| \rangle}{\langle s \rangle} \right) d\lambda ds.$$

We show that II has the same estimate as (3.19). Let $t \geq 0$. As before, we divide the integral region into two parts: $0 \leq s \leq t$ and $-\infty < s \leq 0$.

We first estimate the integral for the region $0 \leq s \leq t$. Using (3.22), we have

$$\begin{aligned} II_+ &\equiv \frac{\|u\|^p}{r} \int_0^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + s \rangle^{\gamma+k}} \left(1 + \log \frac{\langle \lambda + s \rangle}{\langle \lambda - s \rangle} \right)^{\delta_{1k}p} \left(1 + \log \frac{\langle \lambda + s \rangle}{\langle s \rangle} \right) d\lambda ds \\ &\lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^{\gamma+k-1}} \int_0^\alpha \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right)^{\delta_{1k}p} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \alpha - \beta \rangle} \right) d\beta d\alpha. \end{aligned} \quad (3.26)$$

Splitting the β -integral above into two parts and using the fact that $\langle \alpha \rangle / \langle \alpha - \beta \rangle$ is bounded on $(0, \alpha/2)$, we see that

$$\begin{aligned} &\left(\int_0^{\alpha/2} + \int_{\alpha/2}^\alpha \right) \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right)^{\delta_{1k}p} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \alpha - \beta \rangle} \right) d\beta \\ &\lesssim \int_0^{\alpha/2} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right)^{\delta_{1k}p} d\beta + \int_{\alpha/2}^\alpha \left(1 + \log \frac{\langle \alpha \rangle}{\langle \alpha - \beta \rangle} \right) d\beta \\ &\lesssim \int_0^{\alpha/2} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right)^p d\beta \lesssim \langle \alpha \rangle \end{aligned}$$

by Lemma 3.5 (1). Hence from (3.26),

$$II_+ \lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^{\gamma+k-2}} d\alpha.$$

We here note that $\gamma \geq 2$ if $k(p-1) = 3$ since $k(p-1) \geq 5 - \gamma$ by assumption. Thus, by Lemma 3.4, we obtain

$$II_+ \lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^k} d\alpha \lesssim \frac{\|u\|^p}{\langle t+r \rangle^k} \left(1 + \log \frac{\langle t+r \rangle}{\langle t-r \rangle} \right)^{\delta_{1k}}. \quad (3.27)$$

We next treat the integral region $s \leq 0$ when $t \geq 0$ and also the case $t < 0$ to show

$$\begin{aligned} II_- &\equiv \frac{\|u\|^p}{r} \int_{-\infty}^{\min\{0,t\}} \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + |s| \rangle^{\gamma+k}} \left(1 + \log \frac{\langle \lambda + |s| \rangle}{\langle \lambda - |s| \rangle} \right)^{\delta_{1kp}} \\ &\quad \left(1 + \log \frac{\langle \lambda + |s| \rangle}{\langle s \rangle} \right) d\lambda ds \\ &\lesssim \frac{\|u\|^p}{\langle |t| + r \rangle^k} \left(1 + \log \frac{\langle |t| + r \rangle}{\langle |t| - r \rangle} \right)^{\delta_{1k}}. \end{aligned} \quad (3.28)$$

Let $|t| \geq r$. Then changing variables by (3.22), we have

$$\begin{aligned} II_- &\lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \int_{\alpha}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma+k-1}} \\ &\quad \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1kp}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \beta - \alpha \rangle} \right) d\beta d\alpha. \end{aligned} \quad (3.29)$$

Splitting the β -integral above into two parts:

$$\begin{aligned} &\left(\int_{2\alpha}^{\infty} + \int_{\alpha}^{2\alpha} \right) \frac{1}{\langle \beta \rangle^{\gamma+k-1}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1kp}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \beta - \alpha \rangle} \right) d\beta \\ &\equiv J_1 + J_2. \end{aligned} \quad (3.30)$$

Note that $\langle \beta \rangle / \langle \beta - \alpha \rangle$ is bounded for $\beta \geq 2\alpha$, and that $\gamma + k - 1 \geq 1 + k > 1$ since $\gamma \geq 2$ as mentioned before. Then J_1 is estimated by

$$J_1 \lesssim \int_{2\alpha}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma+k-1}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1kp}} d\beta \lesssim \frac{1}{\langle \alpha \rangle^k} \quad (3.31)$$

from Lemma 3.5 (2) and $\gamma \geq 2$.

On the other hand,

$$J_2 \lesssim \int_0^{\alpha} \frac{1}{\langle \alpha + \beta \rangle^{\gamma+k-1}} \left(1 + \log \frac{\langle \alpha + \beta \rangle}{\langle \beta \rangle} \right) d\beta$$

$$\lesssim \frac{1}{\langle \alpha \rangle^{\gamma+k-1}} \int_0^\alpha \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle}\right) d\beta \lesssim \frac{1}{\langle \alpha \rangle^k} \quad (3.32)$$

by Lemma 3.5 (1) and $\gamma \geq 2$. From (3.29)-(3.32) and Lemma 3.4, we obtain

$$II_- \lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \frac{1}{\langle \alpha \rangle^k} d\alpha \lesssim \frac{\|u\|^p}{\langle |t|+r \rangle^k} \left(1 + \log \frac{\langle |t|+r \rangle}{\langle |t|-r \rangle}\right)^{\delta_{1k}}.$$

Thus, (3.28) holds for $|t| \geq r$.

We next assume $r \geq |t|$. As shown in Case 1, we have only to estimate

$$II'_- \equiv \frac{\|u\|^p}{\langle r \rangle} \int_0^{r-t} \int_{r-t}^\infty \frac{1}{\langle \beta \rangle^{\gamma+k-1}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle}\right)^{\delta_{1kp}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \beta - \alpha \rangle}\right) d\beta d\alpha.$$

We split the integral above into two parts:

$$\begin{aligned} II'_- &= \frac{\|u\|^p}{\langle r \rangle} \left(\int_0^{r-t} \int_{\max\{r-t, 2\alpha\}}^\infty + \int_{(r-t)/2}^{r-t} \int_{r-t}^{2\alpha} \right) \\ &\quad \frac{1}{\langle \beta \rangle^{\gamma+k-1}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle}\right)^{\delta_{1kp}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \beta - \alpha \rangle}\right) d\beta d\alpha \\ &\equiv J_3 + J_4. \end{aligned} \quad (3.33)$$

Note that $\langle \beta \rangle / \langle \beta - \alpha \rangle$ is bounded on the integral region of J_3 since $\beta \geq 2\alpha$. Then

$$\begin{aligned} J_3 &\lesssim \frac{\|u\|^p}{\langle r \rangle} \int_0^{r-t} \int_{r-t}^\infty \frac{1}{\langle \beta \rangle^{\gamma+k-1}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle}\right)^{\delta_{1kp}} d\beta d\alpha \\ &\lesssim \frac{\|u\|^p}{\langle r + |t| \rangle \langle r - t \rangle^k} \int_0^{r-t} \left(1 + \log \frac{\langle r - t \rangle}{\langle \alpha \rangle}\right)^{\delta_{1kp}} d\alpha \end{aligned}$$

by Lemma 3.5 (2) and $\gamma \geq 2$. Using Lemma 3.5 (1), we obtain

$$J_3 \lesssim \frac{\|u\|^p}{\langle r + |t| \rangle^k}. \quad (3.34)$$

To estimate J_4 , we note that $\langle \beta \rangle / \langle \alpha \rangle$ is bounded on the integral region of J_4 . Then

$$J_4 \lesssim \frac{\|u\|^p}{\langle r \rangle} \int_{(r-t)/2}^{r-t} \int_{r-t}^{2\alpha} \frac{1}{\langle \beta \rangle^{\gamma+k-1}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \beta - \alpha \rangle}\right) d\beta d\alpha$$

$$\begin{aligned}
&\lesssim \frac{\|u\|^p}{\langle r \rangle} \int_{(r-t)/2}^{r-t} \int_{r-t-\alpha}^{\alpha} \frac{1}{\langle \alpha + \beta \rangle^{\gamma+k-1}} \left(1 + \log \frac{\langle \alpha + \beta \rangle}{\langle \beta \rangle} \right) d\beta d\alpha \\
&\lesssim \frac{\|u\|^p}{\langle r + |t| \rangle \langle r - t \rangle^{\gamma+k-1}} \int_0^{r-t} \int_0^{\alpha} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right) d\beta d\alpha.
\end{aligned}$$

Using Lemma 3.5 (1) yields

$$J_4 \lesssim \frac{\|u\|^p}{\langle r + |t| \rangle \langle r - t \rangle^{\gamma+k-1}} \int_0^{r-t} \langle \alpha \rangle d\alpha \lesssim \frac{\|u\|^p}{\langle r + |t| \rangle^k}. \quad (3.35)$$

Hence, by (3.33)-(3.35), we obtain (3.28) for $r \geq |t|$, and thus it follows from (3.27) and (3.28) that II has the same estimate as (3.19).

Case 3. $k(p-1) < 3$.@ By (3.1), (3.3) and (3.11), $L[(V_2 * G(u))u](t, x)$ is estimated by the following type of integral:

$$\begin{aligned}
III \equiv \frac{\|u\|^p}{r} \int_{-\infty}^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + |s| \rangle^{k(p-1)+\gamma-3+k}} \\
\left(1 + \log \frac{\langle \lambda + |s| \rangle}{\langle \lambda - |s| \rangle} \right)^{\delta_{1k}p} d\lambda ds.
\end{aligned}$$

We show that III has the same estimate as (3.19). We first estimate the integral for the region $0 \leq s \leq t$. Using (3.22) and Lemma 3.5 (1), we have

$$\begin{aligned}
&III_+ \\
&\equiv \frac{\|u\|^p}{r} \int_0^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + s \rangle^{k(p-1)+\gamma-3+k}} \left(1 + \log \frac{\langle \lambda + s \rangle}{\langle \lambda - s \rangle} \right)^{\delta_{1k}p} d\lambda ds \\
&\lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^{kp+\gamma-4}} \int_0^{\alpha} \left(1 + \log \frac{\langle \alpha \rangle}{\langle \beta \rangle} \right)^{\delta_{1k}p} d\beta d\alpha \\
&\lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^{kp+\gamma-5}} d\alpha.
\end{aligned}$$

Note that $kp + \gamma - 5 \geq k$ (or $> k$) by assumption. Thus from Lemma 3.4, we obtain

$$III_+ \lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^k} d\alpha \lesssim \frac{\|u\|^p}{\langle t+r \rangle^k} \left(1 + \log \frac{\langle t+r \rangle}{\langle t-r \rangle} \right)^{\delta_{1k}}. \quad (3.36)$$

We next treat the integral region $s \leq 0$ when $t \geq 0$ and also the case $t < 0$ to show

III_-

$$\begin{aligned}
&\equiv \frac{\|u\|^p}{r} \int_{-\infty}^{\min\{0,t\}} \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{\langle \lambda + |s| \rangle^{kp+\gamma-3}} \left(1 + \log \frac{\langle \lambda + |s| \rangle}{\langle \lambda - |s| \rangle} \right)^{\delta_{1k}p} d\lambda ds \\
&\lesssim \frac{\|u\|^p}{\langle |t| + r \rangle^k} \left(1 + \log \frac{\langle |t| + r \rangle}{\langle |t| - r \rangle} \right)^{\delta_{1k}}.
\end{aligned} \tag{3.37}$$

Let $|t| \geq r$. Then changing variables by (3.22) yields

$$III_- \lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \int_{\alpha}^{\infty} \frac{1}{\langle \beta \rangle^{kp+\gamma-4}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k}p} d\beta d\alpha.$$

Since $kp + \gamma - 5 \geq k$ (or $> k$), using Lemmas 3.4 and 3.5 (2), we obtain

$$III_- \lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \frac{1}{\langle \alpha \rangle^k} d\alpha \lesssim \frac{\|u\|^p}{\langle |t| + r \rangle^k} \left(1 + \log \frac{\langle |t| + r \rangle}{\langle |t| - r \rangle} \right)^{\delta_{1k}},$$

and hence, (3.37) holds for $|t| \geq r$.

We finally assume $r \geq |t|$. As shown in Case 1, we have only to estimate

$$III'_- \equiv \frac{\|u\|^p}{\langle r \rangle} \int_0^{r-t} \int_{r-t}^{\infty} \frac{1}{\langle \beta \rangle^{kp+\gamma-4}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k}p} d\beta d\alpha.$$

Using $kp + \gamma - 5 \geq k$ (or $> k$) and Lemma 3.5 (1), (2), we obtain

$$\begin{aligned}
III'_- &\lesssim \frac{\|u\|^p}{\langle r + |t| \rangle} \int_0^{r-t} \int_{r-t}^{\infty} \frac{1}{\langle \beta \rangle^{1+k}} \left(1 + \log \frac{\langle \beta \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k}p} d\beta d\alpha \\
&\lesssim \frac{\|u\|^p}{\langle r + |t| \rangle \langle r - t \rangle^k} \int_0^{r-t} \left(1 + \log \frac{\langle r - t \rangle}{\langle \alpha \rangle} \right)^{\delta_{1k}p} d\alpha \\
&\lesssim \frac{\|u\|^p}{\langle r + |t| \rangle^k}.
\end{aligned}$$

Thus (3.37) holds also for $r \geq |t|$. Therefore, it follows from (3.36) and (3.37) that III has the same estimate as (3.19).

Proof of Lemma 3.3 for the norm (3.2).

We next treat the case $k(p-1) > 4$ and $\gamma_2 + k \leq 3/2$. We first estimate $\|L[V_1 u]\|$. By (3.2), (3.3) and (3.11), $L[V_1 u](t, x)$ is estimated by the following type of integral:

$$IV \equiv \frac{\|u\|}{r} \int_{-\infty}^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda \langle s \rangle^{\delta-k}}{\langle \lambda \rangle^{\gamma_1} \langle \lambda + |s| \rangle^{\delta}} d\lambda ds.$$

We first estimate the integral for the region $0 \leq s \leq t$ to show

$$IV_+ \equiv \frac{\|u\|}{r} \int_0^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda \langle s \rangle^{\delta-k}}{\langle \lambda \rangle^{\gamma_1} \langle \lambda + s \rangle^\delta} d\lambda ds \quad (3.38)$$

$$\lesssim \frac{\|u\| \langle t \rangle^{\delta-k}}{\langle t+r \rangle^\delta}. \quad (3.39)$$

Using (3.22) in (3.38), we have

$$IV_+ \lesssim \frac{\|u\|}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^k} \int_0^\alpha \frac{1}{\langle \alpha - \beta \rangle^{\gamma_1-1}} d\beta d\alpha \lesssim \frac{\|u\|}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^k} d\alpha$$

since $\gamma_1 > 2$. Lemma 3.4 (2) yields

$$IV_+ \lesssim \frac{\|u\| \langle t \rangle^{\delta-k}}{\langle t+r \rangle^\delta}.$$

Hence (3.39) is proved.

We next treat the integral region $s \leq 0$ when $t \geq 0$ and also the case $t < 0$ to show

$$IV_- \equiv \frac{\|u\|}{r} \int_{-\infty}^{\min\{0,t\}} \int_{|r-t+s|}^{r+t-s} \frac{\lambda \langle s \rangle^{\delta-k}}{\langle \lambda \rangle^{\gamma_1} \langle \lambda + |s| \rangle^\delta} d\lambda ds \lesssim \frac{\|u\| \langle t \rangle^{\delta-k}}{\langle |t| + r \rangle^\delta}. \quad (3.40)$$

Let $|t| \geq r$. Then changing variables by (3.22), we have

$$\begin{aligned} IV_- &\lesssim \frac{\|u\|}{r} \int_{|t|-r}^{|t|+r} \int_\alpha^\infty \frac{\langle \beta - \alpha \rangle^{\delta-k}}{\langle \alpha + \beta \rangle^{\gamma_1-1} \langle \beta \rangle^\delta} d\beta d\alpha \\ &\lesssim \frac{\|u\|}{r} \int_{|t|-r}^{|t|+r} \frac{1}{\langle \alpha \rangle^k} d\alpha \end{aligned}$$

since $\gamma_1 > 2$. Thus we see by Lemma 3.4 (2) that (3.40) holds for $|t| \geq r$.

We finally assume $r \geq |t|$. As shown before, we have only to estimate

$$IV'_- \equiv \frac{\|u\|}{\langle r \rangle} \int_0^{r-t} \int_{r-t}^\infty \frac{\langle \beta - \alpha \rangle^{\delta-k}}{\langle \alpha + \beta \rangle^{\gamma_1-1} \langle \beta \rangle^\delta} d\beta d\alpha.$$

Note that $k + \gamma_1 - 2 \geq \delta$. Then

$$IV'_- \lesssim \frac{\|u\|}{\langle r + |t| \rangle \langle r - t \rangle^{k+\gamma_1-2}} \int_0^{r-t} d\alpha.$$

If $t \leq 0$ or $r \geq 2t \geq 0$, we have

$$IV'_- \lesssim \frac{\|u\|}{\langle r + |t| \rangle^\delta}.$$

If $0 \leq t \leq r \leq 2t$, by $\gamma_1 > 2$,

$$IV'_- \lesssim \frac{\|u\| \langle r - t \rangle^{1-k}}{\langle r + |t| \rangle} \lesssim \frac{\|u\|}{\langle r + |t| \rangle^k} \lesssim \frac{\|u\| \langle t \rangle^{\delta-k}}{\langle r + |t| \rangle^\delta}.$$

Thus (3.40) holds also for $r \geq |t|$. Therefore, it follows from (3.39) and (3.40) that

$$\|L[V_1 u]\| \lesssim \nu_1 \|u\|$$

holds.

We next estimate $\|L[(V_2 * G(u))u]\|$. For simplicity, we denote γ_2 by γ , again. By (3.2), (3.3) and (3.11), $L[V_1 u](t, x)$ is estimated by the following type of integral:

$$V \equiv \frac{\|u\|^p}{r} \int_{-\infty}^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda \langle s \rangle^{\delta-k}}{\langle \lambda + |s| \rangle^{\gamma+\delta} \langle s \rangle^{k(p-1)-3}} d\lambda ds.$$

To prove (3.18) for the norm (3.2), we have only to show that

$$V \lesssim \frac{\|u\|^p \langle t \rangle^{\delta-k}}{\langle |t| + r \rangle^\delta}. \quad (3.41)$$

We first estimate the integral for the region $0 \leq s \leq t$ to show

$$V_+ \equiv \frac{\|u\|^p}{r} \int_0^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda \langle s \rangle^{\delta-k}}{\langle \lambda + |s| \rangle^{\gamma+\delta} \langle s \rangle^{k(p-1)-3}} d\lambda ds \quad (3.42)$$

$$\lesssim \frac{\|u\|^p \langle t \rangle^{\delta-k}}{\langle t + r \rangle^\delta}. \quad (3.43)$$

Using (3.22) in (3.42), we have

$$\begin{aligned} V_+ &\lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^{\gamma-1+k}} \int_0^\alpha \frac{1}{\langle \alpha - \beta \rangle^{k(p-1)-3}} d\beta d\alpha \\ &\lesssim \frac{\|u\|^p}{r} \int_{|t-r|}^{t+r} \frac{1}{\langle \alpha \rangle^k} d\alpha \end{aligned}$$

since $k(p-1) > 4$ and $\gamma \geq 1$. Hence, Lemma 3.4 (2) yields

$$V_+ \lesssim \frac{\|u\|^p \langle t \rangle^{\delta-k}}{\langle t + r \rangle^\delta},$$

and (3.43) is proved.

We next treat the integral region $s \leq 0$ when $t \geq 0$ and also the case $t < 0$ to show

$$\begin{aligned} V_- &\equiv \frac{\|u\|^p}{r} \int_{-\infty}^{\min\{0,t\}} \int_{|r-t+s|}^{r+t-s} \frac{\lambda \langle s \rangle^{\delta-k}}{\langle \lambda + |s| \rangle^{\gamma+\delta} \langle s \rangle^{k(p-1)-3}} d\lambda ds \\ &\lesssim \frac{\|u\|^p \langle t \rangle^{\delta-k}}{\langle |t| + r \rangle^{\delta}}. \end{aligned} \quad (3.44)$$

Let $|t| \geq r$. Then changing variables by (3.22), we have

$$V_- \lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \int_{\alpha}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma-1+k} \langle \beta - \alpha \rangle^{k(p-1)-3}} d\beta d\alpha.$$

Note that $\gamma - 1 + k + k(p-1) - 3 > 1$ since $k(p-1) > 4$ and $\gamma \geq 1$. By Lemma 3.5 (2),

$$V_- \lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \frac{1}{\langle \alpha \rangle^{\gamma-1+k}} d\alpha.$$

Hence, using Lemma 3.4 (2), we obtain

$$V_- \lesssim \frac{\|u\|^p}{r} \int_{|t|-r}^{|t|+r} \frac{1}{\langle \alpha \rangle^k} d\alpha \lesssim \frac{\|u\|^p \langle t \rangle^{\delta-k}}{\langle |t| + r \rangle^{\delta}}.$$

Thus we see that (3.44) holds for $|t| \geq r$.

We finally assume $r \geq |t|$. As shown before, we have only to estimate

$$\begin{aligned} V'_- &\equiv \frac{\|u\|^p}{\langle r \rangle} \int_0^{r-t} \int_{r-t}^{\infty} \frac{(\beta + \alpha) \langle \beta - \alpha \rangle^{\delta-k}}{\langle \beta \rangle^{\gamma+\delta} \langle \beta - \alpha \rangle^{k(p-1)-3}} d\beta d\alpha \\ &\lesssim \frac{\|u\|^p}{\langle r \rangle} \int_0^{r-t} \int_{r-t}^{\infty} \frac{1}{\langle \beta \rangle^{\gamma+\delta-1} \langle \beta - \alpha \rangle^{pk-3-\delta}} d\beta d\alpha. \end{aligned} \quad (3.45)$$

Let $pk - 4 - \delta \neq 0$. By Lemma 3.5 (2), we have

$$V'_- \lesssim \frac{\|u\|^p}{\langle r + |t| \rangle \langle r - t \rangle^{\gamma+\delta-1+[pk-4-\delta]_-}} \int_0^{r-t} d\alpha.$$

Note that

$$\gamma + \delta - 1 + [pk - 4 - \delta]_- = \begin{cases} \gamma + \delta - 1 \geq \delta & (pk - 4 - \delta > 0) \\ pk - 5 + \gamma \geq \delta & (pk - 4 - \delta < 0) \end{cases}$$

by assumption. Hence, proceeding as in the estimate of IV'_- and by $\delta \geq k$, we obtain

$$V'_- \lesssim \frac{\|u\|^p \langle t \rangle^{\delta-k}}{\langle r + |t| \rangle^\delta}. \quad (3.46)$$

If $pk - 4 - \delta = 0$, we split the integral in (3.45) into two parts:

$$\begin{aligned} V'_- &= \frac{\|u\|^p}{\langle r \rangle} \left(\int_0^{r-t} \int_{\max\{r-t, 2\alpha\}}^\infty + \int_{(r-t)/2}^{r-t} \int_{r-t}^{2\alpha} \right) \frac{1}{\langle \beta \rangle^{\gamma+\delta-1} \langle \beta - \alpha \rangle} d\beta d\alpha \\ &\lesssim \frac{\|u\|^p}{\langle r \rangle} \int_0^{r-t} \int_{r-t}^\infty \frac{1}{\langle \beta \rangle^{\gamma+\delta}} d\beta d\alpha \\ &\quad + \frac{\|u\|^p}{\langle r \rangle} \int_{(r-t)/2}^{r-t} \int_{r-t}^{2\alpha} \frac{1}{\langle \beta \rangle^{\gamma+\delta-1} \langle \beta - \alpha \rangle} d\beta d\alpha \\ &\lesssim \frac{\|u\|^p}{\langle r + |t| \rangle \langle r - t \rangle^{\gamma+\delta-1}} \int_0^{r-t} d\alpha \\ &\quad + \frac{\|u\|^p}{\langle r + |t| \rangle \langle r - t \rangle^{\gamma+\delta-1}} \int_{(r-t)/2}^{r-t} \log \frac{\langle \alpha \rangle}{\langle r - t - \alpha \rangle} d\alpha. \end{aligned}$$

To handle the last integral, we note that

$$\int_{\beta/2}^\beta \log \frac{\langle \alpha \rangle}{\langle \beta - \alpha \rangle} d\alpha \leq \int_{\beta/2}^\beta \log \frac{\alpha}{\beta - \alpha} d\alpha = \beta \int_{1/2}^1 \log \frac{z}{1-z} dz = C\beta$$

for all $\beta \geq 0$. Using this fact and proceeding as in the estimate of IV'_- , we now easily get

$$V'_- \lesssim \frac{\|u\|^p \langle t \rangle^{\delta-k}}{\langle r + |t| \rangle^\delta}. \quad (3.47)$$

Thus (3.44) holds also for $r \geq |t|$ from (3.46) and (3.47). Therefore, it follows from (3.43) and (3.44) that

$$\|L[(V_2 * G(u))u]\| \lesssim \|u\|^p$$

holds. This completes the proof of Lemma 3.3. $\square$

4. Proofs of Theorems.

Proof of Theorem 2.1. Let X be the linear space defined by

$$X = \{u(t, x) : \partial_x^\alpha u(t, x) \in C(\mathbf{R} \times \mathbf{R}^3), \|\partial_x^\alpha u\| < \infty \text{ for } |\alpha| \leq 2\},$$

where $\|\cdot\|$ is defined in (3.1) and (3.2). We can verify that X is complete with respect to the norm

$$\|u\|_X = \sum_{|\alpha| \leq 2} \|\partial_x^\alpha u\|.$$

Let us define the sequence of functions by

$$u_0 = u_0^-, \quad u_{n+1} = u_0^- + L[F(u_n)],$$

where $F(u) = V_1(x)u + (V_2 * G(u))u$. Then Lemma 3.1 gives us

$$\|u_0^-\|_X \leq C_k \varepsilon_0.$$

Hence $u_0^- \in X$. Choose ε_0 and ν_1 sufficiently small depending on $p, k, \gamma_1, \gamma_2, \nu_2, A_1$ and A_2 . Proceeding as in [1] and [3], we see that $u_n \in X$ and u_n is a Cauchy sequence in X by Lemma 3.3, and therefore that there exists a unique global solution of (2.2). This completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. We show $\|u - u_0^-\|_e \rightarrow 0$ ($t \rightarrow -\infty$). To prove this fact, as is well known, it is sufficient to show

$$\int_{-\infty}^t \|F(u(s))\|_{L^2(\mathbf{R}^3)} ds \lesssim \langle t \rangle^{-\delta_0}, \quad t \leq 0 \quad (4.1)$$

for some $\delta_0 > 0$, and for that, it suffices to show

$$\|F(u(s))\|_{L^2(\mathbf{R}^3)}^2 \lesssim \langle s \rangle^{-2-\delta}, \quad s \leq 0$$

for some $\delta > 0$. Since $V_1(x) \equiv 0$, we have only to show

$$\|(V_2 * G(u))u(s)\|_{L^2(\mathbf{R}^3)}^2 \lesssim \langle s \rangle^{-2-\delta} \quad (4.2)$$

with some $\delta > 0$. We divide the proof of (4.2) into three cases as in the previous section.

Case 1. $k(p-1) > 3$.@ We subdivide this into two cases.

(i) Let $[k(p-1) > 4$ and $\gamma_2 + k > 3/2]$ or $3 < k(p-1) \leq 4$. Note that the latter case also yields $\gamma_2 + k > 3/2$ since $11/2 - \gamma_2 < pk \leq k + 4$ by assumption.

From (3.1) and Proposition 3.2, we have

$$\begin{aligned} & |(V_2 * G(u))u(s, x)|^2 \\ & \lesssim \frac{\|u\|^{2p}}{\langle |x| + |s| \rangle^{2\gamma_2 + 2k} \langle s \rangle^{2k(p-1)-6}} \left(1 + \log \frac{\langle |s| + |x| \rangle}{\langle |s| - |x| \rangle} \right)^{2\delta_{1k}}. \end{aligned}$$

Integrating over $\mathbf{R}^3$ gives

$$\begin{aligned} & \| (V_2 * G(u))u(s) \|_{L^2(\mathbf{R}^3)}^2 \\ & \lesssim \frac{\|u\|^{2p}}{\langle s \rangle^{2k(p-1)-6}} \int_0^\infty \frac{r^2}{\langle r + |s| \rangle^{2\gamma_2+2k}} \left(1 + \log \frac{\langle |s| + r \rangle}{\langle |s| - r \rangle} \right)^{2\delta_{1k}} dr \\ & \lesssim \frac{\|u\|^{2p}}{\langle s \rangle^{2pk+2\gamma_2-9-\varepsilon\delta_{1k}}} \end{aligned}$$

for any $\varepsilon > 0$. Note that we can choose $\varepsilon > 0$ so small that $-2pk - 2\gamma_2 + 9 + \varepsilon < -2$ since $pk > (11 - 2\gamma_2)/2$. Thus, we obtain (4.2).

(ii) We next treat the remaining case $k(p-1) > 4$ and $\gamma_2 + k \leq 3/2$. From (3.2) and Proposition 3.2, we have

$$|(V_2 * G(u))u(s, x)|^2 \lesssim \frac{\|u\|^{2p} \langle s \rangle^{2\delta-2k}}{\langle |x| + |s| \rangle^{2\gamma_2+2\delta} \langle s \rangle^{2k(p-1)-6}},$$

where $\delta = \min\{2k + \gamma_2 - 1/2, pk - 5 + \gamma_2\}$. Note that $\gamma_2 + \delta > 3/2$. In fact,

$$2k + 2\gamma_2 - \frac{1}{2} \geq 2k + \frac{3}{2} > \frac{3}{2}, \quad \text{and} \quad pk - 5 + 2\gamma_2 > \gamma_2 + \frac{1}{2} \geq \frac{3}{2}$$

since $\gamma_2 \geq 1$ and $pk > (11 - 2\gamma_2)/2$. Integrating over $\mathbf{R}^3$ gives

$$\begin{aligned} \| (V_2 * G(u))u(s) \|_{L^2(\mathbf{R}^3)}^2 & \lesssim \frac{\|u\|^{2p}}{\langle s \rangle^{2pk-2\delta-6}} \int_0^\infty \frac{r^2}{\langle r + |s| \rangle^{2\gamma_2+2\delta}} dr \\ & \lesssim \frac{\|u\|^{2p}}{\langle s \rangle^{2pk+2\gamma_2-9}}. \end{aligned}$$

Since $pk > (11 - 2\gamma_2)/2$, we obtain (4.2).

Case 2. $k(p-1) = 3$. From (3.1) and Proposition 3.2, we have

$$\begin{aligned} & |(V_2 * G(u))u(s, x)|^2 \\ & \lesssim \frac{\|u\|^{2p} (1 + \log \langle |s| + |x| \rangle)^2}{\langle |x| + |s| \rangle^{2\gamma_2+2k}} \left(1 + \log \frac{\langle |s| + |x| \rangle}{\langle |s| - |x| \rangle} \right)^{2\delta_{1k}}. \end{aligned}$$

Integrating over $\mathbf{R}^3$ gives

$$\begin{aligned} & \| (V_2 * G(u))u(s) \|_{L^2(\mathbf{R}^3)}^2 \\ & \lesssim \|u\|^{2p} \int_0^\infty \frac{r^2 (1 + \log \langle r + |s| \rangle)^2}{\langle r + |s| \rangle^{2\gamma_2+2k}} \left(1 + \log \frac{\langle |s| + r \rangle}{\langle |s| - r \rangle} \right)^{2\delta_{1k}} dr \end{aligned}$$

$$\lesssim \frac{\|u\|^{2p}}{\langle s \rangle^{2\gamma_2+2k-3-\varepsilon}}$$

for any $\varepsilon > 0$. Note that we can choose $\varepsilon > 0$ so small that $-2\gamma_2-2k+3+\varepsilon < -2$ since $k+3 = pk > (11-2\gamma_2)/2$. Thus, we obtain (4.2).

Case 3. $k(p-1) < 3$. From (3.1) and Proposition 3.2, we have

$$|(V_2 * G(u))u(s, x)|^2 \lesssim \frac{\|u\|^{2p}}{\langle |x| + |s| \rangle^{2pk+2\gamma_2-6}} \left(1 + \log \frac{\langle |s| + |x| \rangle}{\langle |s| - |x| \rangle} \right)^{2\delta_{1k}}.$$

Integrating over $\mathbf{R}^3$ gives

$$\begin{aligned} & \| (V_2 * G(u))u(s) \|_{L^2(\mathbf{R}^3)}^2 \\ & \lesssim \|u\|^{2p} \int_0^\infty \frac{r^2}{\langle r + |s| \rangle^{2pk+2\gamma_2-6}} \left(1 + \log \frac{\langle |s| + r \rangle}{\langle |s| - r \rangle} \right)^{2\delta_{1k}} dr \\ & \lesssim \frac{\|u\|^{2p}}{\langle s \rangle^{2pk+2\gamma_2-9-\varepsilon\delta_{1k}}} \end{aligned}$$

for any $\varepsilon > 0$, which is the same as in Case 1. Thus we obtain (4.2). This completes the proof of (4.1).

We next define

$$u_0^+(t, x) = u(t, x) - \frac{1}{4\pi} \int_t^\infty \frac{1}{s-t} \int_{|y-x|=s-t} F(u(s, y)) dS_y ds.$$

Then clearly u_0^+ is a C^2 -solution of (2.1) and we can show that $\|u(t) - u_0^+(t)\|_e \rightarrow 0$ ($t \rightarrow +\infty$) as before.

We finally show the existence of the scattering operator. As stated above, we have

$$\|u_0^-\|_X \leq C_k \varepsilon_0.$$

We see from the proof of the existence of the solutions u and u_+ that

$$\|u\|_X \leq 2C_k \varepsilon_0 \quad \text{and} \quad \|u_0^+\|_X \leq 3C_k \varepsilon_0.$$

Let $X(\varepsilon) = \{f \in X : \|f\|_X \leq \varepsilon\}$. We see that there exists the wave operator as a mapping from $X(C_k \varepsilon_0)$ into $X(2C_k \varepsilon_0)$, and the scattering operator as one from $X(C_k \varepsilon_0)$ into $X(3C_k \varepsilon_0)$. This completes the proof of Theorem 2.2. $\square$

Proof of Corollary 1. For the case $[k(p-1) \geq 7/2$ and $\gamma_2 + k > 3/2]$ or $k(p-1) < 7/2$, we use (3.1) with $t \in \mathbf{R}$ replaced by $t \geq 0$ as the weighted norm. For the case $k(p-1) \geq 7/2$ and $\gamma_2 + k \leq 3/2$, we use (3.2) with $\delta = 1 + k$ and $t \in \mathbf{R}$ replaced by $t \geq 0$.

We define

$$L^+[F(u)](t, x) = \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \int_{|y-x|=t-s} F(u(s, y)) dS_y ds. \quad (4.3)$$

Let p, k, γ_1 and γ_2 satisfy the assumptions of Corollary 1. Then

$$\|L^+[F(u)]\| \lesssim \nu_1 \|u\| + \|u\|^p \quad (4.4)$$

holds.

In fact, for the case $[k(p-1) \geq 7/2$ and $\gamma_2 + k > 3/2]$ or $k(p-1) < 7/2$, the proof of (4.4) is the same as that of Lemma 3.3. See the estimates of I_+ , II_+ and III_+ .

For the case $k(p-1) \geq 7/2$ and $\gamma_2 + k \leq 3/2$, we see from the estimates of IV_+ and V_+ in the proof of Lemma 3.3 that

$$IV_+, V_+ \lesssim \frac{\|u\|^p \langle t \rangle^{\delta-k}}{\langle t+r \rangle^\delta}$$

holds even for $\delta = 1 + k$. Thus, we obtain (4.4).

The rest of the proof of the corollary is the same as that of Theorem 2.1. $\square$

Proof of Theorem 2.3. Let u and u_0 be the solutions given in Corollary 1. We first claim that

$$\|u(t) - u_0(t)\|_e \lesssim \begin{cases} \chi_{V_1} \|u\| \langle t \rangle^{1-k+\varepsilon\delta_{1k}} + \|u\|^p \langle t \rangle^{11/2-pk-\gamma_2+\varepsilon} & (11/2 \geq pk + \gamma_2), \\ \chi_{V_1} \|u\| \langle t \rangle^{1-k+\varepsilon\delta_{1k}} + \|u\|^p & (11/2 < pk + \gamma_2) \end{cases} \quad (4.5)$$

for each $t \geq 0$ and any $\varepsilon > 0$, where $\chi_{V_1} = 1$ if $V_1(x) \not\equiv 0$, and $\chi_{V_1} = 0$ if $V_1(x) \equiv 0$.

We show (4.5). From (H2) and (3.1), we first have

$$|V_1(x)u(s, x)| \lesssim \begin{cases} \frac{\|u\|}{\langle x \rangle^{\gamma_1} \langle s + |x| \rangle^k} \left(1 + \log \frac{\langle s + |x| \rangle}{\langle s - |x| \rangle} \right)^{\delta_{1k}} \\ \quad ([k(p-1) \geq 7/2 \text{ and } \gamma_2 + k > 3/2] \\ \quad \text{or } k(p-1) < 7/2), \\ \\ \frac{\|u\| \langle s \rangle}{\langle x \rangle^{\gamma_1} \langle s + |x| \rangle^{1+k}} \\ \quad (k(p-1) \geq 7/2 \text{ and } \gamma_2 + k \leq 3/2). \end{cases}$$

For the case $[k(p-1) \geq 7/2 \text{ and } \gamma_2 + k > 3/2]$ or $k(p-1) < 7/2$,

$$\begin{aligned} \|V_1 u(s)\|_{L^2(\mathbf{R}^3)}^2 &\lesssim \|u\|^2 \int_0^\infty \frac{r^2}{\langle r \rangle^{\gamma_1} \langle s+r \rangle^{2k}} \left(1 + \log \frac{\langle s+r \rangle}{\langle s-r \rangle} \right)^{2\delta_{1k}} dr \\ &\lesssim \frac{\|u\|^2}{\langle s \rangle^{2k-2\varepsilon\delta_{1k}}} \end{aligned}$$

for any $\varepsilon > 0$. For the case $k(p-1) \geq 7/2$ and $\gamma_2 + k \leq 3/2$,

$$\|V_1 u(s)\|_{L^2(\mathbf{R}^3)}^2 \lesssim \|u\|^2 \int_0^\infty \frac{r^2}{\langle r \rangle^{2\gamma_1} \langle s+r \rangle^{2+2k}} dr \lesssim \frac{\|u\|^2}{\langle s \rangle^{2k}}.$$

Thus, we obtain

$$\|V_1 u(s)\|_{L^2(\mathbf{R}^3)} \lesssim \frac{\|u\|}{\langle s \rangle^{k-\varepsilon\delta_{1k}}} \quad (4.6)$$

for either case.

We next estimate $\|(V_2 * G(u))u(s)\|_{L^2(\mathbf{R}^3)}$. For the case $[k(p-1) \geq 7/2 \text{ and } \gamma_2 + k > 3/2]$ or $k(p-1) < 7/2$, from the proof of Lemma 2.2, we have

$$\|(V_2 * G(u))u(s)\|_{L^2(\mathbf{R}^3)}^2 \lesssim \begin{cases} \frac{\|u\|^{2p}}{\langle s \rangle^{2pk+2\gamma_2-9-\varepsilon\delta_{1k}}} \\ \quad ([k(p-1) \geq 7/2 \text{ and } \gamma_2 + k > 3/2], \\ \quad 3 < k(p-1) < 7/2, \text{ or } k(p-1) < 3) \\ \\ \frac{\|u\|^{2p}}{\langle s \rangle^{2\gamma_2+2k-3-\varepsilon}} \quad (k(p-1) = 3) \end{cases}$$

for any $\varepsilon > 0$. For the case $k(p-1) \geq 7/2$ and $\gamma_2 + k \leq 3/2$, we see from (3.2) and Proposition 3.2 that

$$|(V_2 * G(u))u(s, x)| \lesssim \frac{\|u\|^{2p} \langle s \rangle^{2\delta-2k}}{\langle |x| + |s| \rangle^{2\gamma_2+2\delta} \langle s \rangle^{2k(p-1)-6}}.$$

Note that $\gamma_2 + \delta = \gamma_2 + 1 + k \geq 2 + k > 3/2$. Integrating over $\mathbf{R}^3$ gives

$$\|(V_2 * G(u))u(s)\|_{L^2(\mathbf{R}^3)}^2 \lesssim \frac{\|u\|^{2p}}{\langle s \rangle^{2pk+2\gamma_2-9}}.$$

Thus, we obtain

$$\|(V_2 * G(u))u(s)\|_{L^2(\mathbf{R}^3)} \lesssim \frac{\|u\|^p}{\langle s \rangle^{pk+\gamma_2-9/2-\varepsilon/2}} \quad (4.7)$$

for either case. Combining (4.6) with (4.7), we see that

$$\begin{aligned} \|u(t) - u_0(t)\|_e &\lesssim \int_0^t (\|V_1 u(s)\|_{L^2(\mathbf{R}^3)} + \|(V_2 * G(u))u(s)\|_{L^2(\mathbf{R}^3)}) ds \\ &\lesssim \|u\| \langle t \rangle^{1-k+\varepsilon\delta_{1k}} + \|u\|^p \langle t \rangle^{11/2-pk-\gamma_2+\varepsilon/2} \\ &\quad \text{if } 11/2 \geq pk + \gamma_2, \\ \|u(t) - u_0(t)\|_e &\lesssim \|u\| \langle t \rangle^{1-k+\varepsilon\delta_{1k}} + \|u\|^p \quad \text{if } 11/2 < pk + \gamma_2 \end{aligned}$$

for each $t \geq 0$. Thus, (4.5) holds.

Proof of Theorem 2.3 assuming (2.6). The proof is the same as that of Theorem 3 assuming (1.15) in [4].

Proof of Theorem 2.3 assuming (2.7). Suppose that

$$\|u(t) - u_0(t)\|_e \longrightarrow 0 \quad (t \rightarrow \infty). \quad (4.8)$$

Since $2pk > 2k + 2(5 - \gamma_2) > 4$, it is easy to see that $\langle |x| + t \rangle^{-pk} \in L^2(\mathbf{R}^3)$ for all $t \geq 0$. Then

$$H_2(t) = \int \langle |x| + t \rangle^{-pk} \cdot \partial_t(u - u_0) dx \quad (4.9)$$

is well-defined on (T_0, ∞) for some $T_0 > 0$.

$$\begin{aligned} H_2'(t) &= pk \int \langle |x| + t \rangle^{-pk-1} \partial_t(u - u_0) dx \\ &\quad + \int \langle |x| + t \rangle^{-pk} \cdot \Delta(u - u_0) dx \\ &\quad + \int \langle |x| + t \rangle^{-pk} \cdot (V_2 * |u|^{p-1})u(t, x) dx \\ &\equiv B_1 + B_2 + B_3 \end{aligned} \quad (4.10)$$

for all $t > T_0$.

We first estimate B_1 . Hölder's inequality yields

$$\begin{aligned} |B_1| &\lesssim \|u(t) - u_0(t)\|_e \cdot \left(\int \langle |x| + t \rangle^{-2pk-2} dx \right)^{1/2} \\ &\lesssim \|u(t) - u_0(t)\|_e \cdot \langle t \rangle^{-pk+1/2} \end{aligned} \quad (4.11)$$

since $2pk + 2 > 6$ as above.

We next estimate B_2 . Using integration by parts and Hölder's inequality we obtain

$$\begin{aligned} |B_2| &\lesssim \int \langle |x| + t \rangle^{-pk-1} \cdot |\nabla(u - u_0)| dx \\ &\lesssim \|u(t) - u_0(t)\|_e \cdot \left(\int \langle |x| + t \rangle^{-2pk-2} dx \right)^{1/2} \\ &\lesssim \|u(t) - u_0(t)\|_e \cdot \langle t \rangle^{-pk+1/2}, \end{aligned} \quad (4.12)$$

which is the same as (4.11).

Finally, we derive a lower bound for B_3 . By Lemma 9 in [4], we have

$$u(t, x) \gtrsim t \int_{|\xi|=1} \langle x + t\xi \rangle^{-k-1} dS_\xi \gtrsim \frac{t}{\langle |x| + t \rangle^{k+1}} \quad (4.13)$$

for all $t > T_0$. Then we note that (3.11) and (4.13) imply

$$\begin{aligned} V_2 * |u|^{p-1}(t, x) &= \int |x - y|^{-\gamma_2} |u|^{p-1}(t, y) dy \\ &\gtrsim \int_0^\infty \left(\frac{1}{r} \int_{|r-\eta|}^{r+\eta} \rho^{1-\gamma_2} d\rho \right) \eta \left(\frac{t}{\langle \eta + t \rangle^{k+1}} \right)^{p-1} d\eta, \end{aligned}$$

where $r = |x|$.

We claim here that

$$\frac{1}{r} \int_{|r-\eta|}^{r+\eta} \rho^{1-\gamma_2} d\rho \gtrsim \frac{\eta(\eta + r)^{1-\gamma_2}}{\max\{r, \eta\}} \quad \text{for } \gamma_2 \geq 1. \quad (4.14)$$

Indeed, the estimate above is easy to show.

Using (4.14), we obtain the lower bound of $V_2 * |u|^{p-1}$:

$$V_2 * |u|^{p-1}(t, x) \gtrsim t^{p-1} \int_0^\infty \frac{\eta^2(\eta + r)^{1-\gamma_2}}{\max\{r, \eta\} \langle \eta + t \rangle^{(p-1)(k+1)}} d\eta$$

$$\begin{aligned}
&\gtrsim \frac{t^{p-1}}{\langle r+t \rangle^{(p-1)(k+1)} r} \int_0^r \eta^2 (\eta+r)^{1-\gamma_2} d\eta \\
&\gtrsim \frac{t^{p-1} r^{3-\gamma_2}}{\langle r+t \rangle^{(p-1)(k+1)}}.
\end{aligned} \tag{4.15}$$

By (4.10), (4.13) and (4.15), B_3 is estimated by

$$\begin{aligned}
B_3 &\gtrsim t^p \int_{t/3 \leq |x| \leq t/2} \langle |x| + t \rangle^{-pk} \frac{|x|^{3-\gamma_2}}{\langle |x| + t \rangle^{p(k+1)}} dx \\
&\gtrsim \langle t \rangle^{-2pk-\gamma_2+6}
\end{aligned} \tag{4.16}$$

for all $t > T_0 + 1$. Combining (4.10), (4.11), (4.12) and (4.16) together, we have

$$H'_2(t) \geq C_1 \langle t \rangle^{-2pk-\gamma_2+6} - C_2 \|u(t) - u_0(t)\|_e \langle t \rangle^{-pk+1/2}$$

for some constants $C_1, C_2 > 0$ and $t > T_0 + 1$. Note that

$$\langle t \rangle^{-2pk-\gamma_2+6} \geq \langle t \rangle^{-pk+1/2}$$

since $pk \leq (11 - 2\gamma_2)/2$. Moreover, by (4.8),

$$H'_2(t) \geq (C_1 - C_2 \|u(t) - u_0(t)\|_e) \langle t \rangle^{-2pk-\gamma_2+6} \geq C_3 \langle t \rangle^{-2pk-\gamma_2+6}$$

for all $t > T_1$ with sufficiently large $T_1 > 0$. Integrating the last inequality yields

$$H_2(2t) - H_2(t) \geq C_4 \langle t \rangle^{-2pk-\gamma_2+7} \tag{4.17}$$

for $t > T_1$.

On the other hand, from (4.9) and $2pk > 4$,

$$\begin{aligned}
|H_2(s)| &\leq \|\partial_t(u - u_0)\|_{L^2(\mathbf{R}^3)} \left(\int \langle |x| + s \rangle^{-2pk} dx \right)^{1/2} \\
&\leq \|u(s) - u_0(s)\|_e \langle s \rangle^{3/2-pk}
\end{aligned} \tag{4.18}$$

for any $s \geq 0$. Combining (4.17) and (4.18) together,

$$\begin{aligned}
C_4 \langle t \rangle^{-2pk-\gamma_2+7} &\leq \sup_{t \leq s \leq 2t} 2|H_2(s)| \\
&\leq C \langle t \rangle^{3/2-pk} \sup_{t \leq s \leq 2t} \|u(s) - u_0(s)\|_e
\end{aligned}$$

for all $t > T_1$. Since $pk \leq (11 - 2\gamma_2)/2$, we obtain

$$\sup_{t \leq s \leq 2t} \|u(s) - u_0(s)\|_e \geq C_5 \langle t \rangle^{11/2-pk-\gamma_2} \geq C_5 > 0$$

for $t > T_1$. This is a contradiction to (4.8). This completes the proof of Theorem 2.3. $\square$

References

- [1] F. Asakura, Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimensions, *Comm. Partial Differential Equations*, **11**, No.13 (1986), 1459–1487.
- [2] X. Cheng and Y. Gao, Small data global well-posedness for the nonlinear wave equation with nonlocal nonlinearity, *Math. Methods Appl. Sci.*, **36** (2013), 99–112.
- [3] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.*, **28** (1979), 235–268.
- [4] P. Karageorgis and K. Tsutaya, On the asymptotic behavior of nonlinear waves in the presence of a short-range potential, *Manuscripta Math.*, **119**, no. 3 (2006), 323–345.
- [5] P. Karageorgis and K. Tsutaya, Existence and blow up for a Hartree-type wave equation, in preparation.
- [6] H. Kubo, On the point-wise decay estimates for the wave equation and their applications, in *Dispersive Nonlinear Problems in Mathematical Physics* (eds P.D’Ancona and V. Georgiev), Aracne (2005).
- [7] H. Pecher, Scattering for semilinear wave equations with small data in three space dimensions, *Math. Z.*, **198** (1988), 277–289.
- [8] W. A. Strauss and K. Tsutaya, Existence and blow up of small amplitude nonlinear waves with a negative potential, *Discrete and Conti. Dynamical Systems*, **3** (1997), 175–188.
- [9] K. Tsutaya, Global existence and the life span of solutions of semilinear wave equations with data of non compact support in three space dimensions, *Funkcial Ekvac.*, **37** (1) (1994), 1–18.
- [10] K. Tsutaya, Global existence and blow up for a wave equation with a potential and a cubic convolution, *Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday. Vol. 2*, 913–937, Kluwer Acad. Publ., Dordrecht (2003).
- [11] K. Tsutaya, Scattering theory for a wave equation of Hartree type, *Differential & Difference Equations and Applications*, 1061–1065, Hindawi Publ. Corp., New York (2006).

- [12] K. Tsutaya, Weighted estimates for a convolution appearing in the wave equation of Hartree type, *J. Math. Anal. Appl.*, **411** (2014), 719-731.
- [13] K. Tsutaya, Scattering theory for the wave equation of a Hartree type in three space dimensions, *Discrete and Continuous Dynamical Systems*, **34** (2014), 2261-2281.