

HYERS-ULAM STABILITY OF SECOND ORDER LINEAR DIFFERENCE EQUATIONS

A.K. Tripathy

Department of Mathematics
Sambalpur University
Sambalpur, 768019, INDIA

Abstract: In this work, the Hyers-Ulam stability of second order linear difference equations of the form:

$$y_{n+2} + \alpha y_{n+1} + \beta y_n = 0,$$

$$y_{n+2} + \alpha y_{n+1} + \beta y_n = r_n$$

and

$$y_{n+2} - \alpha_n y_{n+1} + \beta_n y_n = r_n$$

are studied, where $\alpha, \beta \in \mathbb{R}$ and r_n, α_n, β_n are sequences of reals.

AMS Subject Classification: 39A45, 39B42

Key Words: Hyers-Ulam stability, difference equation

1. Introduction

In 1940 [23], it is S.M. Ulam who had discussed the number of important unsolved problems in a talk concerning the study of stability problems for various functional equations. Among such problems, a problem concerning the stability of functional equations *Give conditions for a order mapping near approximately linear mapping to exist* is one of them. In 1941, Hyers [3] gave an answer to

the problem on Banach space. Furthermore, the result of Hyers has been generalized by Rassias [19]. After that many researchers have extended the Ulam's stability problems to other functional equations and generalized Hyer's result in various directions(see for example [4], [11], [12], [20]).

Thereafter, Ulam's stability problem for functional equations was replaced by stability of differential equations. The differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) + h(t) = 0$$

has the Hyers-Ulam stability, if for given $\epsilon > 0$, I be an open interval and for any function f satisfying the differential inequality

$$|a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) + h(t)| \leq \epsilon,$$

then there exists a solution $f_0(t)$ of the above equation such that

$$|f(t) - f_0(t)| \leq K(\epsilon) \text{ and } \lim_{\epsilon \rightarrow 0} K(\epsilon) = 0, t \in I.$$

Li and Shen [10] have discussed the Hyers-Ulam stability of the linear second order differential equations of the form:

$$y'' + ay' + by = 0 \tag{1.1}$$

and

$$y'' + ay' + by = f(t), \tag{1.2}$$

where $a, b \in \mathbb{R}$. The objective of this work is to study the Hyers-Ulam stability of discrete analogue of the equations (1.1) and (1.2) as

$$y_{n+2} + \alpha y_{n+1} + \beta y_n = 0 \tag{1.3}$$

and

$$y_{n+2} + \alpha y_{n+1} + \beta y_n = r_n \tag{1.4}$$

where $\alpha, \beta \in \mathbb{R}$ and r_n is a sequence of reals. Also, an effort is made to study the Hyers-Ulam stability of

$$y_{n+2} - \alpha_n y_{n+1} + \beta_n y_n = r_n. \tag{1.5}$$

Definition 1.1. The difference equation

$$a_k(n)y(n+k) + a_{k-1}(n)y(n+k-1) + \dots + a_1(n)y(n+1) + a_0(n)y(n) + h(n) = 0$$

has the Hyers-Ulam stability, if for given $\epsilon > 0$, I be an open interval and for any real valued function f satisfying the inequality

$$|a_k(n)y(n+k) + a_{k-1}(n)y(n+k-1) + \dots + a_1(n)y(n+1) + a_0(n)y(n) + h(n)| \leq \epsilon,$$

then there exists a solution f_0 of the above difference equation such that $|f(n) - f_0(n)| \leq K(\epsilon)$ and $\lim_{\epsilon \rightarrow 0} K(\epsilon) = 0$, for $n \in I \subset N(0) = \{0, 1, 2, 3, \dots\}$.

On Hyers-Ulam stability, several works have been done in the direction of differential equations. Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see for example [16], [20]). After that, Alsina and Ger published their work [1], where they have proved the Hyers-Ulam stability of the differential equation $y'(t) = y(t)$. In this direction we refer some of the works [5], [6], [7], [8], [13], [14], [15], [17], [21], [22] to the readers and the references cited therein. We remark here that absolutely there is no such work on (1.3)/ (1.4)/ (1.5) regarding the Hyers-Ulam stability. However, we refer the reader to some few works (see [2], [9], [18]) and the references cited therein.

Definition 1.2. We say that (1.4) has the Hyers-Ulam stability if there exists a constant $K > 0$ with the property: for every $\epsilon > 0$, y_n, r_n defined for $n \in (a, b+1)$, $0 < a < b < \infty$, if

$$|y_{n+2} + \alpha y_{n+1} + \beta y_n - r_n| \leq \epsilon, \quad (1.6)$$

then there exists some $z_n, n \in (a, b+1)$ satisfying

$$z_{n+2} + \alpha z_{n+1} + \beta z_n = r_n$$

such that $|y_n - z_n| < K\epsilon$. We call such K as a Hyers-Ulam stability constant for (1.4).

2. Stability Results for (1.3) and (1.4)

This section deals with the Hyers-Ulam stability of (1.3) and (1.4).

Theorem 2.1. *Assume that the characteristic equation $m^2 + \alpha m + \beta = 0$ have two different positive roots. Then (1.3) has the Hyers-Ulam stability.*

Proof. Let $\epsilon > 0$ and $y_n, n \in (a, b+1)$ be a solution of (1.3) satisfying the property

$$|y_{n+2} + \alpha y_{n+1} + \beta y_n| \leq \epsilon.$$

Let λ and μ be the positive roots of the characteristic equation. For $n \in (a, b+1)$, define $g_n = y_{n+1} - \lambda y_n$. Then

$$g_{n+1} = y_{n+2} - \lambda y_{n+1}$$

and hence

$$\begin{aligned} |g_{n+1} - \mu g_n| &= |y_{n+2} - \lambda y_{n+1} - \mu y_{n+1} + \lambda \mu y_n| \\ &= |y_{n+2} - (\lambda + \mu) y_{n+1} + \lambda \mu y_n| \\ &= |y_{n+2} + \alpha y_{n+1} + \beta y_n| \leq \epsilon. \end{aligned}$$

Equivalently, g_n satisfies the relation

$$-\epsilon \leq g_{n+1} - \mu g_n \leq \epsilon. \quad (2.1)$$

Upon the choice of λ and μ , we have four possibilities viz.

i) $\lambda > 1, \mu > 1$; ii) $\lambda \leq 1, \mu \leq 1$; iii) $\lambda > 1, \mu \leq 1$; iv) $\lambda \leq 1, \mu > 1$.

Consider Case (i). Then (2.1) can be viewed as

$$-\epsilon \mu^{-(n+1)} \leq \mu^{-(n+1)} [g_{n+1} - \mu g_n] \leq \epsilon \mu^{-(n+1)},$$

that is,

$$-\epsilon \mu^{-(n+1)} \leq \Delta(\mu^{-n} g_n) \leq \epsilon \mu^{-(n+1)}. \quad (2.2)$$

Therefore for $n \in (a, b+1)$, it follows that

$$-\epsilon \sum_{j=n}^b \mu^{-(j+1)} \leq \sum_{j=n}^b \Delta(\mu^{-j} g_j) \leq \epsilon \sum_{j=n}^b \mu^{-(j+1)}.$$

which then implies that

$$\frac{-\epsilon\mu^{-n}}{\mu-1} \leq \mu^{-(b+1)}g_{b+1} - \mu^{-n}g_n \leq \frac{\epsilon\mu^{-n}}{\mu-1}.$$

Consequently,

$$-\epsilon_1 \leq \mu^{-(b-n+1)}g_{b+1} - g_n \leq \epsilon_1,$$

where $\epsilon_1 = \frac{\epsilon}{\mu-1}$. Let $z_n = \mu^{-(b-n+1)}g_{b+1}$. Then $z_{n+1} - \mu z_n = 0$. Now, $|g_n - z_n| \leq \epsilon_1$ implies that

$$-\epsilon_1 \leq y_{n+1} - \lambda y_n - z_n \leq \epsilon_1$$

and hence

$$-\epsilon_1 \lambda^{-(n+1)} \leq \lambda^{-(n+1)}[y_{n+1} - \lambda y_n - z_n] \leq \epsilon_1 \lambda^{-(n+1)}.$$

Proceeding as above, we obtain

$$-\epsilon_1 \frac{\lambda^{-n}}{\lambda-1} \leq \lambda^{-(b+1)}y_{b+1} - \lambda^{-n}y_n - \sum_{j=n}^b \lambda^{-(j+1)}z_j \leq \epsilon_1 \frac{\lambda^{-n}}{\lambda-1},$$

that is,

$$\frac{-\epsilon_1}{\lambda-1} \leq \lambda^{-(b-n+1)}y_{b+1} - y_n - \lambda^n \sum_{j=n}^b \lambda^{-(j+1)}z_j \leq \frac{\epsilon_1}{\lambda-1}.$$

If we denote

$$u_n = \lambda^{-(b-n+1)}y_{b+1} - \sum_{j=n}^b \lambda^{-(j-n+1)}z_j,$$

then $|u_n - y_n| \leq \frac{\epsilon_1}{\lambda-1} = \frac{\epsilon}{(\lambda-1)(\mu-1)}$. It is easy to verify that $u_{n+1} = \lambda u_n + z_n$ and hence

$$u_{n+2} - \lambda u_{n+1} = z_{n+1} = \mu z_n = \mu[u_{n+1} - \lambda u_n]$$

implies that

$$u_{n+2} + \alpha u_{n+1} + \beta u_n = 0.$$

Consequently, (1.3) has the Hyers-Ulam stability with the stability constant $K = \frac{1}{(\lambda-1)(\mu-1)}$.

Next, we consider Case (ii). Assume that there exist positive integers $M, L > 0$ such that $\mu M > 1$ and $\lambda L > 1$. Using the same type of argument as in Case (i), we get (2.2) and hence

$$-\epsilon \sum_{j=n}^b (\mu M)^{-(j+1)} M^{j+1} \leq \sum_{j=n}^b \Delta(\mu^{-j} g_j) \leq \epsilon \sum_{j=n}^b (\mu M)^{-(j+1)} M^{j+1}. \quad (2.3)$$

Let $M^* = \max\{M^{n+1}, M^{n+2}, \dots, M^{b+1}\}$, $a \leq n < b+1$. Then (2.3) becomes

$$-\epsilon M^* \sum_{j=n}^b (\mu M)^{-(j+1)} \leq \mu^{-(b+1)} g_{b+1} - \mu^{-n} g_n \leq \epsilon M^* \sum_{j=n}^b (\mu M)^{-(j+1)},$$

that is,

$$M^* \frac{-\epsilon(\mu M)^{-n}}{(\mu M - 1)} \leq \mu^{-(b+1)} g_{b+1} - \mu^{-n} g_n \leq M^* \frac{\epsilon(\mu M)^{-n}}{(\mu M - 1)}.$$

Consequently,

$$\frac{-\epsilon M^*}{(\mu M - 1) M^a} \leq \mu^{-(b-n+1)} g_{b+1} - g_n \leq \frac{\epsilon M^*}{(\mu M - 1) M^a}.$$

The rest of the proof follows from Case (i). We note that the Hyers-Ulam stability constant is given by

$$K = \frac{\epsilon M^* L^*}{(\mu M - 1)(\lambda L - 1)(ML)^a},$$

where $L^* = \max\{L^{n+1}, L^{n+2}, \dots, L^{b+1}\}$, $a \leq n < b+1$.

Cases (iii) and (iv) follow from Cases (i) and (ii). Hence the proof of the theorem is complete.

Theorem 2.2. Assume that the characteristic equation $m^2 + \alpha m + \beta = 0$ have two different positive roots. Furthermore, assume that (1.6) holds. Then (1.4) has the Hyers-Ulam stability.

Proof. Proceeding as in the proof of Theorem 2.1, we obtain

$$\begin{aligned} |g_{n+1} - \mu g_n - r_n| &= |y_{n+2} - \lambda y_{n+1} - \mu y_{n+1} + \lambda \mu y_n - r_n| \\ &= |y_{n+2} - (\lambda + \mu) y_{n+1} + \lambda \mu y_n - r_n| \end{aligned}$$

$$= |y_{n+2} + \alpha y_{n+1} + \beta y_n - r_n| \leq \epsilon.$$

Equivalently, g_n satisfies the relation

$$-\epsilon \leq g_{n+1} - \mu g_n - r_n \leq \epsilon.$$

Similar to Theorem 2.1, we have four possibilities upon the choices of λ and μ . We consider Case (i) only. Other cases can similarly be dealt with the cases in Theorem 2.1. Hence, similar to (2.2), we have

$$-\epsilon \mu^{-(n+1)} \leq \Delta(\mu^{-n} g_n) - \mu^{-(n+1)} r_n \leq \epsilon \mu^{-(n+1)},$$

and we let $z_n = \mu^{-(b-n+1)} g_{b+1} - \mu^n \sum_{j=n}^b \mu^{-(j+1)} r_j$. Therefore, z_n satisfies

$$z_{n+1} - \mu z_n - r_n = 0,$$

and $|g_n - z_n| \leq \epsilon_1$. Using the same type of argument as in Theorem 2.1, we can show that there exists

$$u_n = \lambda^{-(b-n+1)} y_{b+1} - \sum_{j=n}^b \lambda^{-(j-n+1)} z_j$$

such that $|u_n - y_n| \leq \frac{\epsilon}{(\lambda-1)(\mu-1)}$ and u_n satisfies

$$u_{n+2} + \alpha u_{n+1} + \beta u_n - r_n = 0.$$

This completes the proof of the theorem.

3. Stability Results for (1.5)

In this section, the Hyers-Ulam stability of (1.5) is presented. For this we need the following result:

Lemma 3.1. *Assume that $0 \leq p_n \leq p < \infty$ for every n . Then for $n \in (a, b+1)$, the equation*

$$y_{n+1} - p_n y_n - r_n = 0 \tag{3.1}$$

has the Hyers-Ulam stability.

Proof. Let $\epsilon > 0$ and $y_n, n \in (a, b+1)$ be a solution of (3.1) satisfying the property

$$|y_{n+1} - p_n y_n - r_n| \leq \epsilon,$$

that is,

$$-\epsilon \leq y_{n+1} - p_n y_n - r_n \leq \epsilon.$$

Therefore,

$$-\epsilon \left(\prod_{i=a}^n p_i \right)^{-1} \leq [y_{n+1} - p_n y_n - r_n] \left(\prod_{i=a}^n p_i \right)^{-1} \leq \epsilon \left(\prod_{i=a}^n p_i \right)^{-1}$$

implies that

$$-\epsilon \left(\prod_{i=a}^n p_i \right)^{-1} \leq \Delta \left[y_n \left(\prod_{i=a}^{n-1} p_i \right)^{-1} \right] - r_n \left(\prod_{i=a}^n p_i \right)^{-1} \leq \epsilon \left(\prod_{i=a}^n p_i \right)^{-1}.$$

Summing the above inequality from a to $n-1$, we obtain

$$\begin{aligned} -\epsilon \sum_{j=a}^{n-1} \left(\prod_{i=a}^j p_i \right)^{-1} &\leq \sum_{j=a}^{n-1} \Delta \left[y_j \left(\prod_{i=a}^{j-1} p_i \right)^{-1} \right] - \sum_{j=a}^{n-1} r_j \left(\prod_{i=a}^j p_i \right)^{-1} \\ &\leq \epsilon \sum_{j=a}^{n-1} \left(\prod_{i=a}^j p_i \right)^{-1} \end{aligned}$$

and hence

$$\begin{aligned} -\epsilon \sum_{j=a}^{n-1} \left(\prod_{i=a}^j p_i \right)^{-1} &\leq y_n \left(\prod_{i=a}^{n-1} p_i \right)^{-1} - y_a - \sum_{j=a}^{n-1} r_j \left(\prod_{i=a}^j p_i \right)^{-1} \\ &\leq \epsilon \sum_{j=a}^{n-1} \left(\prod_{i=a}^j p_i \right)^{-1}, \quad (3.2) \end{aligned}$$

where we have used the sign convention $\prod_{i=a}^{a-1} p_i = 1$. We consider two cases, viz. $p < 1$ and $p \geq 1$. Consider the former case. Using the fact

$$\sum_{j=a}^{n-1} \left(\prod_{i=a}^j p_i \right)^{-1} = \frac{1}{p_a} + \frac{1}{p_a p_{a+1}} + \dots + \frac{1}{p_a p_{a+1} \dots p_{n-1}}$$

$$\leq \frac{1}{1-p} \left(\prod_{i=a}^{n-1} p_i \right)^{-1}$$

in (3.2), we obtain that

$$\begin{aligned} -\frac{\epsilon}{1-p} \left(\prod_{i=a}^{n-1} p_i \right)^{-1} &\leq y_n \left(\prod_{i=a}^{n-1} p_i \right)^{-1} - y_a - \sum_{j=a}^{n-1} r_j \left(\prod_{i=a}^j p_i \right)^{-1} \\ &\leq \frac{\epsilon}{1-p} \left(\prod_{i=a}^{n-1} p_i \right)^{-1}, \end{aligned}$$

that is,

$$-\frac{\epsilon}{1-p} \leq y_n - y_a \left(\prod_{i=a}^{n-1} p_i \right)^{-1} - \left(\prod_{i=a}^{n-1} p_i \right)^{-1} \sum_{j=a}^{n-1} r_j \left(\prod_{i=a}^j p_i \right)^{-1} \leq \frac{\epsilon}{1-p}.$$

For $n \in (a, b+1)$, if we define

$$z_n = y_a \left(\prod_{i=a}^{n-1} p_i \right)^{-1} + \left(\prod_{i=a}^{n-1} p_i \right)^{-1} \sum_{j=a}^{n-1} r_j \left(\prod_{i=a}^j p_i \right)^{-1},$$

then it is easy to verify that $z_{n+1} - p_n z_n - r_0 = 0$ and hence $|y_n - z_n| \leq \frac{\epsilon}{1-p}$ implies that (3.1) has the Hyers-Ulam stability with stability constant $K = \frac{1}{1-p}$. For the latter case, we have

$$\begin{aligned} \sum_{j=a}^{n-1} \left(\prod_{i=a}^j p_i \right)^{-1} &= \frac{1}{p_a} + \frac{1}{p_a p_{a+1}} + \dots + \frac{1}{p_a p_{a+1} \dots p_{n-1}} \\ &\leq \frac{p^n - 1}{p - 1} \left(\prod_{i=a}^{n-1} p_i \right)^{-1} \\ &\leq \frac{p^b - 1}{p - 1} \left(\prod_{i=a}^{n-1} p_i \right)^{-1}. \end{aligned}$$

Using the same type of argument as in the former case, we obtain that $|y_n - z_n| \leq \frac{p^b - 1}{p - 1} \epsilon$. This completes the proof of the lemma.

Theorem 3.2. Assume that $\alpha_n, \beta_n > 0$ for every n . Furthermore, assume that c_n is a particular solution of

$$u_{n+1}u_n - \alpha_n u_n + \beta_n = 0 \quad (3.3)$$

such that $0 < c_n \leq c < \infty$ and $d_n = \frac{\beta_n}{c_n} \leq d < \infty$, for $n \in (a, b+1)$. Then (1.5) has the Hyers-Ulam stability.

Proof. Let $\epsilon > 0$ and $y_n, n \in (a, b+1)$ be a solution of (1.5) satisfying the property

$$|y_{n+2} - \alpha_n y_{n+1} + \beta_n y_n - r_n| \leq \epsilon.$$

For $n \in (a, b+1)$, define $v_n = y_{n+1} - c_n y_n$. Then

$$\begin{aligned} |v_{n+1} - d_n v_n - r_n| &= |y_{n+2} - (d_n + c_{n+1})y_{n+1} + c_n d_n y_n - r_n| \\ &= |y_{n+2} - \alpha_n y_{n+1} + \beta_n y_n - r_n| \leq \epsilon, \end{aligned}$$

where we have used the fact that c_n is a particular solution of (3.3). Upon the choice of c and d , we undertake four possibilities viz.

i) $c < 1, d < 1$; ii) $c < 1, d \geq 1$; iii) $c \geq 1, d < 1$; iv) $c \geq 1, d \geq 1$.

We consider Case (i) and the other cases can similarly be dealt with. It follows from Lemma 3.1 that

$$v_{n+1} - d_n v_n - r_n = 0$$

has the Hyers-Ulam stability with the property $|v_n - w_n| \leq \frac{\epsilon}{1-d}$, where

$$w_n = v_a \left(\prod_{i=a}^{n-1} d_i \right) + \left(\prod_{i=a}^{n-1} d_i \right) \sum_{j=a}^{n-1} r_j \left(\prod_{i=a}^j d_i \right)^{-1}.$$

Using v_n in $|v_n - w_n| \leq \frac{\epsilon}{1-d}$, we obtain that

$$|y_{n+1} - c_n y_n - w_n| \leq \frac{\epsilon}{1-d}$$

for $n \in (a, b+1)$. Again by Lemma 3.1,

$$y_{n+1} - c_n y_n - w_n = 0$$

has the Hyers-Ulam stability with the property $|y_n - z_n| \leq \frac{\epsilon}{(1-d)(1-c)}$, where

$$z_n = y_a \left(\prod_{i=a}^{n-1} c_i \right) + \left(\prod_{i=a}^{n-1} c_i \right) \sum_{j=a}^{n-1} w_j \left(\prod_{i=a}^j c_i \right)^{-1}.$$

Consequently, (1.5) has the Hyers-Ulam stability with the stability constant $K = \frac{1}{(1-c)(1-d)}$. Hence the proof of the theorem is complete.

Example 3.3. Consider the difference equation

$$y_{n+2} - \frac{1}{n+2}y_{n+1} + \frac{1}{(n+1)(n+2)}y_n = \frac{n^2 - n + 1}{n(n+1)(n+2)} \quad (3.4)$$

on the interval $I = (1, \infty)$. Let $c_n = \frac{1}{n+1}$, $n \in I$ be a particular solution of

$$u_{n+1}u_n - \frac{2}{n+2}u_n + \frac{1}{(n+1)(n+2)} = 0.$$

By Theorem 3.2, $d_n = \frac{1}{n+2} \leq d < 1$, $\prod_{i=1}^{n-1} d_i = \frac{2}{(n+1)!}$ and $\prod_{i=1}^{n-1} c_i = \frac{1}{(n)!}$. It is easy to verify that $w_n \geq \frac{1}{(n+1)!}$ and $z_n \geq \frac{1}{(n)!}$ for $n \in I$. Indeed, $y_n = \frac{1}{n}$ is a solution of (3.4) and $|y_n - z_n| \leq \frac{\epsilon}{(1-c)(1-d)}$. Hence (3.4) has the Hyers-Ulam stability.

References

- [1] C. Alsina, R. Ger, On some inequalities and stability results related to the exponential function, *J. Inequ. Appl.*, **2** (1998), 373-380.
- [2] J. Brzdek, D. Popa, B. Xu, Remarks on stability of linear recurrence of hogher order, *Appl. Math. Lett.*, **23** (2010), 1459-1463.
- [3] D.H. Hyers, On the stability of the linear functional equations, *Proc. Natl. Acad. Sci.*, **27** (1941), 222-224.
- [4] K.W. Jun, Y.H. Lee, A generalization of the Hyers-Ulam-Rassias stability of the Jenson's equation, *J. Math. Anal. Appl.*, **238** (1999), 305-315.
- [5] S.M. Jung, Hyers-Ulam stability of linear differential equations of first order, *Appl. Math. Lett.*, **17** (2004), 1135-1140.
- [6] S.M. Jung, Hyers-Ulam stability of linear differential equations of first order (III), *J. Math. Anal. Appl.*, **311** (2005), 139-146.

- [7] S.M. Jung, Hyers-Ulam stability of linear differential equations of first order (II), *Appl. Math. Lett.*, **19** (2006), 854-858.
- [8] S.M. Jung, Th. M. Rassias, Ulam's problem for approximate homomorphisms in connection with Bernoulli's differential equation, *Appl. Math. Comp.*, **187** (2007), 223-227.
- [9] S.M. Jung, Hyers-Ulam stability of the first order matrix difference equations, *Adv. Differ. Equ.* (2015), 1-13.
- [10] Y. Li, Y. Sen, Hyers-Ulam stability of linear differential equations of second order, *Appl. Math. Lett.*, **23** (2010), 306-309.
- [11] T. Miura, S.E. Takahasi, H. Choda, On the Hyers-Ulam stability of real continuous function valued differentiable map, *Tokyo J. Math.* (2001), 467-476.
- [12] T. Miura, On the Hyers-Ulam stability of a differentiable map, *Sci. Math. Japan*, **55** (2002), 17-24.
- [13] T. Miura, S. Miyajima, S.E. Takahasi, A characterization of Hyers-Ulam stability of first order linear differential operators, *J. Math. Anal. Appl.*, **286** (2003), 136-146.
- [14] T. Miura, S.M. Jung, S.E. Takahasi, Hyers-Ulam stability of the Banach space valued linear differential equations $y' = \lambda y$, *Korean Math. Soc.*, **41** (2004), 995-1005.
- [15] T. Miura, H. Oka, S.E. Takahasi, N. Niwa, Hyers-Ulam stability of the first order linear differential equations for Banach space valued holomorphic mappings, *J. Math. Ineq.*, **1** (2007), 377-385.
- [16] M. Obloza, Hyers stability of the linear differential equation, *Rocznik Nauk-Dydakt, Proce. Mat.*, **13** (1993), 259-270.
- [17] M. Obloza, Connections between Hyers and Lyapunov stability of the ordinary differential equations, *Rocznik Nauk-Dydakt, Proce. Mat.*, **14** (1997), 141-146.
- [18] D. Popa, Hyers-Ulam-Rassias stability of a linear recurrence, *J. Math. Anal. Appl.*, **309** (2005), 591-597.
- [19] Th.M. Rassias, On the stability of linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

- [20] Th. M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.*, **62** (2000), 23-30.
- [21] S.E. Takahasi, H. Takagi, T. Miura, S. Miyajima, The Hyers-Ulam stability constants of first order linear differential operators, *J. Math. Anal. Appl.*, **296** (2004), 403-409.
- [22] A.K. Tripathy, A. Satapathy, Hyers-Ulam stability of fourth order Euler's differential equations, *J. Comp. Sci. Appl. Math.*, **1** (2015), 49-58.
- [23] S.M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Wiley, New York, 1964.

