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QUALITATIVE PROPERTIES OF SOLUTIONS OF RICCATI'S α-DIFFERENCE EQUATIONS

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Abstract: In this paper, by introducing α-difference equation with the definition of generalized α-difference operator, we discuss the general properties and boundedness behaviour of solutins of the generalized Riccati's α-difference equation

$$p(k)u(k+\ell)+\alpha^2 p(k-\ell)u(k-\ell)=\alpha q(k)u(k), k\in[\ell,\infty), \tag{1}$$

where the real valued functions p and q are defined on $[\ell,\infty)$ and $p(k)>0$ for all $k\in[\ell,\infty)$. Equation (1) equivalently can be written as

$$-\Delta_{\alpha(\ell)}\Big(p(k-\ell)\Delta_{\alpha(\ell)}u(k-\ell)\Big)+\alpha f(k)u(k)=0, k\in[\ell,\infty), \tag{2}$$

where $f(k)=q(k)-p(k)-p(k-\ell)$.

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1. Introduction

Difference equations represent a fascinating mathematical area on its own as well as a rich field of the applications in such diverse disciplines as population dynamics, operations research, ecology, economics, biology etc. For general background as difference equations with many examples from diverse fields, one can refer to [1].

The theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k), \qquad k \in [0,\infty)$. Eventhough many authors ([1],[7]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k), \qquad \ell \in (0,\infty), \tag{3}$$

no significant progress took place on this line. But recently, when we took up the definition of Δ as given in (3), the theory of difference equations are developed in a different direction. For convenience, we labelled the operator Δ defined by (3) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results in number theory were obtained. By extending theory of Δ_ℓ to complex function, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were established for the solutions of difference equations involving Δ_ℓ. The results obtained can be found in [2]-[6].

Qualitative properties of solutions of difference equations assume importance in the absence of closed form solutions. In case the solutions are not expressible in terms of the usual known functions, an analysis of the equation is necessary to find the facts of the solutions.

In this paper, we extend the theory of Δ_ℓ to the more generalized difference operator $\Delta_{\alpha(\ell)}$ and its inverse $\Delta_{\alpha(\ell)}^{-1}$. Also, we present the general properties and boundedness behavior of Riccati's α-difference equation 1.

Throughout this paper, we make use of the following assumptions:

(i) α, ℓ are positive reals and n, r are positive integers,

(ii) $[x]$ is the integer part of x,

(iii) $c, c_0, c_1, c_2, \ldots, c_j$ are constants, and

(iv) $[0,\infty)$ is the set of all nonnegative reals.

2. Basic Definitions

In this section, we present some definitions and preliminary results which are used in our subsequent discussion.

Definition 2.1. Let $u(k), k \in [0,\infty)$ be a real or complex valued function. Then the generalized α-difference operator for $u(k)$ is defined as

$$\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k), \ell \in (0,\infty). \tag{4}$$

Lemma 2.2. *If $v(k)$ and $w(k)$ are two functions, then*

$$\Delta_{\alpha(\ell)}\Big(\frac{v(k)}{w(k)}\Big) = \frac{v(k)\Delta_{\alpha(\ell)} w(k) - \alpha w(k)\Delta_{\ell} v(k)}{v(k)v(k+\ell)}. \tag{5}$$

Definition 2.3. The inverse of generalized α-difference operator denoted by $\Delta^{-1}_{\alpha(\ell)}$ is defined as follows. If $\Delta_{\alpha(\ell)} v(k) = u(k)$, then

$$\Delta^{-1}_{\alpha(\ell)} u(k) = v(k) - \alpha^{\left[\frac{k}{\ell}\right]} c_j,\ j = k - \left[\frac{k}{\ell}\right]\ell \tag{6}$$

and the n^{th} order inverse operator denoted by $\Delta^{-n}_{\alpha(\ell)}$ is defined as,

$$\Delta^{-n}_{\alpha(\ell)} u(k) = \Delta^{-1}_{\alpha(\ell)}(\Delta^{-(n-1)}_{\alpha(\ell)} u(k)).$$

Lemma 2.4. *If $k \in [\ell,\infty)$ and $j = k - \left[\frac{k}{\ell}\right]\ell$, then*

$$\Delta^{-1}_{\alpha(\ell)} u(k)\Big|_j^k = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1} u(k - r\ell), \tag{7}$$

Proof. (7) follows by Definition 2.3 and $\Delta^{-1}_{\alpha(\ell)} v(k) - \Delta^{-1}_{\alpha(\ell)} v(j)$, where $v(k) = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1} u(k - r\ell)$. □

Definition 2.5. [3] If $m \in \mathbb{N}(1)$, then the equation of the form

$$f(k, u(k), u(k+\ell), u(k+2\ell), \ldots, u(k+m\ell)) = 0$$

is called the generalized difference equation and

$$f(\alpha, k, u(k), u(k+\ell), u(k+2\ell), \ldots, u(k+m\ell)) = 0$$

is called generalized α-difference equation.

Lemma 2.6. *Consider the second order generalized α-difference equation*

$$p_2(k)u(k+2\ell)+p_1(k)u(k+\ell)+p_0u(k)=0. \tag{8}$$

If $v(k)$ is one of the solution of the equation (8), then

$$w(k)=\frac{1}{v(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\alpha^{r-1}\frac{z(k-r\ell)}{v(k)(v(k+\ell))},$$

where

$$z(k)=v(k)\Delta_{\alpha(\ell)}w(k)-\alpha w(k)\Delta_{\ell}v(k)$$

is the another solution of the equation (8).

Proof. The proof follows from (5) and (7). □

3. General Properties of Solutions of (1)

In this section, we state and prove some oscillatory and nonoscillatory properties of solutions of equation (1).

Property 3.1. Let $v(k)$ and $w(k)$ are two linearly independent solutions of (1). Then

$$\Delta_{\alpha^2(\ell)}(p(k-\ell)(v(k-\ell)w(k)-v(k)w(k-\ell)))=0 \tag{9}$$

and hence there exists constant $c_j\neq 0$, $j=k-\left[\frac{k}{\ell}\right]\ell$ such that

$$p(k)(v(k)w(k+\ell)-v(k+\ell)w(k))=\alpha^{-\left[\frac{2k}{\ell}\right]}c_j, \tag{10}$$

for all $k\in\mathbb{N}_\ell(j)$.

Proof. From the given condition, we find

$$p(k)v(k+\ell)w(k)+\alpha^2p(k-\ell)v(k-\ell)w(k)$$

$$-p(k)w(k+\ell)v(k)-\alpha^2p(k-\ell)w(k-\ell)v(k)=0. \tag{11}$$

which yield (9). Now (10) follows from (11) and Definition 2.3. □

Example 3.2. For the second order generalized α-difference equation

$$k^2u(k-\ell)+\alpha^2(k-\ell)^2u(k-\ell)=\frac{k[(1+\alpha^2)k^2-\alpha^2\ell^2]}{(k+\ell)}u(k) \tag{12}$$

$v(k)=\dfrac{1}{\alpha k}$ and $w(k)=\dfrac{\alpha}{k}\sum\limits_{r=1}^{\left[\frac{k}{\ell}\right]}\dfrac{(k-r\ell)(k-(r-1)\ell)}{(k-(r+1)\ell)^2}\alpha^{2\left[\frac{k}{\ell}\right]-(r+1)}$ are two linearly independent solutions and satisfying condition (9) and (10).

Property 3.3. Let $v(k)$ and $w(k)$ be the two solutions of (1). If $w(k)$ is of fixed sign on $[k_1,\infty)$, where $k_1\in[\ell,\infty)$ then (10) implies that $\Delta_\ell\Big(\frac{v(k)}{w(k)}\Big), 0<\alpha<1$ or $\alpha>1$ is of fixed sign on $[k_1,\infty)$. i.e., $\dfrac{v(k)}{w(k)}$ is monotonic. However it is not possible if $v(k)$ is oscillatory. Thus, if $v(k)$ is oscillatory then $w(k)$ is also oscillatory and then every solution of (1) is oscillatory (nonoscillatory) then the equation itself is oscillatory (nonoscillatory).

Proof. The proof follows by the relations $\Delta_\ell u(k)=u(k+\ell)-u(k)$ and $\dfrac{v(k+j+\ell)}{w(k+j+\ell)}>\dfrac{v(k+j)}{w(k+j)}$ for $0\le j<\ell$. □

Property 3.4. If $|q(k)|\ge\alpha p(k-\ell)+\frac{1}{\alpha}p(k)$, $k\in[\ell,\infty)$ and $v(k)$ is a solution of (1) such that for some $k_1\in[\ell,\infty)$, $|v(k_1+j+\ell)|\ge\alpha|v(k_1+j)|$, for all $j\in[0,\ell)$ then by an easily induction it follows that $|v(k+\ell)|\ge\alpha|v(k)|$ for all $k\in[k_1,\infty)$. Further, if there exists a function $\epsilon(k)\ge0$, $k\in[\ell,\infty)$ such that $\sum^{\infty}\epsilon(k)=\infty$ and $|q(k)|\ge(1+\epsilon(k))(\alpha p(k-\ell)+\frac{1}{\alpha}p(k))$, then $|v(k+\ell)|\ge(1+\epsilon(k))|v(k)|$ for all $k\in[k_1,\infty)$ and consequently $|v(k)|\to\infty$ as $k\to\infty$.

Proof. Replacing k by $k_1+j+\ell$ in (1) and using the given conditions, we obtain $|v(k_1+j+2\ell)|\ge|v(k_1+j+\ell)|$ for all $0\le j<\ell$. Continuing this process repeatedly, we get the proof of first part.

Replacing k by $k_1+j+\ell$ in (1) and using the given conditions, we obtain

$$|v(k_1+j+2\ell)|\ge[1+\epsilon(k_1+j+\ell)]|v(k_1+j+\ell)|, 0\le j<\ell. \tag{13}$$

By replacing k_1 by $k_1+\ell$ again and again in (13), we find

$$|v(k_1+j+(n+1)\ell)|-|v(k_1+j+n\ell)|\ge\epsilon(k_1+j+n\ell)|v(k_1+j+n\ell)|. \tag{14}$$

Now, the proof follows by taking $k=k_1+j+n\ell$. □

Example 3.5. Consider the generalized α-difference equation

$$\frac{1}{k^2}u(k+\ell)+\alpha^2\frac{1}{(k-\ell)^2}u(k-\ell)=\frac{(k^2-\ell^2)+\alpha^2k^2}{k^3(k-\ell)}u(k),k\in[\ell,\infty). \quad (15)$$

Here $p(k)=\frac{1}{k^2}$, $q(k)=\frac{(k^2-\ell^2)+\alpha^2k^2}{\alpha k^3(k-\ell)}$. Now $v(k)=\alpha k$ and

$$w(k)=\big(\frac{k}{\alpha}\big)\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\frac{(k-\ell)}{(k-(r-1)\ell)}\alpha^{2\left[\frac{k}{\ell}\right]-(r+1)}$$

are the two solutions of equation (15) and satisfying the property 3.4 when $\alpha=\frac{1}{2}$.

Property 3.6. If $q(k)\geq \alpha p(k-\ell)+\frac{1}{\alpha}p(k)$, $k\in[\ell,\infty)$ and $v(k)$ is a solution of (1) such that $v(j)=v(\ell+j)=\ell+j$ for all $j\in[0,\ell)$, then $v(k+\ell)\geq\alpha|v(k)|\geq\ell+j$ for all $k\in[\ell,\infty)$ with $j=k-\left[\frac{k}{\ell}\right]\ell$ and hence $v(k)$ is non-oscillatory. Therefore by property 3.3 the α-difference equation (1) is non-oscillatory.

Proof. Replacing k by $k_1+j+\ell$ in (1) and using the given conditions, we obtain $|v(2\ell+j)|\geq|v(\ell+j)|$ for $0\leq j<\ell$. Continuing this process repeatedly, we get the proof. □

Property 3.7. Let $u(k)$ be a non-oscillatory solution of (2), say positive for all $k\in[k_1,\infty)$ and $f(k)\geq 0$ for all $k\in[\ell,\infty)$. Since for the function $v(k)=\alpha^{-\left[\frac{k}{\ell}\right]}u(k)p(k-\ell)\Delta_{\alpha(\ell)}u(k-\ell)$, $\Delta_{\alpha(\ell)}v(k)=p(k)\Delta_{\alpha(\ell)}u(k)\Delta_{\ell}\alpha^{-\left[\frac{k}{\ell}\right]}u(k)+\alpha^{-\left[\frac{k}{\ell}\right]+1}f(k)u^2(k)\geq 0$, if there exists a $k_2\in[k_1,\infty)$ such that $\Delta_{\alpha(\ell)}u(k_2+j)>0$, for all $j\in[0,\ell)$, then $\Delta_{\alpha(\ell)}u(k)>0$ for all $k\in[k_2,\infty)$ and $\alpha>0$. Therefore either $u(k)$ is eventually increasing or eventually nonincreasing for $\alpha\geq 1$.

Proof. The proof follows as in the proof of property 3.4. □

Lemma 3.8. *$v_1(k)$ is solution of the generalized Riccati's equation $p(k)u(k+\ell)+p(k-\ell)u(k-\ell)=q(k)u(k)$ iff $v(k)=\alpha^{\left[\frac{k}{\ell}\right]}v_1(k)$ is solution of equation (1).*

Proof. The proof is obvious. □

4. Boundedness of Solutions of (1)

Theorem 4.1. *Assume that* $f(k) = q(k) - p(k) - p(k-\ell) \geq 0$ *for all* $k \in [\ell, \infty)$ *and* $f(k) \neq 0$ *for infinitely many* k *in the difference equation (1). Then every solution of (1) is bounded on* $[\ell, \infty)$ *if and only if*

$$\sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{f(s)}{p(r)} < \infty. \tag{16}$$

Proof. Let all solutions of (1) be bounded. For $k_1 \in [\ell, \infty)$, we define a solution $\alpha^{-\left[\frac{k}{\ell}\right]} u(k)$ of (1) by setting $u(k_1 + j) = (\ell + j)\alpha^{-\left[\frac{k_1}{\ell}\right]-1}$ and $u(k_1 + j + \ell) = (2\ell + j)\alpha^{-\left[\frac{k_1}{\ell}\right]}$ for $0 \leq j < \ell$. Thus $\Delta_{\alpha(\ell)} u(k_1 + j + \ell) = \ell\alpha^{-\left[\frac{k_1}{\ell}\right]} > 0$ and from (2), we find

$$\begin{aligned} p(k_1 + j + \ell)\Delta_{\alpha(\ell)} u(k_1 + j + \ell) &= \alpha p(k_1 + j)\Delta_{\alpha(\ell)} u(k_1 + j) \\ &\quad + \alpha f(k_1 + j + \ell) u(k_1 + j + \ell) \\ &> \alpha f(k_1 + j + \ell) u(k_1 + j + \ell) \geq 0. \end{aligned}$$

Since $p(k) > 0, \alpha > 0$ and $f(k) \geq 0$, implies that $\Delta_{\alpha(\ell)} u(k_1 + j + \ell) \geq 0$ which yields $u(k_1 + j + 2\ell) \geq 0$. Thus, by induction $u(k) \geq (\ell + j)\alpha^{-\left[\frac{k}{\ell}\right]-1}$ and $\Delta_{\alpha(\ell)} u(k) > 0$ for all $k \in [k_1, \infty)$. Now since from (2), we have

$$\begin{aligned} u(k_1 + j + n\ell + \ell) &= \alpha u(k_1 + j + \ell) + (\alpha - 1)\sum_{r=1}^{n-1} u(k_1 + j + (r+1)\ell) \\ &\quad + \sum_{r=1}^{n} \frac{\alpha p(k_1 + j)\Delta_{\alpha(\ell)} u(k_1 + j)}{p(k_1 + j + r\ell)} \\ &\quad + \sum_{r=1}^{n} \frac{1}{p(k_1 + j + r\ell)} \sum_{s=1}^{r} \alpha f(k_1 + j + s\ell) u(k_1 + j + s\ell), \end{aligned} \tag{17}$$

it follows that

$$\begin{aligned} u(k_1 + j + n\ell + \ell) &\geq (2\ell + j) + \sum_{r=1}^{n} \frac{\alpha p(k_1 + j)\Delta_{\alpha(\ell)} u(k_1 + j)}{p(k_1 + j + r\ell)} \\ &\quad + \sum_{r=1}^{n} \sum_{s=1}^{r} \frac{\alpha f(k_1 + j + s\ell) u(k_1 + j + s\ell)}{p(k_1 + j + r\ell)} \end{aligned}$$

from which it is clear that if $u(k)$ is bounded, then (16) must be satisfied as α is finite.

Conversely, let $u(k)$ be an unbounded solution of (1) so that by properties 3.4 and 3.7, there exists a $k_2 \in [\ell, \infty)$ such that $u(k) > 0$ and $\Delta_{\alpha(\ell)} u(k) > 0$ for all $k \in [k_2, \infty)$. Then by (2), we get

$$f(k) = \frac{\Delta_{\alpha(\ell)}(\frac{1}{\alpha} p(k-\ell)\Delta_{\alpha(\ell)} u(k-\ell))}{u(k)}$$

$$\geq \frac{\frac{1}{\alpha} p(k)\Delta_{\alpha(\ell)} u(k)}{u(k)} - \frac{p(k-\ell)\Delta_{\alpha(\ell)} u(k-\ell)}{\alpha u(k-\ell)}, \; k \in [k_2+\ell, \infty)$$

which yields

$$\frac{1}{p(k_2+j)} \sum_{r=1}^{n} \alpha f(k_2+j+r\ell) + \frac{p(k_2+j+n\ell)\Delta_{\alpha(\ell)} u(k_2+j)}{p(k_2+j+n\ell)u(k_2+j)}$$

$$\geq \frac{\Delta_{\alpha(\ell)} u(k_2+j+n\ell)}{u(k_2+j+n\ell)}$$

and hence

$$\sum_{r=1}^{n}\sum_{s=1}^{r} \frac{\alpha f(k_2+j+s\ell)}{p(k_2+j+r\ell)} + \frac{p(k_2+j)\Delta_{\alpha(\ell)} u(k_2+j)}{u(k_2+j)} \sum_{r=1}^{n} \frac{1}{p(k_2+j+r\ell)}$$

$$\geq \sum_{r=1}^{n} \frac{\Delta_{\alpha(\ell)} u(k_2+j+r\ell)}{u(k_2+j+r\ell)}. \tag{18}$$

For $k_2+j+r\ell \leq t \leq k_2+j+r\ell+\ell$, define

$$\rho_\alpha(t) = \ell u(k_2+j+r\ell) + [t-(k_2+j+r\ell)]\Delta_{\alpha(\ell)} \frac{1}{\alpha} u(k_2+j+r\ell). \tag{19}$$

Then $\rho'_\alpha(t) = \Delta_{\alpha(\ell)} \frac{1}{\alpha} u(k_2+j+r\ell)$ and $\rho_\alpha(t) \geq \ell u(k_2+j+r\ell)$. Hence, we have

$$\frac{\Delta_{\alpha(\ell)} u(k_2+j+r\ell)}{\ell u(k_2+j+r\ell)} \geq \frac{\alpha \rho'_\alpha(t)}{\rho_\alpha(t)}. \tag{20}$$

From (19) and integrating (20) from $k_2+j+r\ell$ to $k_2+j+r\ell+\ell$, we get

$$\frac{\Delta_{\alpha(\ell)} u(k_2+j+r\ell)}{u(k_2+j+r\ell)} \geq \alpha[\ln u(k_2+j+r\ell+\ell) - \ln u(k_2+j+r\ell)] \tag{21}$$

which yields

$$\sum_{r=1}^{n} \frac{\Delta_{\alpha(\ell)} u(k_2+j+r\ell)}{u(k_2+j+r\ell)} \geq \alpha \sum_{r=1}^{n} [\ln u(k_2+j+r\ell+\ell) - \ln u(k_2+j+r\ell)]. \quad (22)$$

Hence, for $0 \leq j < \ell$ and $n = [\frac{k-k_2}{\ell}]$, we have

$$\sum_{r=1}^{n} \frac{\Delta_{\alpha(\ell)} u(k_2+j+r\ell)}{u(k_2+j+r\ell)} \geq \alpha[\ln u(k_2+j+n\ell+\ell) - \ln u(k_2+j+\ell)]. \quad (23)$$

Since $f(k) \neq 0$ for infinitely many k, there exists $0 \leq j < \ell$ and $m \geq 1$ such that $k_3 \geq k_2 + j + \ell$ and $f(k_3) \neq 0$, which yields

$$\sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{\alpha f(s)}{p(r)} \geq \sum_{t=k_3}^{\infty} \frac{\alpha f(k_3)}{p(k_2+j+r\ell)}. \quad (24)$$

From our assumptions, (18), (23) and (24), we get a contradiction

$$\alpha[\ln u(k_2+j+n\ell) - \ln u(k_2+j+\ell)] < \infty \; as \; n \to \infty.$$

□

Example 4.2. Consider the generalized α-difference equation

$$k^2 u(k+\ell) + \alpha^2 (k-\ell)^2 u(k-\ell) = \alpha k^2 \Big[\alpha + \frac{k}{\alpha(k+\ell)^2}\Big] u(k). \quad (25)$$

Now $v(k) = \dfrac{1}{(\alpha k)^2}$ and $w(k) = \left(\frac{\alpha}{k}\right)^2 \sum\limits_{r=1}^{\left[\frac{k}{\ell}\right]} \left(\frac{(k-\ell)(k-(r-1)\ell)}{(k-(r+1)\ell)}\right)^2 \alpha^{2\left[\frac{k}{\ell}\right]-(r+1)}$ are the solutions of (25). Thus every solutions of (25) is bounded iff

$$\sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{1}{r^2} \left\{ s^2 \Big(\alpha + \frac{s}{\alpha(s+\ell)^2}\Big) - s^2 - (s-\ell)^2 \right\} < \infty.$$

Theorem 4.3. *Assume that* $f(k) = q(k) - p(k) - p(k-\ell) \geq 0$ *for all* $k \in [\ell, \infty)$. *Then for every solution* $u(k)$ *of (1), the function* $\phi(k) = p(k)\Delta_{\alpha(\ell)} u(k)$ *is bounded on* $[\ell, \infty)$ *if and only if*

$$\sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{f(r+\ell)}{p(s)} < \infty. \quad (26)$$

Proof. As in Theorem 4.1, for the solution $u(k)$ of (1) satisfying $u(k_1+j) = (\ell+j)\alpha^{-\left[\frac{k_1}{\ell}\right]-1}$, $u(k_1+j+\ell) = (2\ell+j)\alpha^{-\left[\frac{k_1}{\ell}\right]}$, there exists $k_1 \in [\ell, \infty)$ such that $u(k) > 0$ and $\Delta_{\alpha(\ell)}u(k) > 0$, for all $k \in [k_1, \infty)$. For this $u(k)$, since $\Delta_{\alpha(\ell)}(\frac{1}{\alpha}p(k)\Delta_{\alpha(\ell)}u(k)) = f(k+\ell)u(k+\ell) \geq 0$, we have

$$p(k_1+j+n\ell)\Delta_{\alpha(\ell)}u(k_1+j+n\ell) = p(k_1+j)\Delta_{\alpha(\ell)}u(k_1+j)$$

and hence

$$u(k_1+j+n\ell+\ell) \geq \alpha u(k_1+j) + (\alpha-1)\sum_{r=1}^{n} u(k_1+j+r\ell)$$

$$+p(k_1+j)\Delta_{\alpha(\ell)}u(k_1+j)\sum_{r=1}^{n}\frac{1}{p(k_1+j+r\ell)}.$$

Using this in (2), we obtain

$$\Delta_{\alpha(\ell)}(\frac{1}{\alpha}p(k_1+j+n\ell)\Delta_{\alpha(\ell)}u(k_1+j+n\ell)) = f(k_1+j+n\ell+\ell)u(k_1+j+n\ell+\ell)$$

$$\geq \alpha f(k_1+j+n\ell+\ell)u(k_1+j) + (\alpha-1)f(k_1+j+n\ell+\ell)\sum_{r=1}^{n}u(k_1+j+r\ell)$$

$$+f(k_1+j+n\ell+\ell)p(k_1+j)\Delta_{\alpha(\ell)}u(k_1+j)\sum_{r=1}^{n}\frac{1}{p(k_1+j+r\ell)},$$

which gives that

$$p(k_1+j+n\ell+\ell)\Delta_{\alpha(\ell)}u(k_1+j+n\ell+\ell) \geq p(k_1+j)\Delta_{\alpha(\ell)}u(k_1+j)$$

$$+\alpha^2 u(k_1+j)\sum_{r=1}^{n} f(k+j+(r+1)\ell) + \alpha(\alpha-1)\sum_{r=1}^{n} p(k_1+j+(r+1)\ell)$$

$$\times\sum_{s=1}^{r} u(k_1+j+s\ell) + p(k_1+j)\Delta_{\alpha(\ell)}u(k_1+j)$$

$$+\sum_{r=1}^{n}\sum_{s=1}^{r}\frac{\alpha f(k_1+j+(r+1)\ell)}{p(k_1+j+s\ell)},\ 0 \leq j < \ell.$$

Thus, if $p(k)\Delta_{\alpha(\ell)}u(k)$ is bounded then (26) must hold.

Conversely, we may assume that $u(k)$ is eventually positive. By property 3.7, we may also assume that $u(k)$ is eventually increasing or nonincreasing. If

$\alpha > 1$ and $u(k)$ is eventually nonincreasing, then $p(k)\Delta_{\alpha(\ell)}u(k) \le 0$ for all large k. Further

$$\Delta_{\alpha(\ell)}(\frac{1}{\alpha}p(k)\Delta_{\alpha(\ell)}u(k)) = f(k+\ell)u(k+\ell) \ge 0,$$

which means that $p(k)\Delta_{\alpha(\ell)}u(k)$ is nondecresing as well and hence $p(k)\Delta_{\alpha(\ell)}u(k)$ must be bounded on $[\ell, \infty)$.

Now, assume that $u(k)$ is eventually increasing. Then there exists a $k_2 \in [\ell, \infty)$ such that $u(k) > 0$ and $\Delta_{\alpha(\ell)}u(k) > 0$ for all $k \in [k_2, \infty)$. Thus it follows that

$$\Delta_{\alpha(\ell)}\left(\frac{u(k_2+j+\ell)}{p(k_2+j)\Delta_{\alpha(\ell)}u(k_2+j)}\right)$$

$$= \frac{p(k_2+j)\Delta_{\alpha(\ell)}u(k_2+j)\Delta_{\alpha^2(\ell)}u(k_2+j+\ell)}{p(k_2+j)p(k_2+j+\ell)\Delta_{\alpha(\ell)}u(k_2+j)\Delta_{\alpha(\ell)}u(k_2+j+\ell)}$$

$$-\frac{\alpha^2 f(k_2+j+\ell)u^2(k_2+j+\ell)}{p(k_2+j)p(k_2+j+\ell)\Delta_{\alpha(\ell)}u(k_2+j)\Delta_{\alpha(\ell)}u(k_2+j+\ell)}$$

$$\le \frac{1}{p(k_2+j+\ell)}, \text{ and hence}$$

$$\frac{u(k_2+j+n\ell+\ell)}{p(k_2+j+n\ell)\Delta_{\alpha(\ell)}u(k_2+j+n\ell)} \le \frac{\alpha u(k_2+j+n\ell)}{p(k_2+j)\Delta_{\alpha(\ell)}u(k_2+j)} + (\alpha-1)$$

$$\sum_{r=1}^{n-1}\frac{u(k_2+j+(r+1)\ell)}{p(k_2+j+r\ell)\Delta_{\alpha(\ell)}u(k_2+j+r\ell)} + \sum_{r=1}^{n}\frac{1}{p(k_2+j+(r+1)\ell)}.$$

This implies that

$$\frac{\Delta_{\alpha(\ell)}(\frac{1}{\alpha}p(k_2+j+n\ell)\Delta_{\alpha(\ell)}u(k_2+j+n\ell))}{p(k_2+j+n\ell)\Delta_{\alpha(\ell)}u(k_2+j+n\ell)}$$

$$= \frac{f(k_2+j+n\ell+\ell)u(k_2+j+n\ell+\ell)}{p(k_2+j+n\ell)\Delta_{\alpha(\ell)}u(k_2+j+n\ell)}$$

$$\le \frac{\alpha f(k_2+j+n\ell+\ell)u(k_2+j+n\ell+\ell)}{p(k_2+j)\Delta_{\alpha(\ell)}u(k_2+j)} + (\alpha-1)\times$$

$$\sum_{r=1}^{n-1}\frac{f(k_2+j+n\ell+\ell)u(k_2+j+(r+1)\ell)}{p(k_2+j+r\ell)\Delta_{\alpha(\ell)}u(k_2+j+r\ell)} + \sum_{r=1}^{n}\frac{f(k_2+j+n\ell+\ell)}{p(k_2+j+(r+1)\ell)},$$

which on using an argument similar to the one used in the derivation of (20) leads to

$$\sum_{r=1}^{n}\sum_{s=1}^{r}\frac{f(k_2+j+r\ell+\ell)}{p(k_2+j+s\ell)}$$

$$+(\alpha-1)\sum_{r=1}^{n}\sum_{s=1}^{r-1}\frac{f(k_2+j+(r+1)\ell)u(k_2+j+(r+1)\ell)}{p(k_2+j+s\ell)\Delta_{\alpha(\ell)}u(k_2+j+s\ell)}$$

$$+\alpha\sum_{r=1}^{n}\sum_{s=1}^{r}\frac{f(k_2+j+(r+1)\ell)u(k_2+j+(s+1)\ell)}{p(k_2+j)\Delta_{\alpha(\ell)}u(k_2+j)}$$

$$\geq\sum_{r=1}^{n}\frac{\Delta_{\alpha(\ell)}(\frac{1}{\alpha}p(k_2+j+r\ell)\Delta_{\alpha(\ell)}u(k_2+j+r\ell))}{p(k_2+j+r\ell)\Delta_{\alpha(\ell)}u(k_2+j+r\ell)}$$

$$\geq\frac{1}{\alpha}[\ln(p(k_2+j+r\ell+\ell)\Delta_{\alpha(\ell)}u(k_2+j+r\ell+\ell))$$

$$-\ln(p(k_2+j+\ell)\Delta_{\alpha(\ell)}u(k_2+j+\ell))]. \qquad (27)$$

Furthermore, by reasoning similar to that used in obtaining (24), we get

$$\sum_{r=1}^{\infty}f(k_2+j+(r+1)\ell)<\infty.$$

But this (26) in (27) then implies that $p(k)\Delta_{\alpha(\ell)}u(k)$ is bounded on $[\ell,\infty)$. □

Example 4.4. In the equation 25, taking $p(k)=k^2$, $q(k)=k^2\Big(\alpha+\frac{k}{\alpha(k+\ell)^2}\Big)$, then we obtain $\phi(k)=\frac{1}{\left(1+\frac{\ell}{k}\right)^2}-\alpha$. $\phi(k)$ is bounded iff

$$\sum_{r=1}^{\infty}\sum_{s=1}^{r}\frac{1}{s^2}\left\{(r+\ell)^2\Big(\alpha+\frac{(r+\ell)}{\alpha(r+2\ell)^2}\Big)-(r+\ell)^2-r^2\right\}<\infty.$$

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