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AN OPTIMAL ENERGY CONTROL OF A FLEXIBLE PLATE WITH VISCOELASTIC CONDITIONS

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Abstract: An optimal control problem of a flexible thin plate formulated by partial differential equations with viscoelastic boundary conditions is studied in this paper. The problem is written in standard form of linear infinite dimensional system in an appropriate energy Hilbert space. The semigroup approach of linear operators is adopted in investigating well-posedness of the closed loop system. An optimal energy control is proposed, and existence and uniqueness of the optimal control are demonstrated, and an approximation theorem is proved in terms of semigroup approach and geometric method.

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1. Introduction

The problems of elastic structures with viscoelastic boundary conditions have been studied extensively by many articles (see references [1]-[5]). Motivated by the work on wave and heat equations mentioned above, in this article we are concerned with an elastic thin plate which occupies a bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-smooth boundary Γ. Assume that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where Γ_0 and Γ_1 are relatively open subsets of Γ, $\Gamma_0 \neq \emptyset$ has positive boundary measure,

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and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. If Γ_0 is clamped and the memory effect on Γ_1 is taken into account, the vertical deflection $y(x,t)$ of the thin elastic plate satisfies the following partial differential equation:

$$y_{tt}(x,t) + \Delta^2 y(x,t) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1a}$$

$$y(x,t) = \partial_v y(x,t) = 0, \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \tag{1b}$$

$$\mathcal{B}_1 y(x,t) - \int_0^\infty g'(s)\partial_v[y(x,t) - y(x,t-s)]\mathrm{d}s = 0, \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \tag{1c}$$

$$\mathcal{B}_2 y(x,t) + \int_0^\infty g'(s)[y(x,t) - y(x,t-s)]\mathrm{d}s = u(x,t), \quad \text{on} \Gamma_1 \times \mathbb{R}^+, \tag{1d}$$

$$y(x,0^+) = y_0(x), \quad y_t(x,0^+) = y_1(x), \tag{1e}$$

$$y(x,-s) = \vartheta(x,t), \quad \text{for } 0 < s < \infty, \tag{1f}$$

where g is the relaxation function, u is the boundary control, y_0, y_1, ϑ are the given initial conditions. $\mathcal{B}_1, \mathcal{B}_2$ are the following boundary operators:

$$\mathcal{B}_1 y = \Delta_y + (1-\mu)(2v_1 v_2 \frac{\partial^2 y}{\partial x_1 \partial x_2} - v_1^2 \frac{\partial^2 y}{\partial x_2^2} - v_2^2 \frac{\partial^2 y}{\partial x_1^2}),$$
$$\mathcal{B}_2 y = \partial_v \Delta_y + (1-\mu)\partial_\tau[(v_1^2 - v_2^2)\frac{\partial^2 y}{\partial x_1 \partial x_2} + v_1 v_2(\frac{\partial^2 y}{\partial x_2^2} - \frac{\partial^2 y}{\partial x_1^2})],$$

$v = (v_1, v_2)$ is the unit outer normal vector, $\tau = (-v_2, v_1)$ is the unit tangent vector, and $0 < \mu < \frac{1}{2}$ is the Poisson ratio.

Throughout the article, we assume always that the function $g(\cdot)$ satisfies the following conditions:

(g_1) $g(\cdot) \in C^2[0,\infty)$;

(g_2) $g(t) > 0, \quad g'(t) < 0, \quad g''(t) \geq 0$ for $t \geq 0$;

(g_3) $g(\infty) > 0$;

(g_4) $g'(t) \geq -kg''(t)$ for some $k > 0$ and all $t \geq 0$.

Condition ($g2$) implies that the memory of the boundary is strictly decreasing and the rate of memory loss is also decreasing. From ($g2$), we have also that both $g(\infty)$ and $g'(\infty)$ exist, $g'(\infty) \geq 0$. Condition ($g3$) means that the material behaves like an elastic solid at $t = \infty$. Condition ($g4$) implies that $g'(t)$ decays exponentially, in particular, $g'(\infty) = 0$.

The energy corresponding to the system (1) is defined by

$$E(t) = \frac{1}{2}a(y(\cdot,t)) + \int_\Omega |y_t(x,t)|^2 \mathrm{d}x - \int_0^\infty \int_{\Gamma_1} g'(s)[|\partial_v(y(x,t) - y(x,t-s))|^2$$

$$+|y(x,t)-y(x,t-s)|^2]\mathrm{d}\Gamma\mathrm{d}s, \quad (2)$$

where $a(w)=a(w,w)$ and

$$a(w_1,w_2)=\int_\Omega[\frac{\partial^2 w_1}{\partial x_1^2}\overline{\frac{\partial^2 w_2}{\partial x_1^2}}]+\frac{\partial^2 w_1}{\partial x_2^2}\overline{\frac{\partial^2 w_2}{\partial x_2^2}}+\mu(\frac{\partial^2 w_1}{\partial x_1^2}\overline{\frac{\partial^2 w_2}{\partial x_2^2}}]+\frac{\partial^2 w_1}{\partial x_2^2}\overline{\frac{\partial^2 w_2}{\partial x_1^2}})$$
$$+2(1-\mu)\frac{\partial^2 w_1}{\partial x_1\partial x_2}\overline{\frac{\partial^2 w_2}{\partial x_1\partial x_2}}]\mathrm{d}x, \quad \forall\ w_1,\ w_2\in H^2(\Omega). \quad (3)$$

2. Well-Posedness of the System with Feedback Control

In this section, we shall formulate the system (1) into a standard linear infinite dimensional space with a output feedback control. Let

$$W=\{w\in H^2(\Omega)|w|_{\Gamma_0}=\partial_v w|_{\Gamma_0}=0\},$$

$$\|w\|_W^2=a(w), \quad \forall w\in W,$$

and define the "boundary memory space" by

$$Z=L^2(0,\infty;|g'(.)|;H^1(\Gamma_1)),$$

$$\|z\|_Z^2=\int_0^\infty|g'(s)|[\|\partial_v z(s)\|_{L^2(\Gamma_1)}^2+\|z(s)\|_{L^2(\Gamma_1)}^2]\mathrm{d}s, \qquad \forall z\in Z.$$

Set

$$\mathcal{H}=W\times L^2(\Omega)\times Z$$

equipped with the inner product induced norm

$$\|(w,v,z)\|_{\mathcal{H}}^2=\|w\|_W^2+\|v\|_{L^2_\Omega}^2+\|z\|_Z^2, \quad \forall(w,v,z)\in\mathcal{H}.$$

It is easy to see that $\mathcal{H}$ is a Hilbert space.

Remark. We have that $a(\cdot)^{\frac{1}{2}}$ is an equivalent norm on W since $\Gamma_0\neq\emptyset$ has positive boundary measure (see e.g. Ref.8). Moreover, it is obvious that $(\|\partial_v z\|_{L^2(\Gamma_1)}^2+\|z^2\|_{L^2(\Gamma_1)}^2)^{\frac{1}{2}}$ is an equivalent norm on $H^1(\Gamma_1)$. In fact, if $\|\partial_v z\|_{L^2(\Gamma_1)}^2+\|z^2\|_{L^2(\Gamma_1)}^2=0$, then $z=\partial_v z=0$ on Γ_1. It follows that $\nabla_z=v\partial_v z=0$ on Γ_1. Therefore, $z=0$ in $H^1(\Gamma_1)$.

Next we introduce some operators (Ref.9) as follows:

(i) We set

$$\begin{aligned}&Lz(s) = \textstyle\int_0^\infty g'(s)z(s)\mathrm{d}s,\\ &\mathcal{A}_0 = \Delta^2, \quad \mathcal{D}(\mathcal{A}_0) = \{w \in H^4(\Omega) \cap W | \mathcal{B}_1 w|_{\Gamma_1} = \mathcal{B}_2 w|_{\Gamma_1} = 0\}.\end{aligned}$$

It is easy to know that $\mathcal{A}_0$ is a positive self-adjoint operator on $L^2(\Omega)$.

(ii) The Green operators N_1 and N_2 are introduced to describe the boundary conditions,

$$N_1 g = h \Leftrightarrow \begin{cases} \Delta^2 h = 0, & \text{in} \quad \Omega,\\ h = \partial_v h = 0, & \text{on} \quad \Gamma_0,\\ \mathcal{B}_1 h = g, & \text{on} \quad \Gamma_1,\\ \mathcal{B}_2 h = 0, & \text{on} \quad \Gamma_1,\end{cases}$$

$$N_2 g = h \Leftrightarrow \begin{cases} \Delta^2 h = 0, & \text{in} \quad \Omega,\\ h = \partial_v h = 0, & \text{on} \quad \Gamma_0,\\ \mathcal{B}_1 h = 0, & \text{on} \quad \Gamma_1,\\ \mathcal{B}_2 h = g, & \text{on} \quad \Gamma_1.\end{cases}$$

In terms of the regularity theory for the elliptic equations (Ref.6), we know that

$$\begin{aligned}&N_1 : L^2(\Gamma_1) \to H^{\frac{5}{2}}(\Omega) \quad \text{is continuous},\\ &N_2 : L^2(\Gamma_1) \to H^{\frac{7}{2}}(\Omega) \quad \text{is continuous}.\end{aligned}$$

By these operators defined above, we may rewrite the system (1) as

$$y_{tt}(\cdot,t) + \mathcal{A}_0[y(\cdot,t) - N_1 L_z(\cdot,t,s) + N_2 L_z(\cdot,t,s) - N_2 u(\cdot,t,s)] = 0, \tag{4}$$

where $z(\cdot,t,s) = y(x,t) - y(x,t-s), x \in \Gamma_1$. Considering $L^2(\Omega)$ as the pivot space: $[\mathcal{D}(\mathcal{A}_0)] \subset L^2(\Omega) \subset [\mathcal{D}(\mathcal{A}_0)]'$ and extending the $\mathcal{A}_0$ to be $\widetilde{\mathcal{A}}_0 : L^2(\Omega) \to [\mathcal{D}(\mathcal{A}_0)]'$, we can rewrite (4) as

$$y_{tt}(\cdot,t) = -\widetilde{\mathcal{A}}_0 y(\cdot,t) + \widetilde{\mathcal{A}}_0 N_1 L_z(\cdot,t) - \widetilde{\mathcal{A}}_0 N_2 L_z(\cdot,t) + \widetilde{\mathcal{A}}_0 N_2 u(\cdot,t) \in [\mathcal{D}(\mathcal{A}_0)]'. \tag{5}$$

Thus we can put system $(1a) - (1f)$ into a standard form of linear infinite-dimensional system in $\mathcal{H}$

$$\dot{Y}(t) = \mathcal{A}Y(t) + Bu \tag{6}$$

where

$$Y(t) = \begin{Bmatrix} y(\cdot,t)\\ y_t(\cdot,t)\\ z(\cdot,t,s)\end{Bmatrix}, \quad z(\cdot,t,s) = y(x,t) - y(x,t-s),$$

$$\mathcal{A} = \left\{ \begin{array}{ccc} 0 & I & 0 \\ -\widetilde{\mathcal{A}}_0 & 0 & \widetilde{\mathcal{A}}_0 N_1 L - \widetilde{\mathcal{A}}_0 N_2 L \\ 0 & I & -\frac{\partial}{\partial s} \end{array} \right\}, \quad \mathcal{D}(\mathcal{A}) = \{Y \in \mathcal{H} | \mathcal{A}Y \in \mathcal{H}\},$$

and

$$Bu = \left\{ \begin{array}{c} 0 \\ \widetilde{\mathcal{A}}_0 N_2 u \\ 0 \end{array} \right\}, \quad B : L^2(\Gamma_1) \to [\mathcal{D}(\mathcal{A}^*)]' \text{ is continuous.}$$

Finally, a direct computation gives

$$\begin{aligned} (N_2^* \mathcal{A}_0 f, g)_{L^2(\Gamma_1)} &= (\mathcal{A}_0 f, N_2 g)_{L^2(\Omega)} = (\Delta^2 f, N_2 g)_{L^2(\Omega)} \\ &= \int_\Omega f \overline{\Delta^2 (N_2 g)} \mathrm{d}x - \int_{\Gamma_1} [f \overline{\mathcal{B}_2 (N_2 g)} - \partial_v f \overline{\mathcal{B}_1 (N_2 g)}] \mathrm{d}\Gamma \\ &\quad + \int_{\Gamma_1} [\mathcal{B}_2 f \overline{(N_2 g)} - \mathcal{B}_1 f \overline{\partial_v (N_2 g)}] \mathrm{d}\Gamma \\ &= -\int_{\Gamma_1} f \overline{g} \mathrm{d}\Gamma, \end{aligned}$$

for all $f \in \mathcal{D}(\mathcal{A}_0)$ and $g \in L^2(\Gamma_1)$. Therefore, $N_2^*(\widetilde{\mathcal{A}}_0) f = N_2^* \mathcal{A}_0 f = -f|_{\Gamma_1}$, $f \in \mathcal{D}(\mathcal{A}_0)$. It follows that

$$B^* \left\{ \begin{array}{c} w \\ v \\ z \end{array} \right\} = -v|_{\Gamma_1}, \quad \forall \left\{ \begin{array}{c} w \\ v \\ z \end{array} \right\} \in \mathcal{D}(\mathcal{A}^*). \tag{7}$$

Now, let us consider a feedback control so that the input and output are collocated(Ref.7):

$$u = -kB^*(y, y_t, z)^T = k y_t|_{\Gamma_1}, \quad k \geq 0. \tag{8}$$

The closed-loop system under this output feedback then becomes

$$y_{tt}(x,t) + \Delta^2 y(x,t) = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{9a}$$

$$y(x,t) = \partial_v y(x,t) = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \tag{9b}$$

$$\mathcal{B}_1 y(x,t) - \int_0^\infty g'(s) \partial_v [y(x,t) - y(x,t-s)] \mathrm{d}s = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \tag{9c}$$

$$\mathcal{B}_2 y(x,t) + \int_0^\infty g'(s) [y(x,t) - y(x,t-s)] \mathrm{d}s = k y_t(x,t), \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \tag{9d}$$

$$y(x,0^+) = y_0(x), \quad y_t(x,0^+) = y_1(x), \tag{9e}$$

$$y(x,-s) = \vartheta(x,t). \quad \text{for } 0 < s < \infty, \tag{9f}$$

The initial boundary problem (9) can be written as an evolutionary equation in $\mathcal{H}$:

$$\dot{Y}(t) = \mathcal{T}Y(t), \quad Y(0) = Y_0$$

where $Y = (y, y_t, z), \quad Y_0 = (y_0, y_1, y_0 - \vartheta)$ and

$$\mathcal{T} = \left\{ \begin{array}{ccc} 0 & I & 0 \\ -\Delta^2 & 0 & 0 \\ 0 & I & -\frac{\partial}{\partial s} \end{array} \right\}$$

with the domain

$$\mathcal{D}(\mathcal{T}) = \left\{ (w,v,z) \in \mathcal{H} \left| \begin{array}{l} \Delta^2 w \in L^2(\Omega), v \in W, z(\cdot) \in H^1(0,\infty; |g'(\cdot)|; H^1(\Gamma_1)), \\ z(0) = 0, [\mathcal{B}_1 w - \int_0^\infty g'(s)\partial_v z(s)\mathrm{d}s]_{\Gamma_1} = 0, \\ \; [\mathcal{B}_2 w + \int_0^\infty g'(s)z(s)\mathrm{d}s]_{\Gamma_1} = kv|_{\Gamma_1}, \end{array} \right. \right\},$$

where

$$H^1(0,\infty; |g'(\cdot)|; H^1(\Gamma_1)) = \{z(s) \in Z | \frac{\partial}{\partial s} z(s) \in Z\}.$$

The following theorem ensures that the system (9) is well-posed in $\mathcal{H}$.

Theorem 2.1. *Assume that the function g satisfies $(g1)$ through $(g3)$ and $k \geq 0$. Then the operator $\mathcal{T}$ generates a C_o-semigroup $S(t)$ of contraction on $\mathcal{H}$.*

Proof. We first prove that $\mathcal{R}(I - \mathcal{T}) = \mathcal{H}$. Namely, we need to show that the following system of the equations

$$w - v = f, \tag{10a}$$

$$v + \Delta^2 w = g, \tag{10b}$$

$$z(s) - v + \frac{\partial}{\partial s} z(s) = h(s) \tag{10c}$$

has a solution $(u, v, z) \in \mathcal{D}(\mathcal{T})$ for every $(f, g, h) \in \mathcal{H}$. In fact, it follows from (9) that

$$v = w - f \in W, \tag{11a}$$

$$w + \Delta^2 w = f + g \in L^2(\Omega), \tag{11b}$$

$$z(s) = (1 - e^{-s})w + (1 - e^{-s})f + \int_0^\infty e^{\tau - s} h(\tau)\mathrm{d}\tau \in Z. \tag{11c}$$

Therefore, $v \in W$ and $z(\cdot) \in H^1(0,\infty;|g'(\cdot)|;H^1(\Gamma_1)), z(0) = 0$. Furthermore, by $(11b) - (11c)$ we have that for any $w \in W$ satisfying $\Delta^2 w \in L^2(\Omega)$ and $\mathcal{B}_1 w - \int_0^\infty g'(s)\partial_v z(s)\mathrm{d}s = 0, \mathcal{B}_2 w + \int_0^\infty g'(s)z(s)\mathrm{d}s = kv$, it has for all $\phi \in W$,

$$\int_\Omega w\overline{\phi}\mathrm{d}x + a(w,\phi) + \int_{\Gamma_1}[(kw + Xw)\overline{\phi} + X\partial_v w\overline{\phi}]\mathrm{d}\Gamma$$
$$= \int_\Omega (f+g)\overline{\phi}\mathrm{d}x + \int_{\Gamma_1}[(kf + Xf + \Psi)\overline{\partial_v\phi}]\mathrm{d}\Gamma, \quad (12)$$

where

$$X = -\int_0^\infty g'(s)(1-e^{-s})\mathrm{d}s \geq 0$$

and

$$\Psi = \int_0^\infty g'(s)\int_0^s e^{\tau - s}h(\tau)\mathrm{d}\tau\mathrm{d}s.$$

From the Lax-Milgram theorem (Ref.12), equation (12) admits a unique solution $w \in W$. Combining this with $(11a)$ and (11c), we see that $(w,v,z) \in \mathcal{D}(\mathcal{T})$ solves the equation $(I - \mathcal{T})(w,v,z) = (f,g,h)$.

Next, for any $Y = (w,v,z) \in \mathcal{D}(\mathcal{T})$, it has

$$\begin{aligned} &\mathcal{R}e(\mathcal{T}Y,Y)_{\mathcal{H}} \\ = \; &-k\int_{\Gamma_1}|v|^2\mathrm{d}\Gamma - \tfrac{1}{2}\int_0^\infty\int_{\Gamma_1} g''(s)(|z(s)|^2 + |\partial_v z(s)|^2)\mathrm{d}\Gamma\mathrm{d}s \leq 0. \end{aligned} \quad (13)$$

Hence $\mathcal{T}$ is dissipative. Finally, by theorem 1.4.6 of Ref.13, $\mathcal{D}(\mathcal{T})$ is dense in $\mathcal{H}$. It then follows from the Lumer-Phillips theorem that $\mathcal{T}$ generates a C_o-semigroup of contractions on $\mathcal{H}$. The proof of Theorem 2.1 is complete. □

3. An Optimal Energy Control Problem

In this section, let us discuss an optimal control problem of the following system:

$$\begin{aligned} &\tfrac{dY}{dt} = \mathcal{T}Y(t) + Bu((Y(t),t), \\ &Y(0) = Y_0, \end{aligned} \quad (14)$$

where both state space X and control space Y are Hilbert spaces, the state function $Y(t)$ on $[0,T]$ is valued in X, $\mathcal{T}$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$. B is a bounded linear operator from $L^2([0,T];Y)$ to $L^2([0,T];X)$, $u(Y(t),t)$ is a control of the system.

In this section, we shall discuss a specific optimal control, that is, the minimum energy control of the system (14). We know that the minimum energy control in an abstract space is, in general, the minimum norm control. So, from mathematics point of view, the existence and uniqueness of the optimal control are essential. If these are true, then how to obtain the optimal control is a significant problem. The main content of this paper is to solve these essential and significant issue.

From the theory of operator semigroup, we see that for every control element $u(Y(\cdot),\cdot) \in L^2([0,T],Y)$, the system (14) has an unique mild solution

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)B(u(Y(s),s))ds. \tag{15}$$

Let $\varphi(\cdot)$ be an arbitrary element in C([0,T];X), and

$$\rho = inf_{u\in L^2([0,T];Y)}\|\varphi(t) - S(t)Y_0 - \int_0^t S(t-s)Bu(y(s,s)ds\| \,,$$

define the admissible control set of the system (14) as follows

$$\begin{aligned} U_{ad} = \{u \in L^2([0,T];Y) : \\ \|\varphi(t) - S(t)Y_0 - \int_0^t S(t-s)Bu(Y(s),s)\| \leq \rho + \epsilon\}, \end{aligned} \tag{16}$$

where ϵ is any positive number.

It can be seen from (2.2) that U_{ad} is not empty and contains infinitely many elements related to φ and ϵ. The minimum energy control problem is actually to find the element u, satisfying

$$\|u_0\| = min\{\|u\| : u\epsilon U_{ad}\}, \tag{17}$$

where u_0 is said to be a minimum energy control element.

Lemma 3.1. *The admissible control set U_{ad} defined by (2.2) is a closed convex set in Hilbert space $L^2([0,T];Y)$.*

Proof. Convexity: For any u_1, $u_2 \in U_{ad}$ and a real number λ, $0 < \lambda < 1$, it is easy to see from (2.2) that

$$\|\varphi(t) - S(t)Y_0 - \int_0^t S(t-s)Bu_i(Y(s),s))\| \;\; \leq \;\; \rho + \epsilon, \quad i = 1,2, \tag{18}$$

and hence

$$\begin{aligned}
&\|\varphi(t) - S(t)Y_0 - \int_0^t S(t-s)B(\lambda u_1(Y(s),s) + (1-\lambda)u_2(Y(s),s))ds\| \\
\leq\ & \lambda\|\varphi(t) - S(t)Y_0 - \int_0^t S(t-s)Bu_1(Y(s),s))ds\| \\
&+(1-\lambda)\|\varphi(t) - S(t)Y_0 - \int_0^t S(t-s)Bu_2(Y(s),s)ds\|.
\end{aligned} \tag{19}$$

Since $\lambda u_1 + (1-\lambda)u_2 \in L^2([0,T];Y)$, it follows that $\lambda u_1 + (1-\lambda)u_2 \in U_{ad}$, this implies that U_{ad} is a convex subset of $L^2([0,T];Y)$.

Closedness: Suppose $\{u_n\} \subset U_{ad}$, and $\lim_{n\to\infty} \|u_n - u^*\| = 0$. It can be shown that $u^* \in U_{ad}$. In fact, from the definition of U_{ad} we see that

$$\|\varphi(t) - S(t)y_0 - \int_0^t S(t-s)Bu_n(y(s),s)ds\| \leq \rho + \epsilon, \quad n = 1, 2, \ldots,$$

Since $S(t),\ t \geq 0$ is a C_0-semigroup in Hilbert space X, there is a constant $M > 0$ such that $\sup\limits_{0\leq t\leq T} \|S(t)\| \leq M$. On the other hand, since $Y(s)$ is differentiable on $[0,T]$, it is continuous on $[0,T]$, and hence $\{Y(s) : s \in [0,T]\}$ is a bounded set in $L^2([0,T];Y)$. Thus there is a constant $N > 0$ such that $\|Bu(Y(s),s)\| \leq N\ \ (0 \leq s \leq T)$ and

$$\begin{aligned}
&\|\varphi(t) - S(t)Y_0 - \int_0^t S(t-s)Bu^*(y(s),s)ds\| \\
\leq\ & \|\varphi(t) - S(t)y_0 - \int_0^t u_n(Y(s),s)Bu(Y(s),s)ds\| \\
&+\|\int_0^t S(t-s)B[u_n(Y(s),s) - u^*(Y(s),s)]\| \\
\leq\ & \rho + \epsilon + M\|u_n - u^*\| \cdot NT.
\end{aligned} \tag{20}$$

Letting $n \to \infty$ leads to

$$\|\varphi(t) - S(t)Y_0 - \int_0^t S(t-s)Bu^*(Y(s),s)ds\| \leq \rho + \epsilon\,.$$

Thus, $u^* \in U_{ad}$, and U_{ad} is a closed set. The proof is complete.

Theorem 3.2. *There exists an unique minimum energy control element in the admissible control set U_{ad} of the system (1.1).*

Proof. Since $L^2([0,T],Y)$ is a Hilbert space, it is naturally a strict convex Banach Space. From the preceding Lemma, we have seen that U_{ad} is a closed convex set in $L^2([0,T],Y)$, it follows from [2] that there is an unique element $u_0 \in U_{ad}$ such that

$$\|u_0\| = min\ \{\|u\| : u \in U_{ad}\}$$

According to the definition (2.3), u_0 is just the desired minimum energy control element of the system (1.1). The proof is complete.

Finally, we shall show that the minimum energy control element can be approached.

Theorem 3.3. *Suppose that u_0 is the minimum energy control element of the system (1.1), then there exists a sequence $\{u_n\} \subset U_{ad}$ such that $\{u_n\}$ converges strongly to u_0 in $L^2([0,T];Y)$, namely,*

$$\lim_{n\to\infty} \|u_n - u_0\| = 0$$

Proof. Let $\{u_n\}$ be a minimized sequence in the admissible control set U_{ad}, then it follows that

$$\|u_{n+1}\| \leq \|u_n\|, \quad n = 1, 2, \cdots \tag{21}$$

and

$$\lim_{n\to\infty} \|u_n\| = inf\{\|u\| : u \in U_{ad}\} \tag{22}$$

It is obvious that $\{u_n\}$ is a bounded sequence in $L^2([0,T];Y)$, and so there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to an element $\widetilde{u}$ in $L^2([0,T];Y)$ (see [3]).

Since U_{ad} is a closed convex set in $L^2([0,T];Y)$ (see Lemma 2.1), we see from Mazur's Theorem that U_{ad} is a weakly closed set in $L^2([0,T];Y)$, thus $\tilde{u} \in U_{ad}$. Combining (3.2) and employing the properties of limits of weakly convergent sequence on norm yield

$$\begin{aligned} inf\{\|u\| : u \in U_{ad}\} \leq \|\tilde{u}\| &\leq \underline{\lim}_{k\to\infty} \|u_{n_k}\| \\ &= \lim_{n_k\to\infty} \|u_{n_k}\| = \lim_{n\to\infty} \|u_n\| = \inf\{\|u\|; u \in U_{ad}\}. \end{aligned} \tag{23}$$

Thus, we have

$$\lim_{n\to\infty} \|u_n\| = \|\tilde{u}\| \tag{24}$$

and

$$\|\tilde{u}\| = inf\{\|u\| : u \in U_{ad}\}. \tag{25}$$

Since $\{u_{n_k}\}$ is weakly convergent to $\tilde{u}$, it follows from (3.3) that $\{u_{n_k}\}$ converges to $\tilde{u}$. Therefore, we see in terms of Theorem 3.1 and (3.4) that

$\tilde{u} = u_0$, namely, $\tilde{u}$ is the minimum energy control element. Thus, $\{u_{n_k}\}$ strongly converges to the minimum energy control element in $L^2([0,T];Y)$. Without loss of generality, we can rewrite $\{u_{n_k}\}$ by $\{u_n\}$, then the conclusion of theorem is now obtained.

Theorem 3.2 points out that the minimum energy control element can be approached by a weakly convergent sequence in the control space, which provides the theoretical basis of approximate computation for finding the minimum energy control element.

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