

PROPERTIES OF STABILITY AND HOPF BIFURCATION  
FOR AN HIV/AIDS EPIDEMIC MODEL WITH  
SCREENING AND TIME DELAY

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**Abstract:** In this paper, we consider an HIV/AIDS epidemic model with screening and time delay. We divide the population into four subclasses, one of them is the susceptible population  $S$  and the others are HIV infectives (HIV positives that do not know they are infected)  $I_1$ , HIV positives that know they are infected (by way of medical screening or other ways)  $I_2$  and that of AIDS patients  $A$ . Both the disease-free equilibrium and the infected endemic equilibrium are found and their stability is investigated using stability theory of delay differential equations. The effect of delay on the stability of the endemically infected equilibrium is investigated. It is shown that the introduction of time delay in the model has a destabilizing effect on the system and periodic solutions can arise by Hopf bifurcation when using the delay as a bifurcation parameter. Finally, numerical simulations are presented to illustrate the results.

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## 1. Introduction

Human immunodeficiency virus (HIV) is a lentivirus (a member of the retrovirus family) that causes acquired immunodeficiency syndrome (AIDS). HIV infection in humans is considered pandemic by the World Health Organization (WHO). From its discovery in 1981 to 2006, AIDS killed more than 25 million people. HIV infects about 0.6% of the world's population. In 2009, AIDS claimed an estimated 1.8 million lives. Approximately 260,000 children died of AIDS in 2009. In 2005, it was estimated that HIV would infect 90 million people in Africa, resulting in a minimum estimate of 18 million orphans. Intensified awareness and preventive measures, as well as the natural course of the epidemic, have played a role in reducing both the mortality and the morbidity of HIV infection. Nevertheless, an estimated 2.6 million people were newly infected in 2009. It was noticed that screening of blood products for HIV has largely eliminated transmission through blood transfusions or infected blood products.

The study of HIV transmission dynamics has been of great interest to both applied mathematicians and biologists. Mathematical models have become important tools in analyzing the spread and control of HIV to help improve our understanding of the major contributing factors to the pandemic. Many models available in the literature present the dynamics of the disease by systems of nonlinear differential equations [2,3,10,15,16,18,19]. In particular, Blower [3] showed that incidence rates of HIV will fall as more HIV-positive individuals gain access to treatment, but the underlying assumption is that treated individuals would change their behavior. In [2], Bacher showed that treatment without reduction of risky behavior may even increase the proportion of infected individuals. Srinivasa [18] presented a theoretical frame work for transmission of HIV in India. Tripathi et al. [19] have proposed a nonlinear model to study the effect of screening of unaware infectives on the spread of the disease in a homogeneous population with constant immigration of susceptibles. Several investigations have been conducted to study the effect of screening, see for example [10,15,16]. On the other hand, a number of researchers have considered biological delay systems, for a general reference we refer the reader to [8,12,13]. Hopf bifurcation can be used to show that the introduction of a time delay into the model can destabilize the system and periodic solutions can arise [5,7,9,20,21]. HIV/AIDS models with time delay have recently attracted some researchers [2,4,6,11,14,16,17]. Cai et al. [4] introduced a time delay to describe the time from the start of treatment until treatment effects become visible and investigated the effect of the time delay on the stability of the endemically in-

fect equilibrium. Culshaw and Ruan [6] considered the time delay between infection of CD4+ T-cells and the emission of viral particles on a cellular level to also investigate the effect of the time delay on the stability of the endemically infected equilibrium. Kovacs [11] considered an HIV/AIDS model with delay and used the delay as a bifurcation parameter to study the possibility of periodic solutions. In particular, we are interested in studying the effect of screening and time delay on HIV/AIDS models. Naresh et al. [16] introduced a model to study the effect of time delay in the recruitment of infected persons on the transmission dynamics of HIV/AIDS.

In this paper, we study the model originally suggested in [1] for the spread of HIV/AIDS with screening and we introduce time delay in the screening. We are interested in the effect of the time delay on the stability of the endemically infected equilibrium. The paper is organized as follows. In the next section we present the HIV/AIDS model with time delay. In Section 3 some results are given for the model without delay. In Section 4 we analyze the stability of the equilibrium points. In Section 5 we investigate the existence of Hopf bifurcation with some numerical simulations. The paper ends with a conclusion in Section 6.

## 2. Mathematical Model

In deriving our model equations, see [11], we divided the population into four subclasses, the susceptible  $S(t)$ , the infectives that do not know they are infected  $I_1(t)$ , the infectives that know they are infected  $I_2(t)$  (by way of medical screening or otherwise) and that of the AIDS population  $A(t)$ .

Taking the above considerations, the model dynamics is assumed by the following system of ordinary differential equations:

$$\begin{aligned} \frac{dS}{dt} &= Q_0 - (\beta_1 I_1 + \beta_2 I_2)S - \mu S, \\ \frac{dI_1}{dt} &= (\beta_1 I_1 + \beta_2 I_2)S - (\theta + \mu + \delta)I_1, \\ \frac{dI_2}{dt} &= \theta I_1 - (\mu + \delta)I_2, \\ \frac{dA}{dt} &= \delta(I_1 + I_2) - (\mu + d)A, \end{aligned} \tag{1}$$

where:

$Q_0$  = constant rate of immigration of susceptibles,

$\beta_i (i = 1, 2)$  are the per capita contact rates for susceptibles individuals with (unaware, aware) infectives respectively,

$\mu$  = the natural mortality rate unrelated to AIDS,

$\theta$  = the rate of unaware infectives to become aware infectives by screening,

$\delta$  = the rate by which both types of infectives develop AIDS,

$d$  = the AIDS related death rate.

Since the variable  $A$  of system (1) does not appear in the first three equations, in the subsequent analysis, we only consider the subsystem:

$$\begin{aligned}\frac{dS}{dt} &= Q_0 - (\beta_1 I_1 + \beta_2 I_2)S - \mu S \\ \frac{dI_1}{dt} &= (\beta_1 I_1 + \beta_2 I_2)S - (\theta + \mu + \delta)I_1 \\ \frac{dI_2}{dt} &= \theta I_1 - (\mu + \delta)I_2\end{aligned}\tag{2}$$

By introducing time delay to model (2), we investigate the effect of the time delay on the stability of the infected equilibrium. Let  $\tau$  be the time from screening of HIV until the result of screening become available. Thus, we consider the following model

$$\begin{aligned}S' &= Q_0 - (\beta_1 I_1 + \beta_2 I_2)S - \mu S \\ I_1' &= (\beta_1 I_1 + \beta_2 I_2)S - (\mu + \delta)I_1 - \theta I_1(t - \tau) \\ I_2' &= \theta I_1(t - \tau) - (\mu + \delta)I_2\end{aligned}\tag{3}$$

with initial values

$$S(0) = S_0 > 0, \quad i_1(u) = I_{10} \geq 0$$

for all  $u \in [-\tau, 0]$ ,  $I_2(0) = I_{20} \geq 0$ .

### 3. Model Without Time Delay

Model (2) was investigated in [1]. It was shown that the model has two equilibria, the disease free equilibrium  $E_0 = (\frac{Q_0}{\mu}, 0, 0)$  and the endemic equilibrium  $E^*(S^*, I_1^*, I_2^*)$ , where

$$S^* = \frac{(\mu + \delta)(\mu + \delta + \theta)}{\beta_1(\mu + \delta) + \beta_2\theta} = \frac{Q_0/\mu}{R_0},$$

$$I_1^* = \frac{Q_0}{\mu + \delta + \theta} \left[ 1 - \frac{\mu}{Q_0} S^* \right] = \frac{Q_0}{\mu + \delta + \theta} \left[ 1 - \frac{1}{R_0} \right]$$

and

$$I_2^* = \frac{\theta}{\mu + \delta} I_1^*.$$

The basic reproduction number was found to be

$$R_0 = \frac{Q_0[\beta_1(\mu + \delta) + \beta_2\theta]}{\mu(\mu + \delta)(\mu + \delta + \theta)}.$$

It was proved that disease-free equilibrium  $E_0$  is locally asymptotically stable when  $R_0 < 1$ . Liapunov function is used to show the global stability of  $E_0$  when  $R_0 \leq 1$ . The positive infected equilibrium  $E^*$  is shown to be locally asymptotically stable when  $R_0 > 1$ . Also it was proved that  $E^*$  is globally asymptotically stable when  $R_0 > 1$  By showing that this model has no periodic solutions, homoclinic loops and oriented phase polygons inside the invariant region

$$\Gamma^* = \left\{ (S, I_1, I_2) \in \Gamma : S + \frac{\mu + \delta}{\mu} I_1 + \frac{\mu + \delta}{\mu} I_2 = \frac{Q_0}{\mu} \right\}.$$

#### 4. Equilibria and their Stability

Clearly, model (3) also has the disease free equilibrium point  $E_0(\frac{Q_0}{\mu}, 0, 0)$  and the infected equilibrium point  $E^*$ , and has the same reproduction number  $R_0$  as in model (2). We will show here that the incorporation of a delay will not change the stability of  $E_0$ .

By linearizing system (3) at  $E_0$ , we find the following jacobian matrix:

$$J(E_0) = \begin{bmatrix} -\mu & -\beta_1 \frac{Q_0}{\mu} & -\beta_2 \frac{Q_0}{\mu} \\ 0 & \beta_1 \frac{Q_0}{\mu} - \mu - \delta - \theta e^{-\lambda\tau} & \beta_2 \frac{Q_0}{\mu} \\ 0 & \theta e^{-\lambda\tau} & -\mu - \delta \end{bmatrix},$$

which yields the following characteristic equation:

$$(-\mu - \lambda) \left[ (\mu + \delta - \beta_1 \frac{Q_0}{\mu} + \theta e^{-\lambda\tau} + \lambda)(\mu + \delta + \lambda) - \beta_2 \theta \frac{Q_0}{\mu} e^{-\lambda\tau} \right] = 0.$$

The eigenvalues are given by,  $\lambda_1 = -\mu < 0$  and

$$\lambda^2 + (2\mu + 2\delta - \beta_1 \frac{Q_0}{\mu} + \theta e^{-\lambda\tau})\lambda + (\mu + \delta)(\mu + \delta - \beta_1 \frac{Q_0}{\mu} + \theta e^{-\lambda\tau}) - \beta_2 \theta \frac{Q_0}{\mu} e^{-\lambda\tau} = 0,$$

which can be written as

$$\lambda^2 + 2(\mu + \delta)\lambda + \theta e^{-\lambda\tau}(\mu + \delta + \lambda) + (\mu + \delta)^2 = \frac{Q_0}{\mu}[\beta_1(\mu + \delta + \lambda) + \beta_2\theta e^{-\lambda\tau}].$$

So, two eigenvalues satisfy the equation

$$\frac{Q_0[\beta_1(\mu + \delta + \lambda) + \beta_2\theta e^{-\lambda\tau}]}{\mu[\lambda^2 + 2(\mu + \delta)\lambda + \theta e^{-\lambda\tau}(\mu + \delta + \lambda) + (\mu + \delta)^2]} = 1. \quad (4)$$

Let

$$h(\lambda) = \frac{Q_0[\beta_1(\mu + \delta + \lambda) + \beta_2\theta e^{-\lambda\tau}]}{\mu[\lambda^2 + 2(\mu + \delta)\lambda + \theta e^{-\lambda\tau}(\mu + \delta + \lambda) + (\mu + \delta)^2]}.$$

So, if  $\tau > 0$  and  $\text{Re}(\lambda) \geq 0$ , then

$$|h(\lambda)| \leq \frac{Q_0[\beta_1(\mu + \delta) + \beta_2\theta]}{\mu(\mu + \delta)(\mu + \delta + \theta)} = R_0.$$

It follows that if  $R_0 < 1$ , then for all  $\text{Re}(\lambda) \geq 0$ , the absolute value of the left side of (4) at  $E_0$  is no longer equals unity and thus there can be no root of equation (4) with  $\text{Re}(\lambda) \geq 0$  and thus  $E_0$  is locally asymptotically stable if  $R_0 < 1$ .

**Theorem 1.** *If  $R_0 < 1$ , then the disease free equilibrium point of system (3) is locally asymptotically stable for all delay  $\tau \geq 0$ .*

Now we investigate the effect of the time delay on the stability of the endemic equilibrium  $E^*$ . The jacobian matrix for the linearized system of (3) at  $E^*$  is

$$J(E^*) = \begin{bmatrix} -\mu - k\frac{I_1^*}{S^*} & -\beta_1 S^* & -\beta_2 S^* \\ k\frac{I_1^*}{S^*} & \beta_1 S^* - \mu - \delta - \theta e^{-\lambda\tau} & \beta_2 S^* \\ 0 & \theta e^{-\lambda\tau} & -\mu - \delta \end{bmatrix},$$

with  $\beta_1 I_1^* + \beta_2 I_2^* = k\frac{I_1^*}{S^*}$ . The characteristic equation is given by

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \quad (5)$$

where

$$\begin{aligned} P(\lambda) &= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \\ Q(\lambda) &= b_1\lambda^2 + b_2\lambda + b_3, \\ a_1 &= 3\mu + 2\delta + k\frac{I_1^*}{S^*} - \beta_1 S^*, \end{aligned}$$

$$\begin{aligned}
 a_2 &= 3\mu^2 + 4\mu\delta - 2\mu\beta_1 S^* + 2(\mu + \delta)k\frac{I_1^*}{S^*} - k\beta_1 I_1^* + \delta^2 - \beta_1 \delta S^*, \\
 a_3 &= (\mu + \delta)^2(\mu + k\frac{I_1^*}{S^*}) - \mu\beta_1 S^*(\mu + \delta), \\
 b_1 &= \theta, \\
 b_2 &= 2\mu\theta + \delta\theta + k\theta\frac{I_1^*}{S^*} - \beta_2\theta S^*, \\
 b_3 &= \theta(\mu + \delta)(\mu + k\frac{I_1^*}{S^*}) + \mu\beta_2\theta S^*.
 \end{aligned}$$

Let  $\tau = 0$  in (5), then we get

$$\lambda^3 + (a_1 + b_1)\lambda^2 + (a_2 + b_2)\lambda + a_3 + b_3 = 0$$

The roots of this characteristic equation have negative real parts [1]. But we want to show that equation (5) have negative real part if  $\tau > 0$  also (and we do this by the continuity of  $\tau$ ). Let  $\lambda = \chi(\tau) + i\omega(\tau)$  ( $\omega > 0$ ) be the eigenvalue of the characteristic equation (5), where  $\chi(\tau)$  and  $\omega(\tau)$  depend on the delay  $\tau$ . Since the infected equilibrium  $E^*$  of the ODE model is stable when  $\tau = 0$ , it follows that  $\chi(\tau) < 0$  when  $\tau > 0$  is sufficiently small (by continuity of  $\tau$ ) and  $E^*$  is still stable, If by increasing  $\tau$  the real part of the root of (5) reaches zero at  $\tau_0$  i.e.  $\chi(\tau_0) = 0$  for a certain value  $\tau_0 > 0$  we see that  $\lambda = i\omega(\tau_0)$ , so we have a purely imaginary root which mean that equation (5) could get a root with positive real parts (because of continuity of  $\tau$ ), and  $E^*$  becomes unstable. If such  $\omega(\tau_0)$  does not exist which means that equation (5) does not have purely imaginary roots for all delay, then  $E^*$  is always stable.

Now, we shall determine if (5) has purely imaginary roots, from which we then shall be able to find conditions for all eigenvalues to have negative real parts.

When  $R_0 > 1$  and  $\tau > 0$ , assuming  $\lambda = i\omega$  with  $\omega > 0$  in (5) we get:

$$-i\omega^3 - a_1\omega^2 + ia_2\omega + a_3 + (-b_1\omega^2 + ib_2\omega + b_3)(\cos \omega\tau - i \sin \omega\tau) = 0.$$

Separating the real and imaginary parts,we get:

$$\begin{aligned}
 a_1\omega^2 - a_3 &= (b_3 - b_1\omega^2) \cos \omega\tau + b_2\omega \sin \omega\tau \\
 \omega^3 - a_2\omega &= b_2\omega \cos \omega\tau - (b_3 - b_1\omega^2) \sin \omega\tau,
 \end{aligned} \tag{6}$$

by squaring equations (6), we have

$$\omega^6 + (a_1^2 - 2a_2 - b_1^2)\omega^4 + (a_2^2 - b_2^2 - 2a_1a_3 + 2b_1b_3)\omega^2 + a_3^2 - b_3^2 = 0.$$

Let

$$y = \omega^2, p = a_1^2 - 2a_2 - b_1^2, q = a_2^2 - b_2^2 - 2a_1a_3 + 2b_1b_3, r = a_3^2 - b_3^2.$$

Thus, we have:

$$G(y) = y^3 + py^2 + qy + r = 0. \quad (7)$$

By Routh-Herwitz criteria this polynomial has roots with negative real parts if  $p > 0, r > 0, pq > r$ . But  $y = \omega^2 > 0$ . So, our assumption that  $\lambda = i\omega$  is

a root of (5) is wrong which means that (5) has no positive roots and the real parts of all eigenvalues are negative for all delay  $\tau \geq 0$ . From this analysis we can derive the following theorem

**Theorem 2.** *If  $R_0 > 1$  and  $p > 0, r > 0$  and  $pq > r$ , the infected equilibrium  $E^*$  is locally asymptotically stable for all delay  $\tau \geq 0$ .*

## 5. Hopf Bifurcation

If  $r < 0$ , then  $G(0) = r < 0$  and  $\lim_{y \rightarrow \infty} G(y) = \infty$ . Then there exist at least a positive root satisfying equation (7), so the characteristic equation (5) has at least a pair of purely imaginary roots of the form  $\pm i\omega_0$ . Eliminating  $\sin \omega\tau$  from (6) we get

$$\cos \omega\tau = \frac{(b_3 - b_1\omega^2)(a_1\omega^2 - a_3) + b_2\omega(\omega^2 - a_2\omega)}{(b_3 - b_1\omega^2)^2 + (b_2\omega)^2}.$$

Therefore,  $\tau_n^*$  corresponding to  $\omega_0$  is given by:

$$\tau_n^* = \frac{1}{\omega_0} \cos^{-1} \left[ \frac{(b_3 - b_1\omega_0^2)(a_1\omega_0^2 - a_3) + b_2\omega_0(\omega_0^2 - a_2\omega_0)}{(b_3 - b_1\omega_0^2)^2 + (b_2\omega_0)^2} \right] + \frac{2n\pi}{\omega_0}.$$

For  $\tau = 0$ , the positive equilibrium  $E^*$  is stable when  $R_0 > 1$  [1]. Hence  $E^*$  remains stable for  $\tau < \tau_0$  where  $\tau_0 = \tau_0^*$  as  $n = 0$ . So, if (5) has a pair of purely imaginary roots, then it has roots with positive real part (by continuity in  $\tau$ ). Hence,  $E^*$  lose its stability and periodic solutions may happen [5] i.e Hopf bifurcation occur if  $\left. \frac{d(\text{Re}\lambda)}{d\tau} \right|_{\tau=\tau_0} > 0$ .

Differentiating (5) with respect to  $\tau$ , we get

$$[(3\lambda^2 + 2a_1\lambda + a_2) + e^{-\lambda\tau}(2b_1\lambda + b_2) - \tau e^{-\lambda\tau}(b_1\lambda^2 + b_2\lambda + b_3)] \frac{d\lambda}{d\tau}$$



$$= \lambda e^{-\lambda\tau}(b_1\lambda^2 + b_2\lambda + b_3),$$

Which gives:

$$\begin{aligned} \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{(3\lambda^2 + 2a_1\lambda + a_2) + e^{-\lambda\tau}(2b_1\lambda + b_2) - \tau e^{-\lambda\tau}(b_1\lambda^2 + b_2\lambda + b_3)}{\lambda e^{-\lambda\tau}(b_1\lambda^2 + b_2\lambda + b_3)} \\ &= \frac{3\lambda^2 + 2a_1\lambda + a_2}{\lambda e^{-\lambda\tau}(b_1\lambda^2 + b_2\lambda + b_3)} + \frac{2b_1\lambda + b_2}{\lambda(b_1\lambda^2 + b_2\lambda + b_3)} - \frac{\tau}{\lambda} \\ &= \frac{3\lambda^2 + 2a_1\lambda + a_2}{-\lambda(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3)} + \frac{2b_1\lambda + b_2}{\lambda(b_1\lambda^2 + b_2\lambda + b_3)} - \frac{\tau}{\lambda} \\ &= \frac{2\lambda^3 + a_1\lambda^2 - a_3}{-\lambda^2(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3)} + \frac{b_1\lambda^2 - b_3}{\lambda^2(b_1\lambda^2 + b_2\lambda + b_3)} - \frac{\tau}{\lambda}. \end{aligned}$$

Thus

$$\begin{aligned} \text{sign} \left\{ \frac{d(\text{Re}\lambda)}{d\tau} \right\}_{\lambda=i\omega_0} &= \text{sign} \left\{ \text{Re} \left( \frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=i\omega_0} \\ &= \text{sign} \left\{ \text{Re} \left[ \frac{2\lambda^3 + a_1\lambda^2 - a_3}{-\lambda^2(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3)} \right. \right. \\ &\quad \left. \left. + \frac{b_1\lambda^2 - b_3}{\lambda^2(b_1\lambda^2 + b_2\lambda + b_3)} - \frac{\tau}{\lambda} \right] \right\}_{\lambda=i\omega_0} \\ &= \text{sign} \left\{ \text{Re} \left[ \frac{-2i\omega_0^3 - a_1\omega_0^2 - a_3}{\omega_0^2(-i\omega_0^3 - a_1\omega_0^2 + ia_2\omega_0 + a_3)} \right. \right. \\ &\quad \left. \left. + \frac{-b_1\omega_0^2 - b_3}{-\omega_0^2(-b_1\omega_0^2 + ib_2\omega_0 + b_3)} - i\frac{\tau}{\omega} \right] \right\} \\ &= \text{sign} \left\{ \text{Re} \left[ \frac{(-a_1\omega_0^2 - a_3) - 2i\omega_0^3}{\omega_0^2[(a_3 - a_1\omega_0^2) + (a_2\omega_0 - \omega_0^3)i]} \right. \right. \\ &\quad \left. \left. + \frac{b_3 + b_1\omega_0^2}{\omega_0^2[(b_3 - b_1\omega_0^2) + ib_2\omega_0]} - i\frac{\tau}{\omega} \right] \right\} \\ &= \text{sign} \left\{ \text{Re} \left[ \frac{1}{\omega_0^2} \left[ \frac{(a_3 + a_1\omega_0^2) + 2i\omega_0^3}{(a_1\omega_0^2 - a_3) + (\omega_0^3 - a_2\omega_0)i} \right. \right. \right. \\ &\quad \times \frac{(a_1\omega_0^2 - a_3) - (\omega_0^3 - a_2\omega_0)i}{(a_1\omega_0^2 - a_3) - (\omega_0^3 - a_2\omega_0)i} \\ &\quad \left. \left. + \frac{b_3 + b_1\omega_0^2}{(b_3 - b_1\omega_0^2) + ib_2\omega_0} \times \frac{(b_3 - b_1\omega_0^2) - ib_2\omega_0}{(b_3 - b_1\omega_0^2) - ib_2\omega_0} \right] - i\frac{\tau}{\omega} \right] \right\} \\ &= \frac{1}{\omega_0^2} \text{sign} \left\{ \frac{(a_3 + a_1\omega_0^2)(a_1\omega_0^2 - a_3) + 2\omega_0^3(\omega_0^3 - a_2\omega_0)}{(a_1\omega_0^2 - a_3)^2 + (\omega_0^3 - a_2\omega_0)^2} \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{(b_3 + b_1\omega_0^2)(b_3 - b_1\omega_0^2)}{(b_3 - b_1\omega_0^2)^2 + (b_2\omega_0)^2} \} \\
= & \frac{1}{\omega_0^2} \text{sign} \left\{ \frac{(a_3 + a_1\omega_0^2)(a_1\omega_0^2 - a_3) + 2\omega_0^3(\omega_0^3 - a_2\omega_0) + (b_3 + b_1\omega_0^2)(b_3 - b_1\omega_0^2)}{(b_3 - b_1\omega_0^2)^2 + (b_2\omega_0)^2} \right\} \\
= & \frac{1}{\omega_0^2} \text{sign} \left[ \frac{(a_1\omega_0^2)^2 - a_3^2 + 2\omega_0^6 - 2a_2\omega_0^4 + b_3^2 - (b_1\omega_0^2)^2}{(b_3 - b_1\omega_0^2)^2 + (b_2\omega_0)^2} \right] \\
= & \frac{1}{\omega_0^2} \text{sign} \left[ \frac{2\omega_0^6 + (a_1^2 - 2a_2 - b_1^2)\omega_0^4 + (b_3^2 - a_3^2)}{(b_3 - b_1\omega_0^2)^2 + (b_2\omega_0)^2} \right].
\end{aligned}$$

We see that if  $a_1^2 - 2a_2 - b_1^2 > 0$ ,  $b_3^2 - a_3^2 > 0$ , which mean  $p > 0$  and  $r < 0$ , then we have

$$\frac{d(\text{Re}\lambda)}{d\tau} \Big|_{\tau=\tau_0, \omega=\omega_0} > 0.$$

So we have at least one eigenvalue with positive real part for  $\tau > \tau_0$  and the conditions for Hopf bifurcation are satisfied yielding periodic solutions at  $\tau = \tau_0$ ,  $\omega = \omega_0$ . From the above analysis we can obtain the following theorem

**Theorem 3.** *If  $R_0 > 1$  and  $p > 0$ ,  $r < 0$ , the infected equilibrium  $E^*$  remains stable for  $\tau < \tau_0$  and unstable when  $\tau > \tau_0$ , a Hopf bifurcation occurs as  $\tau$  passes through  $\tau_0$ . That is, system (4.1) has a branch of periodic solutions bifurcating from the positive equilibrium  $E^*$  near  $\tau = \tau_0$ .*

### 5.1. Numerical Simulations

Now, we present some numerical results of system (3) at different values of delay  $\tau$ . To see the dynamical behavior of the model, system (3) is integrated numerically by *dde23* using the following parameters:  $Q_0 = 30$ ,  $\delta = 0.04$ ,  $\beta_1 = 0.0009$ ,  $\beta_2 = 0.00027$ ,  $d = 1$ ,  $\mu = 0.02$ ,  $\theta = 0.8$ , with the initial values  $S(0) = 6320$ ,  $I_1(0) = 5400$ ,  $I_2(0) = 4500$ ,  $A(0) = 1800$ .

From figure (1) we observed that the unique positive equilibrium point  $E^*$  is asymptotically stable for  $\tau \in [0, \tau_0)$  i.e.  $\tau = 1.79$ . Figure (2) shows the effect of delay parameter  $\tau$ . It is observed that when  $\tau = \tau_0 = 1.82$  periodic solutions of system (3) bifurcates from the positive equilibrium  $E^*$ .

The numerical simulation supports the analytical result of the research.

### 6. Conclusion

In this paper, we analyzed a mathematical model to study the effect of time delay on the transmission dynamics of HIV/AIDS. The stability of both the dis-

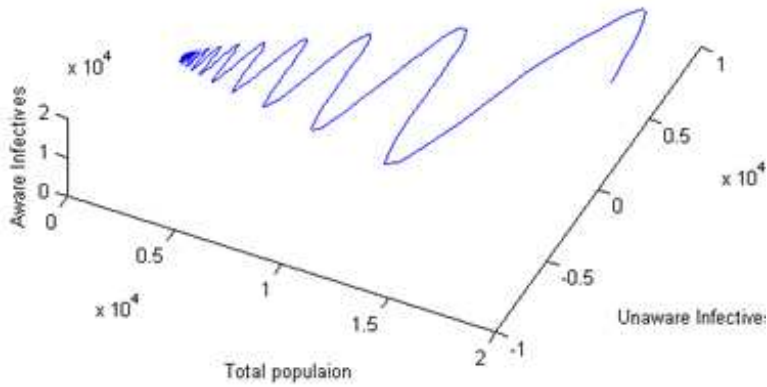


Figure 1:  $\tau = 1.79$  – the endemic equilibrium is asymptotically stable

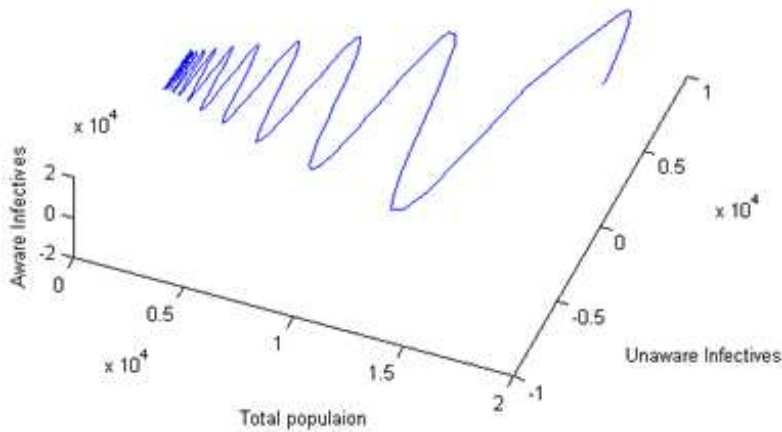


Figure 2:  $\tau = 1.82$  – bifurcating periodic solutions occur

ease free equilibrium and the endemically infected equilibrium is investigated. It is shown that introducing time delay in the model destabilizes the system. By using the time delay as a bifurcating parameter, it was shown that as the time delay  $\tau$  increases (for  $\tau > \tau_0$ ), the positive endemic equilibrium loses its stability and Hopf bifurcation occurs, i.e. a family of periodic solutions bifurcates from  $E^*$ . Finally some numerical simulations are performed to illustrate the analytical results found.

### References

- [1] S. Al-Sheikh, F. Musali, M. Alsolami, Stability analysis of an HIV/AIDS epidemic model with screening, *Int. Math. Forum*, **6**, No. 66 (2011), 3251-3273.
- [2] M. Bacher, A. Dorfmayr, HIV treatment models with time delay, *C.R. Biologies*, **327** (2004), 983-994.
- [3] S. Blower, Calculating the consequences: HAART and risky sex, *AIDS*, **15** (2001), 1309-1310.
- [4] L. Cai, X. Li, M. Ghosh, B. Guo, Stability analysis of an HIV/AIDS epidemic model with treatment, *Journal of Computational and Applied Mathematics*, **229** (2009), 313-323.
- [5] K. Cooke, R. Elderkin, W. Huang, Predator-prey interactions with delays due to juvenile maturation, *SIAM J. Appl. Math.*, **66**, No. 3 (2006), 1050-1079.
- [6] R.V. Culshaw, S. Ruan, A delay-differential equation model of HIV infection of CD4+ T-cells, *Math. Biosci.*, **165** (2000), 27-39.
- [7] G. Fan, M. Min-oo, G.S.K. Wolkowicz, Hopf bifurcation of delay differential equations with delay dependent parameters, *Canadian Applied Mathematics Quarterly*, **17**, No. 1 (2009), 37-60.
- [8] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic Publishers (1992).
- [9] S. Gourley, Y. Kuang, Stage structured predator-prey model and its dependence on maturation delay and death rate, *J. Math. Biol.*, **49** (2004), 188-200.
- [10] S. Hove-Musekwa, F. Nyabadza, The dynamics of an HIV/AIDS model with screened disease carriers, *Comp. Math. methods in Med.*, **10**, No. 4 (2009), 287-305.
- [11] S. Kovacs, Dynamics of HIV/AIDS model – The effect of time delay, *Appl. Math. Comp.*, **188**, No. 2 (2007), 1597-1609.
- [12] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press (1993).

- [13] Z. Ma, Y. Zhou, W. Wang, Z. Jin, *Mathematical Models and Dynamics of Infectious Diseases*, China Science Press, Beijing (2004).
- [14] Z. Mukandavire, W. Garira, C. Chiyaka, Asymptotic properties of an HIV/AIDS model with a time delay, *J. Math. Anal. Appl.*, **330** (2007), 916-933.
- [15] R. Naresh, A. Tripathi, D. Sharma, Modelling and analysis of the spread of AIDS epidemic with immigration of HIV infectives, *Math. Comp. Model.*, **49** (2009), 880-892.
- [16] R. Naresh, A. Tripathi, D. Sharma, A nonlinear AIDS epidemic model with screening and time delay, *Appl. Math. Comp.*, **217** (2011), 4416-4426.
- [17] X. Song, X. Zhou, X. Zhao, Properties of stability and Hopf bifurcation for an HIV infection model with time delay, *Appl. Math. Mod.*, **34** (2010), 1511-1523.
- [18] A.S.R. Srinivasa Rao, Mathematical modelling of AIDS epidemic in India, *Curr. Sci.*, **84** (2003), 1192-1197.
- [19] A. Tripathi, R. Naresh, D. Sharma, Modelling the effect of screening of unaware infectives on the spread of HIV infection, *Appl. Math. Comp.*, **184** (2007), 1053-1068.
- [20] J. Wei, X. Zou, Bifurcation analysis of a population model and the resulting SIS epidemic model with delay, *J. Comp. Appl. Math.*, **197** (2006), 169-187.
- [21] X. Xu, H. Hua, H. Wang, Stability switches, Hopf bifurcation and chaos of a neuron model with delay-dependent parameters, *Phys. Lett. A*, **354** (2006), 126-136.

