

GLOBAL STABILITY OF A DELAYED SIR EPIDEMIC MODEL WITH DIFFUSION

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Abstract: Some mathematical epidemic equation with diffusion, which appears as a model for the spread of disease-causing, is treated. The permanence and global asymptotic properties of the diffusive equation is studied by applying the strong maximum principle and luxury Liapunov functionals.

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In this paper, we shall consider the following diffusive system with boundary condition

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$$\begin{aligned}
\frac{\partial S}{\partial t}(t, x) &= d\Delta S(t, x) - \beta S(t, x) \int_0^\infty I(t-s, x) d\eta(s) \\
&\quad - \mu_1 S(t, x) + b - aN_1(t, x), & t > 0, x \in \Omega, \\
\frac{\partial I}{\partial t}(t, x) &= d\Delta I(t, x) + \beta S(t, x) \int_0^\infty I(t-s, x) d\eta(s) \\
&\quad - \mu_2 I(t, x) - \lambda I(t, x), & t > 0, x \in \Omega, \\
\frac{\partial R}{\partial t}(t, x) &= d\Delta R(t, x) + \lambda I(t, x) - \mu_3 R(t, x), & t > 0, x \in \Omega, \\
\frac{\partial S}{\partial n}(t, x) &= \frac{\partial I}{\partial n}(t, x) = \frac{\partial R}{\partial n}(t, x) = 0, & t > 0, x \in \partial\Omega,
\end{aligned} \tag{1}$$

where $S(t, x) + I(t, x) + R(t, x) \equiv N_1(t, x)$, which denotes the number of a population at time t and space x . Here Δ is the Laplacian in R^m , $\Omega \subset R^m$ is a bounded domain with smooth boundary $\partial\Omega$ and $\partial/\partial n$ is the outward normal derivative to $\partial\Omega$. For the equation (1), $1 \gg d > 0$ is the diffusion coefficient of small constant, $S := S(t, x)$ denotes the number of the population susceptible to the disease, $I := I(t, x)$ denotes the number of infectious individual and $R := R(t, x)$ denotes the member who have been removed from the possibility of infection through full immunity. It is assumed that all newborns are susceptible. The positive constants $\mu_1, \mu_2 = \mu_1 + p, \mu_3 = \mu_1 + q$, p and q are nonnegative constants, represent the death rates of susceptible, infective and recovered, respectively. It is biologically natural to assume that

$$\mu_1 \leq \min\{\mu_2, \mu_3\}.$$

In addition, the positive constants b, a and λ represent the birth and death rates of the population and the recovery rate of infective, respectively. The positive constant β is the average number of contacts per infective per day.

For the after theorems, if $I(t, x)$ have no influence from past time, we say the strong fading memory property for time t as follows.

(H_0) For any small $\epsilon^* > 0$ and any $r > 0$, there exists a $h = h(\epsilon^*, r) > 0$ such that

$$\int_h^\infty I(t-s, x) d\eta(s) \leq \epsilon^* \quad \text{for all } t \in R^+ = [0, \infty),$$

whenever $|x| \leq r$ for all $x \in \bar{\Omega}$, and

$$\int_0^h I(t-s, x) d\eta(s) \leq I(t, x) \int_0^h d\eta(s)$$

for $t > 0, x \in \bar{\Omega}$ and finite time delay $h \geq 0$, where the function $\eta(s) : [0, h] \rightarrow R = (-\infty, +\infty)$ is nondecreasing and has bounded variation such that

$$\int_0^h d\eta(s) = \eta(h) - \eta(0) = 1.$$

The term $\beta S(t, x) \int_0^\infty I(t-s, x) d\eta(s)$ can be considered as the force of infection at time t . For the detailed biological meanings, we refer to [1], [6] and [7].

In 1979, for the ordinary differential equation (without time delay), Anderson and May [1] have studied the asymptotic stability of the following epidemic differential equation

$$\begin{aligned} \frac{dS(t)}{dt} &= -\beta S(t)I(t) - \mu S(t) + \mu, \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - \mu I(t) - \lambda I(t), \\ \frac{dR(t)}{dt} &= \lambda I(t) - \mu R(t), \quad t \geq 0, \end{aligned} \quad (2)$$

where β, μ and λ are positive constants. In (2), it is assume that the total number of the population $N(t)$ is constant, that is $N(t) = 1$ for all $t \geq 0$, and that the birth and the death rates of population are the same value.

On the other hand, for the integro-differential equation (with time delay), Ma, Takeuchi et al. [7] have shown the permanence of the solution $(S(t), I(t), R(t))$ of

$$\begin{aligned} \frac{dS(t)}{dt} &= -\beta S(t) \int_0^h I(t-s) d\eta(s) - \mu_1 S(t) + b, \\ \frac{dI(t)}{dt} &= \beta S(t) \int_0^h I(t-s) d\eta(s) - \mu_2 I(t) - \lambda I(t), \\ \frac{dR(t)}{dt} &= \lambda I(t) - \mu_3 R(t), \quad t \geq 0, \end{aligned} \quad (3)$$

which describe the spread within a population of infectious disease.

Recently, Hamaya and Arai [4] have studied the permanence of the solution $(S(t, x), I(t, x), R(t, x))$ of equation (3) with diffusion.

In this article, we consider the permanence and global asymptotic properties of the solution of diffusive equation with infinite delay which based on [5].

We next observe that $R(t, x)$ can be immediately obtained once $I(t, x)$ are know, so the system (1) can be reduced to

$$\begin{aligned}
\frac{\partial S}{\partial t}(t, x) &= d\Delta S(t, x) - \beta S(t, x) \int_0^\infty I(t-s, x) d\eta(s) \\
&\quad - \mu_1 S(t, x) + b - aN(t, x), & t > 0, x \in \Omega, \\
\frac{\partial I}{\partial t}(t, x) &= d\Delta I(t, x) + \beta S(t, x) \int_0^\infty I(t-s, x) d\eta(s) - \mu_2 I(t, x) \\
&\quad - \lambda I(t, x), & t > 0, x \in \Omega, \\
\frac{\partial S}{\partial n}(t, x) &= \frac{\partial I}{\partial n}(t, x) = 0, & t > 0, x \in \partial\Omega.
\end{aligned} \tag{4}$$

Functions $S, I, R \in C([0, \infty) \times \bar{\Omega}, R)$ is called a (classical) solution of (1) if $\partial S/\partial t, \partial S/\partial x, \partial^2 S/\partial x^2, \partial I/\partial t, \partial I/\partial x, \partial^2 I/\partial x^2, \partial R/\partial t, \partial R/\partial x$ and $\partial^2 R/\partial x^2$ belong to the space $C((0, \infty) \times \Omega)$, $\partial S/\partial n, \partial I/\partial n$ and $\partial R/\partial n$ exist on $(0, \infty) \times \partial\Omega$ and (1) is identically satisfied. From [9, Chapter 6] and [11], we can show that the existence of solution is guaranteed for (1) whenever the initial function

$$S(\theta, x) = \phi_1(\theta, x) \geq 0, x \in \bar{\Omega}, \quad I(\theta, x) = \phi_2(\theta, x) \geq 0, x \in \bar{\Omega}$$

and $R(\theta, x) = \phi_3(\theta, x) \geq 0, x \in \bar{\Omega}$ belong to $(\theta, x) \in (-\infty, 0] \times C^1(\bar{\Omega})$.

For any parameters h, β, a, b, λ and $\mu_i (i = 1, 2, 3)$, it is easy to check that the equilibrium solution $(S(t, x), I(t, x), R(t, x))$ of (1) with the initial condition (5) exists and is a unique for all $t \geq t_0$.

(i) If $b > 0$, then equation (1) always has a disease free equilibrium $E_{S_0^*} = (S_0^*, 0, 0)$, where $S_0^* = b/(\mu_1 + a)$.

(ii) Furthermore, if

$$S_0^* > S^* \equiv \frac{\mu_2 + \lambda}{\beta},$$

then, equation (1) also has a unique positive endemic equilibrium $E^+ = (S^*, I^*, R^*)$, where

$$\begin{aligned}
S^* &= \frac{\mu_2 + \lambda}{\beta}, \\
I^* &= \frac{\mu_3(b - (\mu_1 + a)S^*)}{\mu_3(\mu_2 + \lambda) + a(\mu_3 + \lambda)}, \\
R^* &= \frac{\lambda}{\mu_3} I^* = \frac{\lambda(b - (\mu_1 + a)S^*)}{\mu_3((\mu_2 + \lambda) + a(\mu_3 + \lambda))}.
\end{aligned}$$

Remark 1. It is clear for equation (4) that:

(i') If $b > 0$, then equation (4) always has a disease free equilibrium $E_{S_0^*} = (S_0^*, 0)$, where $S_0^* = b/(\mu_1 + a)$.

(ii') Furthermore, if

$$(H_1) \quad S_0^* > S^* \equiv \frac{\mu_2 + \lambda}{\beta},$$

then equation (4) also has a unique positive endemic equilibrium $E^+ = (S^*, I^*)$, where

$$S^* = \frac{\mu_2 + \lambda}{\beta}, \quad I^* = \frac{b - (\mu_1 + a)S^*}{(\mu_2 + \lambda) + a}.$$

In particular, for parameters a and b , we can only set from view of mathematical conditions as following:

If $b = 0$, then equation (1) always has a trivial equilibrium $E_0 = (0, 0, 0)$.

If $b = 0$ and $0 > a = -\mu_1$, then for any $S > 0$, $E_S = (S, 0, 0)$ is the boundary equilibrium (the disease free equilibrium) of (1).

If $b = 0$ and $0 > a = \mu_1 = \mu_3(\mu_2 + \lambda)/(\mu_3 + \lambda)$, then for any $I > 0$ and $R > 0$ such that $\lambda I = \mu_3 R$, $E_{IR} = (S_0^*, I, R)$ is the positive equilibrium (the endemic equilibrium) of equation (1), where $S_0^* = (\mu_2 + \lambda)/\beta$ (cf. pp.154 in [6]).

If $b > 0$ and $a = 0$, then $E_{S_0} = (S_0, 0, 0)$, where $S_0 = b/\mu_1$ (cf. [4] and [6]).

In this paper, we do not need to treat these conditions since our assumptions have $a > 0$ and $b > 0$.

We discuss the large time behavior of the solution of equation (1) (cf.[4]).

Definition 1. The equation (1) is said to be permanence if there are positive constants ν_i and $M_i (i = 1, 2, 3)$ such that

$$\begin{aligned} \nu_1 &\leq \liminf_{t \rightarrow +\infty} [\inf_{x \in \bar{\Omega}} S(t, x)] \leq \limsup_{t \rightarrow +\infty} [\sup_{x \in \bar{\Omega}} S(t, x)] \leq M_1, \\ \nu_2 &\leq \liminf_{t \rightarrow +\infty} [\inf_{x \in \bar{\Omega}} I(t, x)] \leq \limsup_{t \rightarrow +\infty} [\sup_{x \in \bar{\Omega}} I(t, x)] \leq M_2, \\ \nu_3 &\leq \liminf_{t \rightarrow +\infty} [\inf_{x \in \bar{\Omega}} R(t, x)] \leq \limsup_{t \rightarrow +\infty} [\sup_{x \in \bar{\Omega}} R(t, x)] \leq M_3 \end{aligned}$$

hold for any solution of (1) with the initial condition (5). Here ν_i and M_i ($i = 1, 2, 3$) are independent of (5).

Before main theorem, we mention the following theorem (Strong Maximum Principle in [10]), and then the main results of our paper are stated as follows.

Theorem A. *Let $w \in C^{1,2}(D_T)$ and that*

$$\begin{aligned} w_t - d \nabla^2 w + cw &\geq 0, \quad \text{in } D_T = (0, T] \times \Omega, \\ Bw &= 0, \quad \text{on } S_T = (0, T] \times \partial\Omega, \\ w(0, x) &\geq 0, \quad \text{in } \bar{\Omega}, \end{aligned}$$

where B is Neumann type boundary condition and $c \equiv c(t, x)$ is a bounded function in D_T . If w attains a maximum value M at some point in D_T , then $w \equiv M$ throughout D_T .

To show the following Theorem 1, 2 and 3, we need the assumption (H_0) of the strong fading memory property for time t .

Theorem 1. *Under the above assumptions of parameters and (H_0) , if*

$$S_0^* \equiv \frac{b}{\mu_1 + a} > S^* \equiv \frac{\mu_2 + \lambda}{\beta}, \quad (5)$$

then, for each nonnegative continuous initial function, equation (1) is permanence.

In the rest of this paper, we will report results only for system (4). Before proofs of Theorem 1, we prepare lemmas.

Lemma 1. *The solution $(S(t, x), I(t, x))$ of equation (4) with (5) except for $R(\theta, x)$ satisfies for $t \geq 0$, the following inequality*

$$0 < N(t, x) \leq \max\left\{\sup_{x \in \bar{\Omega}} N(0, x), \frac{b}{\mu_1 + a}\right\} := K, \quad t > 0, x \in \bar{\Omega}, \quad (6)$$

where $N(t, x) = S(t, x) + I(t, x)$ and $N(0, x) = S_0(x) + I_0(x)$.

Proof. For the first inequality of (7), it is sufficient to prove that if for any small $\epsilon > 0$, $N(t_0, x) > \epsilon$ for some $t_0 > 0$ and $x \in \bar{\Omega}$, then $N(t, x) > \epsilon/2$ for $t > t_0, x \in \bar{\Omega}$. If it is not true, then

$$N(t, x) < \frac{\epsilon}{2} \quad \text{for } t > t_1, x \in \bar{\Omega} \quad \text{and} \quad N(t_1, x_1) = \frac{\epsilon}{2} \quad \text{for some } t_1 > t_0, x_1 \in \bar{\Omega}$$

with t_1 being the smallest among all such points (t_1, x_1) . If we set $w_0(t, x) = N(t, x) - \epsilon/2$, then $w_0(t, x) < 0$ ($t > t_1, x \in \bar{\Omega}$), $w_0(t_1, x_1) = 0$ and $\sup_{x \in \bar{\Omega}} w_0(t_0, x) > 0$, hence the function $w_0(t, x)$ takes a nonnegative minimum on $[t_0, t_1] \times \bar{\Omega}$. On the other hand, we have

$$\begin{aligned} \partial w_0 / \partial t &= \partial N / \partial t \\ &= d\Delta N - \mu_1 S(t, x) - \mu_2 I(t, x) - aN + b - \lambda I(t, x) \\ &= d\Delta w_0 - (\mu_1 + a)(w_0 + \frac{\epsilon}{2}) - pI + (b - \lambda I) \end{aligned}$$

and consequently

$$\begin{aligned} d\Delta w_0 - \partial w_0 / \partial t - (\mu_1 + a)w_0 &= \lambda I + pI + (\mu_1 + a)\frac{\epsilon}{2} - b \\ &\leq (\lambda + p)N + (\mu_1 + a)\frac{\epsilon}{2} - b \\ &\leq (\lambda + \mu_1 + a + p)\frac{\epsilon}{2} - b < 0 \end{aligned}$$

on $(t_1, \infty) \times \Omega$. Then there arises a contradiction by the strong maximum principle (cf. [3],[8],[10]). Indeed, if $x_1 \in \Omega$, then $d\Delta w_0 - \partial w_0 / \partial t - (\mu_1 + a)w_0$ must be nonnegative at (t_1, x_1) . This is a contradiction. We thus obtain that $x_1 \in \partial\Omega$ and $w_0(t, x) > w_0(t_1, x_1)$ for all $(t, x) \in [t_0, t_1] \times \Omega$, and hence $\partial w_0 / \partial n \leq 0$ at (t_1, x_1) . This is a contradiction, again (cf.[10]). It is clear that, by the initial point $N(0, x) \geq 0$ and the reduction of the above, $N(t, x) > 0$ for $(t, x) \in (0, t_0) \times \bar{\Omega}$. Therefore we must have the first inequality.

Let $K < K_0$ for some $K_0 > 0$. We claim that $N(t, x) \leq K_0, [0, \infty) \times \bar{\Omega}$. If it is true, by letting $K_0 \rightarrow K$, we hold the second inequality of this lemma. If now this is not true, then there exists $(t_2, x_2) \in (0, \infty) \times \bar{\Omega}$ such that $N(t_2, x_2) > K_0$. If we set $w(t, x) = N(t, x) - K_0$, then $w(t_2, x_2) > 0$. and $\sup_{x \in \bar{\Omega}} w(0, x) \leq 0, (t_2 > 0)$; hence the function $w(t, x)$ takes a positive maximum on $[0, t_2] \times \bar{\Omega}$. On the other hand, we have

$$\begin{aligned} \partial w / \partial t &= \partial N / \partial t \\ &\leq d\Delta N - \mu_1 N - aN + (b - \lambda I) \\ &= d\Delta w - (\mu_1 + a)(w + K_0) + (b - \lambda I) \end{aligned}$$

and consequently

$$d\Delta w - \partial w / \partial t - (\mu_1 + a)w \geq \lambda I + ((\mu_1 + a)K_0 - b) \geq 0.$$

Then there arises a contradiction by the strong maximum principle (cf.[3],[8],[10]). Indeed, if $x_2 \in \Omega$, then $d\Delta w - \partial w / \partial t - (\mu_1 + a)w$ must be negative at (t_2, x_2) .

This is a contradiction. We thus obtain that $x_2 \in \partial\Omega$ and $w(t, x) < w(t_2, x_2)$ for all $(t, x) \in [0, t_2] \times \Omega$, and hence $\partial w / \partial n > 0$ at (t_2, x_2) . This is a contradiction, again (cf.[10]). Therefore we must have (7).

Lemma 2. *Under the assumptions (H_0) and (H_1) , the solution $(S(t, x), I(t, x))$ of equation (4) with (5) except for $R(\theta, x)$ satisfies the following inequality*

$$\liminf_{t \rightarrow \infty} [\inf_{x \in \bar{\Omega}} S(t, x)] \geq \frac{\mu_1 b}{(\mu_1 + a)^2 + \beta b} \equiv \nu_1 > 0. \quad (7)$$

Proof. For some $t_3 > t_0$, we can show that

$$\hat{S}(t) \leq S(t, x), \quad t > t_3, \quad x \in \bar{\Omega}, \quad (8)$$

where $\hat{S}(t)$ is the solution of ordinary differential equation

$$\begin{aligned} \frac{d}{dt} \hat{S}(t) &= -(\mu_1 + a + \beta(\frac{b}{\mu_1 + a} + \epsilon^*)) \hat{S}(t) \\ &\quad - \frac{ab}{\mu_1 + a} + b - \epsilon, \quad t > t_3, \\ \hat{S}(t_3) &= \hat{S}_3 \quad \text{and} \quad \hat{S}_3 \geq \inf_{x \in \bar{\Omega}} S(t_2, x) \geq \frac{\mu_1 b}{(\mu_1 + a)^2 + \beta b}, \end{aligned} \quad (9)$$

where small $\epsilon^* > 0$ is one in the assumption (H_0) . To see this, we consider the function $w_1(t, x) := S(t, x) - \hat{S}(t)$ on $[t_3, \infty) \times \bar{\Omega}$. Then $w_1(0, x) = S(0, x) - \hat{S}(0) \leq 0$ for $x \in \bar{\Omega}$, and moreover, since $S + I(= N) \leq S + \frac{b}{\mu_1 + a}$ by Lemma 1,

$$\begin{aligned} \partial w_1 / \partial t &= \partial S / \partial t - d\hat{S}(t) / dt \\ &\geq d\Delta S - \beta S(\frac{b}{\mu_1 + a} + \epsilon^*) - \mu_1 S + b - aN \quad (\text{by } (H_0)) \\ &\quad - (-(\mu_1 + a)\hat{S} - \beta(\frac{b}{\mu_1 + a} + \epsilon^*)\hat{S} - \frac{ab}{\mu_1 + a} + b - \epsilon) \\ &\geq d\Delta w_1 - \beta(w_1 + \hat{S})(\frac{b}{\mu_1 + a} + \epsilon^*) - \mu_1(w_1 + \hat{S}) - a(w_1 + \hat{S}) \\ &\quad - \frac{ab}{\mu_1 + a} + b - (-(\mu_1 + a)\hat{S} - \beta(\frac{b}{\mu_1 + a} + \epsilon^*)\hat{S} - \frac{ab}{\mu_1 + a} + b - \epsilon) \\ &\geq d\Delta w_1 - (\beta(\frac{b}{\mu_1 + a} + \epsilon^*) + \mu_1 + a)w_1 + \epsilon. \end{aligned}$$

Hence,

$$d\Delta w_1 - \partial w_1 / \partial t - (\beta(\frac{b}{\mu_1 + a} + \epsilon^*) + \mu_1 + a)w_1 \leq -\epsilon \leq 0 \quad \text{on} \quad [t_3, \infty) \times \bar{\Omega}.$$

Therefore, by the same reasoning as the one for $w_0(t, x)$ of Lemma 1, one can see that $w_1(t, x) \leq 0$ on $[t_3, \infty) \times \bar{\Omega}$. Thus, we must have (9).

Moreover, by setting $M = \mu_1 + a + \beta(\frac{b}{\mu_1 + a} + \epsilon^*)$ in (10), we have

$$\frac{d}{dt}\hat{S} = -M\hat{S} + \frac{\mu_1 b}{\mu_1 + a} - \epsilon. \quad (10)$$

By solving equation (11), we obtain that

$$\begin{aligned} \hat{S} &= \frac{\mu_1 b / (\mu_1 + a) - \epsilon}{M} + \hat{C} e^{-Mt} \\ \text{and} \\ \hat{C} &= e^{Mt_3} (\hat{S}_3 - \frac{\mu_1 b / (\mu_1 + a) - \epsilon}{M}). \end{aligned}$$

Therefore, we have

$$\frac{\mu_1 b - (\mu_1 + a)\epsilon}{(\mu_1 + a)^2 + \beta(b + (\mu_1 + a)\epsilon^*)} \leq \hat{S}, \quad t \geq t_3$$

for small ϵ and ϵ^* . Thus, we obtain

$$\frac{\mu_1 b - (\mu_1 + a)\epsilon}{(\mu_1 + a)^2 + \beta(b + (\mu_1 + a)\epsilon^*)} \leq S$$

on $t \geq t_3, x \in \Omega$. By taking infimum, $t \rightarrow \infty$ and later letting ϵ and $\epsilon^* \rightarrow 0$ in the above inequality, we obtain

$$\frac{\mu_1 b}{(\mu_1 + a)^2 + \beta b} \leq \liminf_{t \rightarrow \infty} [\inf_{x \in \bar{\Omega}} S(t, x)].$$

This completes the proof of Lemma 2.

Lemma 3. *Under the assumptions (H_0) and (H_1) , the solution $(S(t, x), I(t, x))$ of equation (4) with (5) except for $R(\theta, x)$ satisfies the following inequality*

$$\limsup_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} I(t, x)] \leq M_2, \quad (11)$$

for some $M_2 > 0$.

Proof. For some $t_4 > t_0$, we can show that

$$I(t, x) \leq \hat{I}(t) \quad t > t_4, x \in \bar{\Omega}. \quad (12)$$

Here, $\hat{I}(t)$ is the solution of ordinary differential equation

$$\begin{aligned} \frac{d}{dt}\hat{I}(t) &= \beta\left(\frac{b}{\mu_1 + a} - \hat{I}(t)\right)\left(\frac{b}{\mu_1 + a} - \frac{\mu_1 b}{(\mu_1 + a)^2 + \beta b} + \epsilon^*\right) \\ &\quad - (\mu_2 + \lambda)\hat{I}(t) + \epsilon, \quad t > t_4, \\ \hat{I}(t_4) &= \hat{I}_4 \quad \text{and} \quad 0 < \hat{I}_4 \leq \sup_{x \in \bar{\Omega}} I(t_4, x) \leq \frac{B}{A}, \end{aligned} \quad (13)$$

where $A = \beta\frac{b}{\mu_1 + a} - \beta(\nu_1 - \epsilon^*) + (\mu_2 + \lambda) > 0$ and $B = \beta\frac{b}{\mu_1 + a}\left(\frac{b}{\mu_1 + a} - \nu_1\right) + \epsilon > 0$ by $\frac{b}{\mu_1 + a} - \nu_1 > 0$, and any $\epsilon^* > 0$ is near zero. Here $\nu_1 = \frac{\mu_1 b}{(\mu_1 + a)^2 + \beta b}$. To see this, we consider the function $w_2(t, x) := I(t, x) - \hat{I}(t)$ on $[t_4, \infty) \times \bar{\Omega}$. Then $w_2(0, x) = I(0, x) - \hat{I}(0) \leq 0$ for $x \in \bar{\Omega}$, and moreover, since $\beta S \int_0^\infty I(t - s, x) d\eta(s) \leq \beta(N - I)(I(t, x) + \epsilon^*) = \beta(N - I)((N - S + \epsilon^*)$,

$$\begin{aligned} \partial w_2 / \partial t &= \partial I / \partial t - d\hat{I}(t) / dt \\ &= d\Delta I + \beta S \int_0^\infty I(t - s, x) d\eta(s) - \mu_2 I - \lambda I \\ &\quad - \left(\beta\left(\frac{b}{\mu_1 + a} - \hat{I}(t)\right)\left(\frac{b}{\mu_1 + a} - \frac{\mu_1 b}{(\mu_1 + a)^2 + \beta b} + \epsilon^*\right) - (\mu_2 + \lambda)\hat{I}(t) + \epsilon\right) \\ &\leq d\Delta I + \beta\left(\frac{b}{\mu_1 + a} - I\right)\left(\frac{b}{\mu_1 + a} - \frac{\mu_1 b}{(\mu_1 + a)^2 + \beta b} + \epsilon^*\right) \\ &\quad - \mu_2 I - \lambda I \quad (\text{by } (H_0)) \\ &\quad - \left(\beta\left(\frac{b}{\mu_1 + a} - \hat{I}(t)\right)\left(\frac{b}{\mu_1 + a} - \frac{\mu_1 b}{(\mu_1 + a)^2 + \beta b} + \epsilon^*\right) - (\mu_2 + \lambda)\hat{I}(t) + \epsilon\right). \end{aligned}$$

Here, for the simplicity we write to use ν_1 and since $\frac{b}{\mu_1 + a} - \nu_1 > 0$,

$$\begin{aligned} \partial w_2 / \partial t &\leq d\Delta I + \beta\left(\frac{b}{\mu_1 + a} - I\right)\left(\frac{b}{\mu_1 + a} - \nu_1 + \epsilon^*\right) - \mu_2 I - \lambda I \\ &\quad - \left(\beta\left(\frac{b}{\mu_1 + a} - \hat{I}\right)\left(\frac{b}{\mu_1 + a} - \nu_1 + \epsilon^*\right) - (\mu_2 + \lambda)\hat{I} + \epsilon\right) \end{aligned}$$

$$\begin{aligned}
&= d\Delta I + \beta \frac{b^2}{(\mu_1 + a)^2} - \beta \frac{b}{\mu_1 + a} \nu_1 - \beta \frac{b}{\mu_1 + a} I + \beta \nu_1 I \\
&+ \beta \epsilon^* \left(\frac{b}{\mu_1 + a} - I \right) \\
&- (\mu_2 + \lambda) I - \beta \frac{b^2}{(\mu_1 + a)^2} + \beta \frac{b}{\mu_1 + a} \nu_1 + \beta \frac{b}{\mu_1 + a} \hat{I} \\
&- \beta \nu_1 \hat{I} - \beta \epsilon^* \frac{b}{\mu_1 + a} + \beta \epsilon^* \hat{I} + (\mu_2 + \lambda) \hat{I} - \epsilon \\
&= d\Delta I - \beta \frac{b}{\mu_1 + a} I + \beta \nu_1 I + \beta \epsilon^* \left(\frac{b}{\mu_1 + a} - I \right) - (\mu_2 + \lambda) I \\
&+ \beta \frac{b}{\mu_1 + a} \hat{I} - \beta \nu_1 \hat{I} - \beta \epsilon^* \frac{b}{\mu_1 + a} + \beta \epsilon^* \hat{I} + (\mu_2 + \lambda) \hat{I} - \epsilon.
\end{aligned}$$

Since $w_2(t, x) = I(t, x) - \hat{I}(t)$,

$$\begin{aligned}
\partial w_2 / \partial t &\leq d\Delta w_2 - \beta \frac{b}{\mu_1 + a} (w_2 + \hat{I}) + \beta \nu_1 (w_2 + \hat{I}) \\
&+ \beta \epsilon^* \left(\frac{b}{\mu_1 + a} - (w_2 + \hat{I}) \right) - (\mu_2 + \lambda) (w_2 + \hat{I}) \\
&+ \beta \frac{b}{\mu_1 + a} \hat{I} - \beta \nu_1 \hat{I} - \beta \epsilon^* \frac{b}{\mu_1 + a} + \beta \epsilon^* \hat{I} + (\mu_2 + \lambda) \hat{I} - \epsilon \\
&= d\Delta w_2 - \beta \frac{b}{\mu_1 + a} w_2 + \beta \nu_1 w_2 - \beta \epsilon^* w_2 - (\mu_2 + \lambda) w_2 - \epsilon.
\end{aligned}$$

Hence,

$$d\Delta w_2 - \partial w_2 / \partial t - \left(\beta \frac{b}{\mu_1 + a} - \beta(\nu_1 - \epsilon^*) + (\mu_2 + \lambda) \right) w_2 \geq \epsilon > 0.$$

Therefore, by the same reasoning as the one for $w_0(t, x)$, of Lemma 1, one can see that $w_2(t, x) \leq 0$ on $[t_4, \infty) \times \bar{\Omega}$. Thus, we must have (13).

Moreover, from (14),

$$\frac{d}{dt} \hat{I}(t) = -\left(\beta \frac{b}{\mu_1 + a} - \beta(\nu_1 - \epsilon^*) + (\mu_2 + \lambda) \right) \hat{I} + \beta \frac{b^2}{(\mu_1 + a)^2} - \beta \frac{b}{\mu_1 + a} \nu_1 + \epsilon.$$

Here, for the simplicity, we use A and $B > 0$ in the above equation, then we obtain

$$\frac{d}{dt} \hat{I} = -A\hat{I} + B. \quad (14)$$

By solving equation (15), we have

$$\begin{aligned}\hat{I} &= \frac{B}{A} + \hat{C}^* e^{-At} \\ \text{and} \\ \hat{C}^* &= e^{At_4}(\hat{I}_4 - \frac{B}{A}).\end{aligned}\tag{15}$$

Therefore, we obtain

$$\hat{I}(t) \leq \hat{I}(t_4)\tag{16}$$

for $t \geq t_4$. By (13),(17), we have

$$I(t, x) \leq \hat{I}(t_4)$$

on $[t_4, \infty) \times \bar{\Omega}$. By taking supremum, $t \rightarrow \infty$ and later letting ϵ and $\epsilon^* \rightarrow 0$ in the above inequality, we obtain (12)

$$\limsup_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} I(t, x)] \leq M_2,$$

where the positive number $M_2 = \frac{B'}{A'}$, $A' = \beta \frac{b}{\mu_1 + a} - \beta \nu_1 + (\mu_2 + \lambda)$ and $B' = \beta \frac{b}{\mu_1 + a} (\frac{b}{\mu_1 + a} - \nu_1)$. This completes the proof of Lemma 3.

Proof of Theorem 1. From Lemma 1, we have

$$N(t, x) \leq \frac{b}{\mu_1 + a},$$

where $N(t, x) = S(t, x) + I(t, x)$. Since, by Lemma 3,

$$\limsup_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} I(t, x)] \leq M_2$$

for $M_2 > 0$, we hold that the solution $(S(t, x), I(t, x))$ of Eq.(4) with initial condition (5) except for $R(\theta, x)$ satisfies

$$\limsup_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} S(t, x)] \leq M_1$$

for some $M_1 > 0$.

We can show that the solution $(S(t, x), I(t, x))$ of Eq.(4) with initial condition (5) except for $R(\theta, x)$ satisfies

$$\liminf_{t \rightarrow \infty} [\inf_{x \in \bar{\Omega}} I(t, x)] \geq \nu_2\tag{17}$$

for some $\nu_2 > 0$ which does not depend on the initial function in (5). To see this, it is sufficient to prove

$$I(t, x) \rightarrow \frac{\beta}{\sigma} \left(\frac{b}{\mu_1 + a} \right)^2 \quad \text{as } t \rightarrow \infty, \quad x \in \bar{\Omega}, \quad (18)$$

where $\sigma = \mu_2 + \lambda$. We define the function

$$f(t, x) = \frac{I(t, x)}{\sigma} - \frac{\beta}{\sigma^2} \left(\frac{b}{\mu_1 + a} \right) \left(\frac{b}{\mu_1 + a} + \epsilon^* \right).$$

Then

$$\begin{aligned} \frac{\partial f}{\partial t} &\leq \frac{1}{\sigma} (d\Delta I + \beta \left(\frac{b}{\mu_1 + a} \right) \left(\frac{b}{\mu_1 + a} + \epsilon^* \right) - \sigma I) && \text{(by } (H_0)) \\ &= \frac{d}{\sigma} \Delta I - I + \frac{\beta}{\sigma} \left(\frac{b}{\mu_1 + a} \right) \left(\frac{b}{\mu_1 + a} + \epsilon^* \right). \end{aligned}$$

We thus have the following differential inequality of

$$\frac{\partial f}{\partial t} \leq d\Delta f - \sigma f. \quad (19)$$

Then, by letting $\epsilon^* \rightarrow 0$, we can see that

$$f(t, x) \rightarrow 0 \quad \text{as } t \rightarrow \infty, x \in \bar{\Omega}.$$

If we set $W(t) := W(f)(t) = \int_{\Omega} f^2(t, x) dx, t \geq 0$, then, $W(t) \geq 0$ and we have

$$\begin{aligned} \frac{dW(t)}{dt} &= 2 \int_{\Omega} f \frac{\partial f}{\partial t} dx \\ &\leq 2 \int_{\Omega} f (d\Delta f - \sigma f) dx \\ &\leq -2d \int_{\Omega} \left(\frac{\partial f}{\partial x} \right)^2 dx - 2\sigma \int_{\Omega} f^2 dx. \end{aligned} \quad (20)$$

H_1 and H_2 be defined by the following

$$H_1 = 2d, \quad H_2 = 2\sigma.$$

Then $H_1 > 0$ and $H_2 > 0$. It follows from (21) that for $t > 0$,

$$W(t) + H_1 \int_0^t \int_{\Omega} \left(\frac{\partial f(s, x)}{\partial x} \right)^2 dx ds + H_2 \int_0^t \int_{\Omega} f^2(s, x) dx ds \leq W(0). \quad (21)$$

Since $W(t) \geq 0$, we have from (22) that

$$\begin{aligned} \int_0^t \left[\int_{\Omega} \left(\frac{\partial f(s, x)}{\partial x} \right)^2 dx \right] ds &< \frac{W(0)}{H_1}, \\ \int_0^t \left[\int_{\Omega} f^2(s, x) dx \right] ds &< \frac{W(0)}{H_2}. \end{aligned} \quad (22)$$

Thus, we conclude from (21), (22) and (23) that $W(t) \in L^1[0, \infty)$ and $\frac{dW(t)}{dt} \in L^1[0, \infty)$. By Barbalate's lemma [2, Lemma 1.2.2.], we obtain $W(t) \rightarrow 0$ and thus, $f \rightarrow 0$ in L^2 as $t \rightarrow \infty$, that is

$$\|f(t, \cdot)\|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (23)$$

where $\|\cdot\|_{L^2}$ denotes the L^2 -norm of functions on Ω . We next prove that

$$\sup_{x \in \bar{\Omega}} |f(t, x)| \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (24)$$

To do this (cf.[8]), we take notice of the boundedness of $f(t, x)$ by (7) in Lemma 1. Thus, we see that the orbit for meaning of differential equation (except for inequality ; $<$) in (20), that is $\{f(t, \cdot) | t \geq 0\}$ is relatively compact. The assertion (25) follows from this fact. Indeed, if (25) is not true, then there exist sequences $\{t_n\}, t_n \rightarrow \infty$ as $n \rightarrow \infty$, and $\{x_n\} \subset \bar{\Omega}$ such that $|f(t_n, x_n)| \geq \epsilon > 0, n = 1, 2, \dots$ for some $\epsilon > 0$. We can assume that $x_n \rightarrow x_0$ and $f(t_n, x_n) \rightarrow \tilde{f}(x_0)$ uniformly on $\bar{\Omega}$ for some $x_0 \in \bar{\Omega}$ and $\tilde{f} \in C(\bar{\Omega})$ as $n \rightarrow \infty$, if necessary taking a subsequence of these. In particular, we get $|\tilde{f}(x_0)| \geq \epsilon$. This is a contradiction, because $\int_{\Omega} \tilde{f}^2(x) dx = \lim_{n \rightarrow \infty} \|f(t_n, \cdot)\|_{L^2}^2 = 0$ by (24). Thus, we must have (25). From the definition of f , we have (19). Thus, $I \rightarrow \frac{\beta}{\sigma} \left(\frac{b}{\mu_1 + a} \right)^2 > 0$, by letting any $\epsilon^* \rightarrow 0$. Thus, (18) holds. Moreover, from Eq.(1) and (18), we easily have that

$$0 < \nu_3 \leq \liminf_{t \rightarrow +\infty} [\inf_{x \in \bar{\Omega}} R(t, x)]$$

for some $\nu_3 > 0$. Thus, equation (1) is permanent by Lemmas 1, 2 and 3. This proves Theorem 1.

Theorem 2. *If $S_0^* < S^*$, the disease free equilibrium E_0 of (4) satisfies*

$$\lim_{t \rightarrow \infty} [\sup_{x \in \Omega} I(t, x)] = 0$$

and

$$\lim_{t \rightarrow \infty} [\sup_{x \in \Omega} |S(t, x) - \frac{b}{\mu_1 + a}|] = 0$$

whenever the assumption (H_0) holds.

Proof. If $b/(\mu_1 + a) \leq N(0, x) \leq K$, then we can show that

$$N(t, x) \leq \hat{N}(t) \quad t > 0, x \in \bar{\Omega}, \quad (25)$$

where $\hat{N}(t)$ is the solution of ordinary differential equation

$$\begin{aligned} \frac{d}{dt} \hat{N}(t) &= -(\mu_1 + a)\hat{N}(t) + b \quad t > 0 \\ \text{and} \\ \hat{N}(0) &= K. \end{aligned}$$

To see this, we consider the function $w_1(t, x) := N(t, x) - \hat{N}(t)$ on $[0, \infty) \times \bar{\Omega}$. Then $w_1(0, x) = N(0, x) - \hat{N}(0) \leq 0$ for $x \in \bar{\Omega}$, and moreover

$$\begin{aligned} \partial w_1 / \partial t &= \partial N / \partial t - d\hat{N}(t) / dt \\ &= d\Delta N - \mu_1 N - aN - pI + b - \lambda I + (\mu_1 + a)\hat{N} - b \\ &= d\Delta w_1 - (\mu_1 + a)(w_1 + \hat{N}) - (\lambda + p)I + (\mu_1 + a)\hat{N} \end{aligned}$$

and hence

$$d\Delta w_1 - \partial w_1 / \partial t - (\mu_1 + a)w_1 = (\lambda + p)I \geq 0.$$

Therefore, by the same reasoning as the one for $w(t, x)$, one can see that $w_1(t, x) \leq 0$; Thus, we must have (26). Since $\hat{N}(t) = \hat{C}e^{-(\mu_1 + a)t} + b/(\mu_1 + a)$, $\hat{C} = K - b/(\mu_1 + a)$, by letting $t \rightarrow \infty$ in the above inequality (26), we obtain

$$\limsup_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} N(t, x)] \leq \frac{b}{\mu_1 + a}.$$

Hence for the discussion of the asymptotic behavior of solutions as $t \rightarrow +\infty$ we can (without loss of generality) assume that

$$N(t, x) \leq b/(\mu_1 + a) \quad t > 0, x \in \bar{\Omega}. \quad (26)$$

We next define

$$f(t, x) = \frac{Q^* I(t, x)}{\sigma} - \frac{\beta \epsilon^*}{\sigma} \frac{b}{\mu_1 + a},$$

where $\sigma = \mu_2 + \lambda$ and $Q^* = \frac{(\mu_1+a)\sigma-\beta b}{\mu_1+a} > 0$. Then

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{Q^*}{\sigma} (d\Delta I + \beta S \int_0^\infty I(t-s, x) d\eta(s) - \sigma I) \\ &\leq \frac{dQ^*}{\sigma} \Delta I + Q^* \left\{ \left(\frac{\beta}{\sigma} \frac{b}{\mu_1+a} - 1 \right) I + \frac{\beta \epsilon^*}{\sigma} \frac{b}{\mu_1+a} \right\} \quad (\text{by } (H_0)) \\ &\leq \frac{dQ^*}{\sigma} \Delta I - \frac{Q^*}{\sigma} (Q^* I - \beta \epsilon^* \frac{b}{\mu_1+a}). \end{aligned}$$

We thus have the following differential inequality of

$$\frac{\partial f}{\partial t} \leq d\Delta f - Q^* f. \quad (27)$$

Then, we can see that

$$f(t, x) \rightarrow 0 \quad \text{as } t \rightarrow \infty, x \in \bar{\Omega},$$

by the same argument of the proof in Theorem 1. From the definition of f and letting any $\epsilon^* \rightarrow 0$ ($t \rightarrow \infty$), we have

$$I(t, x) \rightarrow 0 \quad \text{as } t \rightarrow \infty, x \in \bar{\Omega}. \quad (28)$$

Since S is bounded and I has the strong fading memory property (H_0) in theorem,

$$\beta S \int_0^\infty I(t-s, x) d\eta(s) \rightarrow 0 \quad \text{as } t \rightarrow \infty, x \in \bar{\Omega}.$$

We next claim that

$$S \rightarrow \frac{b}{\mu_1+a} \quad \text{as } t \rightarrow \infty, x \in \bar{\Omega}. \quad (29)$$

By (29), for any small $\epsilon > 0$, there exists a large time $t_2 > 0$ such that $I(t, x) \leq \epsilon$ for $t \geq t_2, x \in \bar{\Omega}$. Then, it is sufficient for (30) to prove

$$N(t, x) \geq \tilde{N}(t) \quad t \geq t_2, x \in \bar{\Omega}, \quad (30)$$

where $\tilde{N}(t)$ is the solution of ordinary differential equation

$$\begin{aligned} \frac{d}{dt} \tilde{N}(t) &= -(\mu_1 + a) \tilde{N}(t) + b - (\lambda + p) \epsilon \quad t > t_2, \\ \tilde{N}(t_2) &= \tilde{N}_2 \quad \text{and} \quad 0 < \tilde{N}_2 \leq \sup_{x \in \bar{\Omega}} N(t_2, x) \leq \frac{b}{\mu_1 + a}. \end{aligned}$$

Then, we have

$$\tilde{N}(t) \leq N(t, x) \leq \frac{b}{\mu_1 + a} \quad t \geq t_2, x \in \bar{\Omega}.$$

Since $\tilde{N}(t) = \tilde{C}e^{-(\mu_1+a)t} + b/(\mu_1 + a) - (\lambda + p)\epsilon/(\mu_1 + a)$, $\tilde{C} = e^{(\mu_1+a)t_2}(\tilde{N}_2 - b/(\mu_1 + a) + (\lambda + p)\epsilon/(\mu_1 + a))$, by letting $t \rightarrow \infty$ and later letting $\epsilon \rightarrow 0$ in the above inequality, we obtain

$$\begin{aligned} \frac{b}{\mu_1 + a} &\leq \liminf_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} N(t, x)] \leq \lim_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} N(t, x)] \\ &\leq \lim_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} S(t, x)] + \lim_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} I(t, x)] = \lim_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} S(t, x)] \\ &\leq \limsup_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} N(t, x)] \leq \frac{b}{\mu_1 + a}. \end{aligned}$$

To see (31), we consider the function $w_2(t, x) := \tilde{N}(t) - N(t, x)$ on $[t_2, \infty) \times \bar{\Omega}$. Then $w_2(t_2, x) = \tilde{N}_2 - N(t_2, x) \leq 0$ for $x \in \bar{\Omega}$, and moreover

$$\begin{aligned} \partial w_2 / \partial t &= d\tilde{N}(t)/dt - \partial N / \partial t \\ &= -(\mu_1 + a)\tilde{N} + b - (\lambda + p)\epsilon - d\Delta N + (\mu_1 + a)N - b + (\lambda + p)I \\ &= d\Delta w_2 + (\mu_1 + a)(\tilde{N} - w_2) - (\mu_1 + a)\tilde{N} + (\lambda + p)(I - \epsilon) \end{aligned}$$

and hence

$$d\Delta w_2 - \partial w_2 / \partial t - (\mu_1 + a)w_2 = (\lambda + p)(\epsilon - I) \geq 0.$$

Therefore, by the same reasoning as the one for $w(t, x)$, one can see that $w_2(t, x) \leq 0$, that is (31) holds. Thus, we have (30). This completes the proof of Theorem 2.

We have show also that the following theorem.

Theorem 3. *If $S_0^* > S^*$ in the assumption (H_1) and $I_0(x) \neq 0$, then, for each nonnegative continuous initial function, there is a unique positive equilibrium (S^*, I^*) of (4) satisfies*

$$\lim_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} |I(t, x) - I^*|] = 0$$

and

$$\lim_{t \rightarrow \infty} [\sup_{x \in \bar{\Omega}} |S(t, x) - S^*|] = 0$$

whenever the assumption (H_0) holds.

Proof of Theorem 3. In order to prove this theorem, we need the following Corollary in [6, pp.148-153]. As there are complete comments and references of this result for ordinary differential equations in [cf. 6, pp.159-160], we omit the proof of this for simplicity.

Corollary. *If $S_0^* > S^*$ and $I_0(x) \neq 0$, then there is a unique positive endemic equilibrium (S^*, I^*) .*

With $N = S + I$, the system (3) drives to

$$\frac{\partial N}{\partial t} = d\Delta N - (\mu_1 + a)N - (\lambda + p)I + b \quad (31)$$

$$\frac{\partial I}{\partial t} \leq d\Delta I + \beta(I + \epsilon^*)[(N - I) - \frac{\sigma}{\beta}] \quad (\text{by } (H_0)). \quad (32)$$

This system has the positive equilibrium (N^*, I^*) where $N^* = S^* + I^*$. We can rewrite (32) in the form

$$\frac{\partial N}{\partial t} = d\Delta N - (\mu_1 + a)(N - N^*) - (\lambda + p)(I - I^*),$$

because $-(\mu_1 + a)N^* - (\lambda + p)I^* + b = 0$. For also (33),

$$\begin{aligned} \frac{\partial I}{\partial t} &\leq d\Delta I + \beta(I + \epsilon^*)\{(N - I) + (-N^* + I^*)\} \\ &= d\Delta I + \beta(I + \epsilon^*)\{G(N) - (I - I^*)\}, \end{aligned}$$

where

$$G(N) = N - N^*.$$

Then, $G(N) > 0$ for $N > N^*$ and $G(N) < 0$ for $N < N^*$. We now define a function $V(t)$ by

$$\begin{aligned} V(t) := V(N, I)(t) &= \int_{\Omega} \left(\frac{\beta}{\lambda + p} \int_{N^*}^N G(s) ds + (I + \epsilon^*) - (I^* + \epsilon^*) \right. \\ &\quad \left. - (I^* + \epsilon^*) \log \frac{I + \epsilon^*}{I^* + \epsilon^*} \right) dx. \end{aligned} \quad (33)$$

Then $V(N^*, I^*)(t) = 0$ and $V(N, I)(t) > 0$ for other admissible (N, I) . Furthermore, we calculate dV/dt along the solution of (32) and (33).

$$\frac{dV(t)}{dt} = \int_{\Omega} \left\{ \frac{\beta}{\lambda + p} G(N) \frac{\partial N}{\partial t} + \frac{\partial I}{\partial t} - (I^* + \epsilon^*) \frac{\partial I / \partial t}{I + \epsilon^*} \right\} dx$$

$$\begin{aligned}
&\leq \int_{\Omega} \left\{ \frac{\beta}{\lambda + p} G(N) (d\Delta N - (\mu_1 + a)(N - N^*) - (\lambda + p)(I - I^*)) \right. \\
&+ d\Delta I + \beta(I + \epsilon^*) \{ (N - N^*) - (I - I^*) \} \\
&- (I^* + \epsilon^*) \left\{ d \frac{\Delta I}{I + \epsilon^*} + \beta(G(N) - (I - I^*)) \right\} dx \\
&= \frac{\beta d}{\lambda + p} \int_{\Omega} \Delta N G(N) dx - \frac{\beta(\mu_1 + a)}{\lambda + p} \int_{\Omega} G(N)(N - N^*) dx \\
&- \beta \int_{\Omega} G(N)(I - I^*) dx \\
&+ d \int_{\Omega} \Delta I \frac{I - I^*}{I + \epsilon^*} dx + \beta \int_{\Omega} G(N)(I - I^*) dx - \beta \int_{\Omega} (I - I^*)^2 dx \quad (34) \\
&< 0
\end{aligned}$$

whenever $(N, I) \neq (N^*, I^*)$. To drive this, we continue to estimate for (35) in more detail.

$$\begin{aligned}
\frac{\beta d}{\lambda + p} \int_{\Omega} \Delta N G(N) dx &= \frac{\beta d}{\lambda + p} \left\{ \left[\frac{\partial N}{\partial x} G(N) \right]_{\Omega} - \int_{\Omega} \frac{\partial N}{\partial x} \frac{\partial G(N)}{\partial x} dx \right\} \\
&= - \frac{\beta d}{\lambda + p} \int_{\Omega} \frac{\partial N}{\partial x} \frac{\partial G(N)}{\partial x} dx. \quad (35)
\end{aligned}$$

Here

$$\frac{\partial G(N)}{\partial x} = \frac{\partial}{\partial x} \{ (N - N^*) \} = \frac{\partial N}{\partial x}.$$

Thus, expression (36) is:

$$- \frac{\beta d}{\lambda + p} \int_{\Omega} \left(\frac{\partial N}{\partial x} \right)^2 dx < 0.$$

Moreover, we have

$$- \frac{\beta(\mu_1 + a)}{\lambda + p} \int_{\Omega} G(N)(N - N^*) dx = - \frac{\beta(\mu_1 + a)}{\lambda + p} \int_{\Omega} (N - N^*)^2 dx < 0.$$

Similarly, we can check

$$\begin{aligned}
d \int_{\Omega} \Delta I \frac{I - I^*}{I + \epsilon^*} dx &= d \int_{\Omega} \Delta I \left(1 - \frac{I^* + \epsilon^*}{I + \epsilon^*} \right) dx \\
&= d \left[\frac{\partial I}{\partial x} \left(1 - \frac{I^* + \epsilon^*}{I + \epsilon^*} \right) \right]_{\Omega} - d(I^* + \epsilon^*) \int_{\Omega} \frac{(\partial I / \partial x)^2}{(I + \epsilon^*)^2} dx < 0.
\end{aligned}$$

Thus, $V(t)$ of (34) is non increasing in t that is there exists a constant $c_1 \geq 0$ such that $V(t) \rightarrow c_1$ as $t \rightarrow \infty$.

Since $I(t, x) \leq b/(\mu_1 + a)$, $I(t, x)$ is uniformly bounded on $[0, \infty) \times \bar{\Omega}$. Thus, we see that for any $h > 0$, there exists $C(h) > 0$ such that $|I(t+h, \cdot) - I(t, \cdot)| \leq C(h)$ for $t \geq 0$. From (35), we have $\dot{V}(t) \leq -W(N, I)(t) \leq 0$ (included equilibrium point case), where $W(N, I)(t)$ is the function of right hand side in (35). Suppose that $\dot{V}(t) \neq 0$. For any sequence $\{t_k\}$, $t_k \rightarrow \infty$ as $k \rightarrow \infty$ and some positive number γ , there exists $\delta > 0$ such that

$$\dot{V}(t) < -\gamma \quad (37)$$

if $|I(t + t_k, \cdot) - I(t, \cdot)| \leq \delta$, $0 \leq t \leq \delta$ and k is sufficient large. For regions $[t_k, t_k + \delta]$, we can see that

$$V(t_k + \delta) \leq V(t_k) - \gamma\delta \quad (38)$$

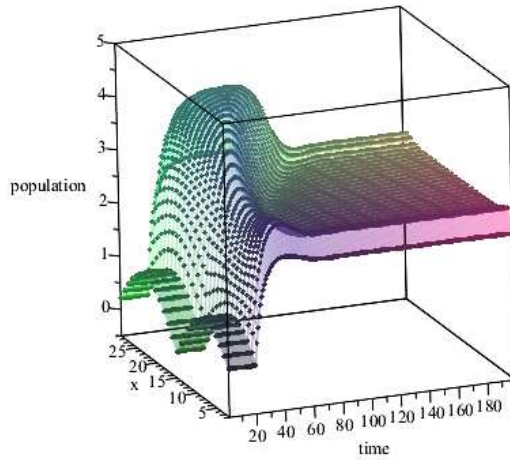
to integral on $[t_k, t_k + \delta]$ for the both sides of (37). Since (38) is true for all large number k and $\lim_{t \rightarrow \infty} V(t) = c_1 \geq 0$, it contradicts by $\gamma\delta$ is positive. This shows that $\dot{V}(t) = 0$. Then, we have $W(N, I)(t) = 0$. We thus obtain $N \rightarrow N^*$ and $I \rightarrow I^*$ by continuity of V and W . The asymptotic behavior of S now follows from the above result on the behavior of N and I . Thus, it is clear from $S = N - I$ that $S \rightarrow S^*$. This completes the proof.

Example. We consider the following equation of

$$\begin{aligned} \frac{\partial S}{\partial t}(t, x) &= 0.1\Delta S(t, x) - 0.1S(t, x) \int_0^h \left(\frac{e^{-s}}{1 - e^{-h}}\right) I(t-s, x) ds \\ &\quad - 0.1S(t, x) + 0.5 - 0.02N(t, x) \quad t > 0, x \in \Omega, \\ \frac{\partial I}{\partial t}(t, x) &= 0.1\Delta I(t, x) + 0.1S(t, x) \int_0^h \left(\frac{e^{-s}}{1 - e^{-h}}\right) I(t-s, x) ds - 0.4I(t, x) \\ &\quad t > 0, x \in \Omega, \end{aligned} \quad (39)$$

where, in equation (4), $d = 0.1$, $h = 20$, $\beta = 0.1$, $\mu_1 = 0.1$, $b = 0.5$, $a = 0.02$ and $\sigma = \mu_2 + \lambda = 0.4$. Thus, we have

$$\begin{aligned} S_0^* &= \frac{b}{\mu_1 + a} = \frac{0.5}{0.1 + 0.02} = 4.1\dot{6}, \\ S^* &= \frac{\mu_2 + \lambda}{\beta} = \frac{\sigma}{\beta} = \frac{0.4}{0.1} = 4.0, \quad \text{therefore } S_0^* > S^*. \\ E_{S_0^*} &= (S_0^*, 0) = (4.1\dot{6}, 0) \quad \text{and} \quad E^+ = (S^*, I^*) \approx (4.0, 0.048), \end{aligned}$$

Figure 1: $S(t, x)$

where

$$I^* = \frac{b - (\mu_1 + a)S^*}{(\mu_2 + \lambda) + a} = \frac{0.5 - 0.12 * 4.0}{0.4 + 0.02} \approx 0.048 > 0.$$

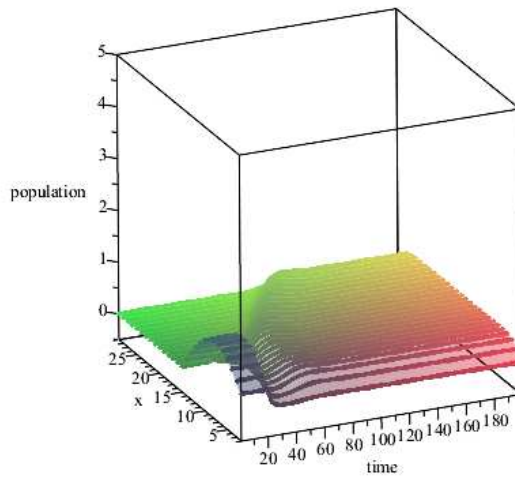
The initial functions are

$$S(\theta, x) = \phi_1(\theta, x) \equiv 1 > 0, \quad x \in \bar{\Omega} \quad \text{and}$$

$$I(\theta, x) = \phi_2(\theta, x) \equiv 1 > 0, \quad x \in \bar{\Omega}$$

$$\text{belong to the } (\theta, x) \in [-20, 0] \times C^1(\bar{\Omega}).$$

Figures illustrate our theorem and show that, for large time delay h , the endemic equilibrium E^+ of equation (4) is globally asymptotically stable if assumptions H_0 and H_1 hold. In figures, the line of x appears space x of $S(t, x)$ and $I(t, x)$ respectively, the axis line is meaning time t of $S(t, x)$ and $I(t, x)$ also respectively, and vertical line is individual of $S(t, x)$ and $I(t, x)$ respectively, in the graph of the trajectory of equation (39).

Figure 2: $I(t, x)$

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References

- [1] R.M. Anderson and R.M. May, Population biology of infectious diseases, Part 1, *Nature*, **280** (1979), 361-367.
- [2] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic Publishers, Dordrecht, 1992.
- [3] Y. Hamaya, On the asymptotic behavior of a diffusive epidemic model (AIDS), *Nonlinear Analysis*, **36** (1999), 685-696.
- [4] Y. Hamaya and T. Arai, Permanence of an SIR epidemic model with diffusion, *Nonlinear Studies*, **17** (2010), 69-79

- [5] Y. Hamaya and K. Saito, *Global asymptotic stability of an SIR epidemic model with diffusion*, to appear.
- [6] H. Inaba, *Mathematical Models for Demography and Epidemics*, University of Tokyo Press, 2002.
- [7] W. Ma, Y. Takeuchi, T. Hara and E. Beretta, Permanence of an SIR epidemic model with distributed time delays, *Tohoku Math. J.*, **54** (2002), 581-591.
- [8] S. Murakami and Y. Hamaya, Global attractivity in an integrodifferential equation with diffusion, *Differential Equations and Dynamical Systems*, **3** (1995), 35-42.
- [9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Math. Sci. Springer-Verlag, New York, 1983.
- [10] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag New York Inc., 1984.
- [11] R. Redlinger, On Volterra's population equation with diffusion, *SIAM J. Math. Anal.*, **16** (1985), 135-142.

