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NUMERICAL STUDY OF FRACTIONAL DIFFERENTIAL EQUATIONS OF LANE-EMDEN TYPE BY THE LEAST SQUARE METHOD

Mohammed S. Mechee[1 §], Norazak Senu[2]

[1]Institute of Mathematical Sciences
University of Malaya
50603, Kuala Lumpur, MALAYSIA

[1]Department of Mathematics
Faculty of Mathematics and Computer Sciences
University of Kufa
Najaf, IRAQ

[2]Department of Mathematics
Institute for Mathematical Research
Universiti Putra Malaysia
43400, UPM Serdang Selangor, MALAYSIA

Abstract: Lane-Emden differential equations of order fractional has been studied. The numerical solution of this type is considered by the least square method. Some of examples are illustrated and a comparison between numerical and analytic methods has been introduced.

Key Words: fractional calculus, fractional differential equation, Lane-Emden equation, the Least square method

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[§]Correspondence author

1. Introduction

Lane-Emden Differential Equations have the following form [16]

$$y''(t) + \frac{k}{t} y'(t) + f(t, y) = g(t), \quad 0 < t \leq 1, \quad k \geq 0, \tag{1}$$

with the initial condition

$$y(0) = A, \quad y'(0) = B,$$

where A, B are constants, $f(t, y)$ is a continuous real valued function and $g(t) \in C[0, 1]$. Where A and B are constants, f(x, y) is a continues real valued function and g(x). Lane-Emden differential equations are singular initial value problems relating to second order differential equations (ODEs) which have been used to model several phenomena in mathematical physics and astrophysics.

In this paper we generalize the definition of Lane-Emden equations up to fractional order as following:

$$D^{\alpha} y(t) + \frac{k}{t^{\alpha-\beta}} D^{\beta} y(t) + f(t, y) = g(t), \\ 0 < t \leq 1, \quad k \geq 0, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \tag{2}$$

with the initial condition

$$y(0) = A, \quad y'(0) = B,$$

where A, B are constants, $f(t, y)$ is a continuous real-valued function and $g(t) \in C[0, 1]$. The theory of singular boundary value problems has become an important area of investigation in the past three decades (see [1-5]). One of the equations describing this type is the Lane-Emden equation. Lane-Emden type equations, first published by Jonathan Homer Lane in 1870 [6], and further explored in detail by Emden [7], represents such phenomena and having significant applications, is a second- order ordinary differential equation with an arbitrary index, known as the polytropic index, involved in one of its terms. The Lane-Emden equation describes a variety of phenomena in physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents [8].

The solution of the Lane-Emden problem, as well as other various linear and nonlinear singular initial value problems in quantum mechanics and astrophysics, is numerically challenging because of the singularity behavior at the origin. The approximate solutions to the Lane-Emden equation were given by homotopy perturbation method [9], variational iteration method [10], and Sinc-Collocation method [11], an implicit series solution [12]. Recently, Parand et. al [13] proposed an approximation algorithm for the solution of the nonlinear Lane-Emden type equation using Hermite functions collocation method. Moreover, Adibi and Rismani [14] introduced a modified Legendre-spectral method. while, Bhrawy and Alofi [15] imposed a Jacobi-Gauss collocation method for solving nonlinear Lane-Emden type equations, Finally, Yigider [16] introduced numerical study of Lane-Emaden Type using Pade Approximation.

2. Fractional Calculus

Fractional calculus and its applications (that is the theory of derivatives and integrals of any arbitrary real or complex order) has importance in several widely diverse areas of mathematical physical and engineering sciences. It generalized the ideas of integer order differentiation and n-fold integration. Fractional derivatives introduce an excellent instrument for the description of general properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids and rocks, and in many other fields (see [17, 18]).

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators [18], Erdlyi-Kober operators [19], Weyl-Riesz operators [20], Caputo operators [21] and Grnwald-Letnikov operators [22], have appeared during the past three decades. The existence of positive solution and multi-positive solutions for nonlinear fractional differential equation are established and studied [23]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are suggested and posed in [24, 25].

One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators (see[22]). The Riemann-Liouville fractional derivative could hardly pose the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. Moreover, this operator possesses advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms (see[26]).

Definition 2.1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^{\alpha} f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau.$$

When $a = 0$, we write $I_a^{\alpha} f(t) = f(t) * \phi_{\alpha}(t)$, where $(*)$ denoted the convolution product (see [22]), $\phi_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_{\alpha}(t) = 0,\ t \leq 0$ and $\phi_{\alpha} \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2. The fractional (arbitrary) order derivative of the function f of order $0 \leq \alpha < 1$ is defined by

$$D_a^{\alpha} f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Remark 2.1. From Definition 2.1 and Definition 2.2, we have

$$D^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \ \ \mu > -1;\ 0 < \alpha < 1$$

and

$$I^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \ \ \mu > -1;\ \alpha > 0.$$

In this note, we consider the fractional Lane-Emden equations of the in Equation (2).

3. Analytic Solution

Consider that we are given a power series representing the solution of fractional Lane-Enden differential equations:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \tag{3}$$

hence

$$D^{\gamma} y(t) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)} t^{n-\gamma} \tag{4}$$

Theorem. *The analytic solution of the IVP(2) satisfied the following equation:*

$$\frac{a_1}{\Gamma(2-\beta)}t^{1-\alpha} + \sum_{n=2}^{\infty} a_n(\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} + \frac{\Gamma(n+1)}{\Gamma(n+1-\beta})t^{n-\alpha}$$
$$+ f(t, \sum_{n=0}^{\infty} a_n t^n) = g(t). \quad (5)$$

Proof. Substitute (3) and (4) into equation (2), we obtain the desired equation.

The method of power series depends to find the coefficients a_{n+k} as a function of n and a_n.

3.1. Linear Lane-Emden Fractional Differential Equation

Consider $f(t, y) = \frac{1}{t^{\alpha-2}}y(t)$ in equation (2), thus

$$D^{\alpha}y(t) + \frac{k}{t^{\alpha-\beta}}D^{\beta}y(t) + \frac{1}{t^{\alpha-2}}y(t) = g(t) \quad (6)$$

with the initial condition

$$y(0) = A, \quad y'(0) = B,$$

Equation (5) convert to the following equation

$$a_0 t^{2-\alpha} + a_1(\frac{t^{1-\alpha}}{\Gamma(1-\beta} + t^{3-\alpha}) + \sum_{n=2}^{\infty} a_n(\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} + k\frac{\Gamma(n+1)}{\Gamma(n+1-\beta})t^{n-\alpha}$$
$$+ \sum_{n=2}^{\infty} a_{n-2}t^{n-\alpha} = g(t) \quad (7)$$

In case $g(t) = 0$, we obtain $a_0 = a_1 = 0$ and in general

$$a_n = \frac{\Gamma(n+1-\alpha)\Gamma(n+1-\beta)}{\Gamma(n+1)(\Gamma(n+1-\alpha) + k\Gamma(n+1-\beta))}a_{n-2} \quad for \quad n = 2, 3, ... \quad (8)$$

3.1.1. Examples

Example 3.1.1.1. Let $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$, we pose the linear FDE

$$D^{\frac{3}{2}}y(t) + \frac{k}{t}D^{\frac{1}{2}}y(t) + t^{\frac{1}{2}}y(t) = g(t) \quad (9)$$

with the initial condition

$$y(0) = A, \quad y'(0) = B,$$

Consider the solution of FDE is $y(t) = \sum_{n=0}^{\infty} a_n t^n$

Consequently, we have

$$a_0 t^{\frac{1}{2}} + a_1 (\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + t^{\frac{3}{2}}) + \sum_{n=2}^{\infty} a_n (\frac{\Gamma(n+1)}{\Gamma(n+-\frac{1}{2})} + k\frac{\Gamma(n+1)}{\Gamma(n++\frac{1}{2}})t^{n-\frac{3}{2}} + \sum_{n=2}^{\infty} a_{n-2} t^{n-\frac{3}{2}} = g(t). \quad (10)$$

Hence

$$a_n = \frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})}{\Gamma(n+1)(\Gamma(n-\frac{1}{2}) + k\Gamma(n+\frac{1}{2}))} a_{n-2} \quad \text{for} \quad n = 2, 3, \dots . \quad (11)$$

Example 3.1.1.2. Let $\alpha = \frac{3}{2}, \beta = 1$, we get the linear FDE

$$D^{\frac{3}{2}} y(t) + \frac{k}{t^{\frac{1}{2}}} Dy(t) + t^{\frac{1}{2}} y(t) = g(t) \quad (12)$$

with the initial condition

$$y(0) = A, \quad y'(0) = B.$$

Consider the solution of FDE is $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

Consequently, we have

$$a_0 t^{\frac{1}{2}} + a_1 (t^{-\frac{1}{2}} + t^{\frac{3}{2}}) + \sum_{n=2}^{\infty} a_n (\frac{\Gamma(n+1)}{\Gamma(n-\frac{1}{2})} + k\frac{\Gamma(n+1)}{\Gamma(n)})t^{n-\frac{3}{2}} + \sum_{n=2}^{\infty} a_{n-2} t^{n-\frac{3}{2}} = g(t), \quad (13)$$

with

$$a_n = \frac{\Gamma(n-\frac{1}{2})\Gamma(n)}{\Gamma(n+1)(\Gamma(n-\frac{1}{2}) + k\Gamma(n))} a_{n-2} \quad \text{for} \quad n = 2, 3, \dots . \quad (14)$$

4. The Least Square Method

The least square method for solving differential equations [28, 29] is one of the most powerful approximate methods for solving fractional differential equations. This method is based on the approximate solution of FDE by a series of complete sequence of functions where a complete sequence of functions means a sequence of linearly independent functions that have no non zero function perpendicular to this sequence of functions. In general, $y(t)$ is approximated by

$$y(t) = \sum_{i=0}^{n} a_i \Theta_i(t) \tag{15}$$

Where a_i for $i = 0, 1, 2, ..., n$ are arbitrary constants to be evaluated and $\Theta_i(t)$ for $i = 0, 1, 2, ..., n$ is given of a set of functions. Therefore, the problem in equation (6) of evaluating $y(t)$is approximated by (15) then is reduced to the problem of evaluating the coefficients a_i for $i = 0, 1, 2, ..., n$.

Let $\{t_0, t_1, t_2, ..., t_n\}$ is a partition to interval [0, 1] and $t_j = jh$ for $j = 0, 1, 2, ..., n$ and $h = \frac{1}{n}$.

Consider

$$D_x^{\alpha} y(x) = g(x) \quad a \leq t \leq b \tag{16}$$

as a fractional differential equation with the initial condition

$$y(0) = A, \quad y'(0) = B,$$

where A, B are constants and $D_x^{\alpha}\phi(x)$is a fractional operator.

The least square method transforming the problem of evaluating the coefficients in approximation form(15) $a_0, a_1, ..., a_n$, gives the best approximation to the solution of fractional differential equation(16). However, to find out the coefficients during minimizing the functional error J, the following should be noted:

1. Define the residual error $E(x) = D_x^{\alpha} y(x) - g(x)$
2. Define the functional $J = \int_a^b (D_x^{\alpha} y(x) - g(x))^2 dx$
3. Consider the solution of equation (16) as following

$$y(x) = A + Bx + \sum_{i=2}^{n} \alpha_i x^i \tag{17}$$

Then,

$$J(a_2, a_3, ..., a_n) = \int_a^b (D_x^{\alpha}(A + Bx + \sum_{i=2}^{n} a_i x^i) - g(x))^2) dx \tag{18}$$

4. Evaluate the values of a_k for k=2, 3, ..., n, as minimized J, by solving linear system $Ax = b$, which is already obtained by letting $\frac{\partial J}{\partial a_k} = 0$ for k=2, 3, ..., n operate by D_x^α to obtain

$$D_x^\alpha(y(x)) = AD_x^\alpha(1) + BD_x^\alpha(x) + \sum_{i=2}^{n} a_i D_x^\alpha(x^i)$$

5. Put $x = x_j$ to get $\sum_{i=2}^{n} a_i D_x^\alpha(t_j^i) = g(t_j) - AD_x^\alpha(1) - BD_x^\alpha(x)$

Then, a linear system $Ax = b$ of $n-1$ equations in $n-1$ variables is obtained, where $a_{ij} = D_x^\alpha(x_j^i)$, $b_j = g(t_j)$ for $i, j = 2, 3, ..., n$.

Hence, from equation(18) the linear system Ax=b is obtained and can be solved by using any numerical method for solving linear systems of algebraic equations

4.1. Numerical Examples

To implement our examples, we used Matlab R2009b on Intel(R)core TM2Duo processor with 3.00 GHZ and 3 GB RAM

Example 4.1.1.

$$D^\alpha y(t) + \frac{k}{t^{\alpha-\beta}} D^\alpha y(t) + \frac{1}{t^{\alpha-2}} y(t) = (6t(\frac{\Gamma(4-\beta) + k(\Gamma(4-\alpha))}{\Gamma(4-\beta)\Gamma(4-\alpha)} + \frac{t^2}{6}) - 2(\frac{\Gamma(3-\beta) + k(\Gamma(3-\alpha)}{\Gamma(3-\beta)\Gamma(3-\alpha)} + \frac{t^2}{2}))t^{2-\alpha}, \quad (19)$$

with the initial condition $y(0) = 0, \quad y'(0) = 0$ Hence

$$D_t^\alpha(t^i) = \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} t^{i-\alpha} + k\frac{\Gamma(i+1)}{\Gamma(i+1-\beta)} t^{i-\alpha} + x^{i+2-\alpha}.$$

See Table1 and Figure 1, where the exact solution is $y(t) = t^3 - t^2$ and $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$

Example 4.1.2.

$$D^\alpha y(t) + \frac{k}{t^{\alpha-\beta}} D^\alpha y(t) + \frac{1}{t^{\alpha-2}} y(t) = (2(\frac{\Gamma(3-\beta) + k(\Gamma(3-\alpha)}{\Gamma(3-\beta)\Gamma(3-\alpha)} + \frac{t^2}{2}) - 6t(\frac{\Gamma(4-\beta) + k(\Gamma(4-\alpha))}{\Gamma(4-\beta)\Gamma(4-\alpha)} + \frac{t^2}{6}))t^{2-\alpha}, \quad (20)$$

with the initial condition $y(0) = 0, \quad y'(0) = 0.$

Hence

$$D_t^{\alpha}(t^i) = \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} t^{i-\alpha} + k\frac{\Gamma(i+1)}{\Gamma(i+1-\beta)} t^{i-\alpha} + x^{i+2-\alpha}.$$

See Table 2 and Figure 2, where the exact solution is $y(t) = t^2 - t^3$ and $\alpha = \frac{3}{2}$, $\beta = 1$.

5. Conclusion

From above, we imposed the Lane-Emden differential equation of fractional order. The generality of definition of Lane-Emden as a fractional order is more importance in applied mathematics, mathematical physics and in astrophysics as well. Basically the order appeared in two different fractional powers where an approximate solution is obtained by employing the method of power series.and a numerical solution is established by the least squares method for these equations.

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Appendix

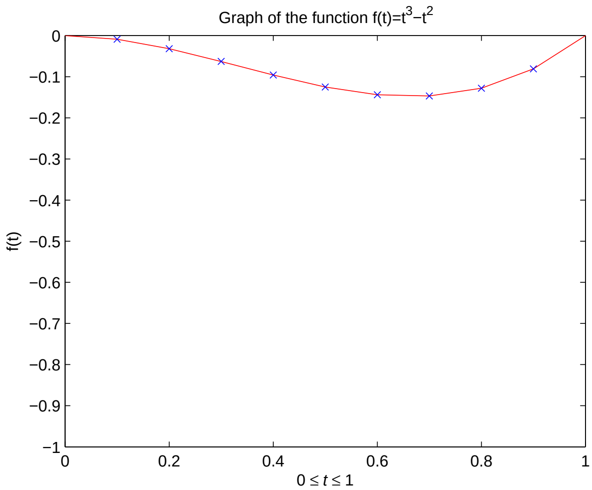

Figure 1: Numerical and analytic graph of solution of Example 4.1

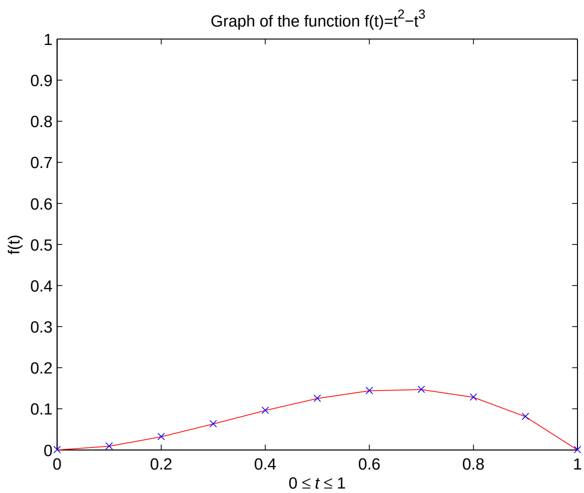

Figure 2: Numerical and analytic graph of solution of Example 4.2

$n \setminus t$	0	0.25	0.5	0.75	1
5	0	1.1670e-3	0.0023	5.1239e-3	2.6119e-3
10	0	1.7483e-5	2.7837e-5	2.5362e-6	1.2734e-5
50	0	2.9321e-7	1.1271e-7	3.1145e-7	3.5362e-7
100	0	2.7848e-8	3.7483e-8	5.0102e-8	5.4345e-7

Table 1: Absolute error of numerical solution of Example 4.1

$n \setminus t$	0	0.25	0.5	0.75	1
5	0	1.1291e-3	0.0120	5.0112e-3	3.1121e-3
10	0	1.1231e-5	1.0234e-5	2.6351e-5	3.1212e-5
50	0	2.0127e-6	1.2267e-7	5.1234e-7	2.3322e-6
100	0	4.7124e-8	2.0123e-8	3.9811e-8	6.1141e-7

Table 2: Absolute error of numerical solution of Example 4.2