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ON THE VERHULST GROWTH MODEL WITH "POLYNOMIAL VARIABLE TRANSFER". SOME APPLICATIONS

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ABSTRACT: Verhulst model [1] makes an extensive use of the logistic sigmoidal function $S(t) = \frac{a}{1+e^{-kt}}$. Studying "Canteloup growth", Pearl et al. [2]–[3] empirically found that one should generalized the logistic map in order to reproduce better the data. We consider a new class of growth curves, generated by reaction networks, based on the insertion of "correcting amendments" of polynomial–type: $M(t) = \frac{1}{1+e^{-F(t)}}$ where $F(t) = \sum_{i=0}^{n} a_i t^i$. We will call this family the **"Verhulst curve of growth with polynomial variable transfer" (VCGPVT)**. The new coronavirus [29], SARS-CoV-2, is the reason for a new disease, Covid-19. As of 22th February, there have been more than 77,977 cases, including 54,298 laboratory confirmed cases. There have been 2,366 deaths [28]. Below we look at some comparisons between the Verhulst model and the new model (VCGPVT), as well as the ability to approximate specific population dynamics data, including "Data Corona Virus". Some numerical examples with real data, using *CAS MATHEMATICA* illustrating our results are given.

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Key Words: Verhulst logistic model, Verhulst growth model with polynomial variable transfer, Heaviside step–function, Hausdorff distance

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1. INTRODUCTION

Verhulst model [1] makes an extensive use of the logistic sigmoidal function

$$s(t) = \frac{a}{1+e^{-kt}}. \tag{1}$$

More precise bounds for the H–distance d between the Heaviside step function h_0 and the sigmoid Verhulst function $s_0 = 1/(1+e^{-kt})$ can be found in [5]:

Theorem A. For the H–distance [7] $d = \rho(h_0, s_0)$ between the Heaviside step function h_0 and the sigmoid Verhulst function s_0 the following inequalities hold for $k \geq 2$:

$$\begin{aligned} d_l = \frac{\ln(k+1)}{k+1} - \frac{\ln\ln(k+1)}{(k+1)\left(1+\frac{1}{\ln(k+1)}\right)} < d < \\ < \frac{\ln(k+1)}{k+1} + \frac{\ln\ln(k+1)}{(k+1)\left(\frac{\ln\ln(k+1)}{1-\ln(k+1)}-1\right)} = d_r. \end{aligned} \tag{2}$$

Studying "Canteloup growth", Pearl et al. [2]–[3] empirically found that one should generalized the logistic map in order to reproduce better the data. Rather than the mere logistic, they propose a form like

$$y(t) = \frac{r}{1+e^{a_0+a_1t+a_2t^2+\ldots}},$$

where y, is the number of seedlings of the canteloups. An extensive overview on the topic can be found in [4], see also [5]–[6]). In this paper we consider a class of growth curves, generated by reaction networks, based on the insertion of "correcting amendments" of polynomial–type.

2. MAIN RESULTS

1. Formally, we will define the following class of growth curves:

$$M(t) = \frac{1}{1+e^{-F(t)}} \tag{3}$$

where

$$F(t) = \sum_{i=0}^{n} a_i t^i. \tag{4}$$

We will call this family the **"Verhulst curve of growth with formally polynomial variable transfer" (VCGFPVT)**. The question of finding precise two-sided estimates for the magnitude of the Hausdorff approximation of the Heaviside function with classes of the indicated family $M(t)$ remains open. The task is greatly complicated

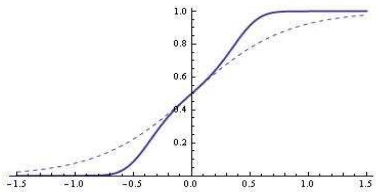

Figure 1: Comparison between Verhulst's model $s_0(t)$ - (dashed) and model (VCGFPVT) - (thick) at fixed $k = 2.5$, $n = 4$, $a_0 = -0.00002$, $a_1 = 2.5$, $a_2 = -0.001$, $a_3 = 8.9$, $a_4 = -0.001$.

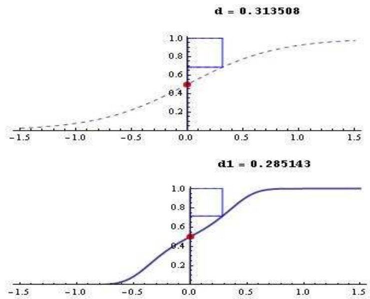

Figure 2: a) Hausdorff distance $d = 0.313508$; b) "Hausdorff saturation" $d_1 = 0.285143$.

by the intrinsic properties of the generated class of functions, as well as by the type and location of zeros of polynomial $F(t)$. Methods for simultaneous approximation of all roors of a given polinomial of degree n can be found in [8]–[9]. Below we look at some comparisons between the Verhulst model and the new model (VCGFPVT), as well as the ability to approximate specific population dynamics data. Some comparisons between Verhulst's model $s_0(t)$ and the model (VCGFPVT) are visualized on Fig. 1 – Fig. 4.

Example 1. We examine the dataset, was proposed by Musa in [10]. For the first 12 hours of testing, the number of failures each hour is given in Fig. 5.

Below, we will illustrate the fitting of this data, for example, with the $M^*(t) = \omega M(t)$ model. The fitted model $M^*(t)$ based on the dataset for the estimated parameters: $\omega = 104$, $n = 4$, $a_0 = -2$, $a_1 = 0.972572$, $a_2 = -0.100246$, $a_3 = 0.00425456$, $a_4 = -0.00004281021$ is plotted on Fig. 6.

The fitted Verhulst model $s(t) = \frac{\omega}{1+be^{-kt}}$ based on the dataset for the estimated parameters: $\omega = 104$, $k = 0.389731$ and $b = 3.19675$ is plotted on Fig. 7. Constructive

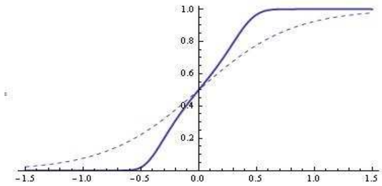

Figure 3: Comparison between Verhulst's model $s_0(t)$ - (dashed) and model (VCGFPVT) - (thick) at fixed $k = 2.5$, $n = 6$, $a_0 = -0.00002$, $a_1 = 3.5$, $a_2 = -0.1$, $a_3 = 10$, $a_4 = -0.1$, $a_5 = 20.1$, $a_6 = 15.9$.

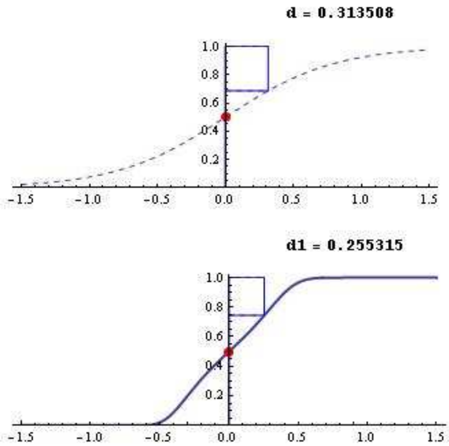

Figure 4: a) Hausdorff distance $d = 0.313508$; b) "Hausdorff saturation" $d_1 = 0.255315$.

Hour	*Number of Failures*	*Cumulative failures*
1	27	27
2	16	43
3	11	54
4	10	64
5	11	75
6	7	82
7	2	84
8	5	89
9	3	92
10	1	93
11	4	97
12	7	104

Figure 5: Dataset [10].

approximation theory by superposition of sigmoidal functions can be found in [11]. For contemporary applicable study on sigmoids and some of their applications see the

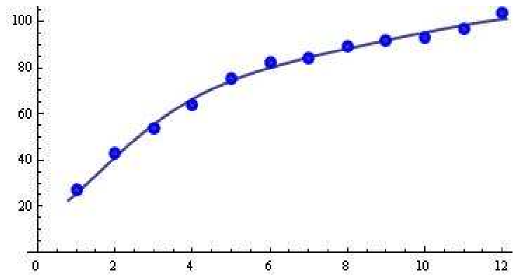

Figure 6: The model $M^*(t)$ with $\omega = 104$, $n = 4$, $a_0 = -2$, $a_1 = 0.972572$, $a_2 = -0.100246$, $a_3 = 0.00425456$, $a_4 = -0.00004281021$.

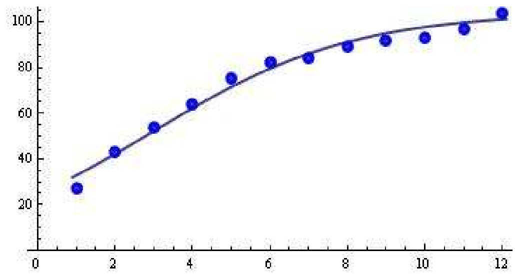

Figure 7: The model $s(t) = \frac{\omega}{1+be^{-kt}}$ with $\omega = 104$, $k = 0.389731$ and $b = 3.19675$.

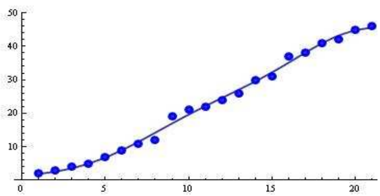

Figure 8: The fitted model $M^*(t)$ for ”On–line IBM entry software package data” .

monographs [6], [12].

Example 2. We examine the following data. (The small on–line data entry software package test data, available since 1980 in Japan [13], is shown in Table 4.1. For more details, see [14]).

The fitted model $M^*(t)$ based on the data of Table 1 for the estimated parameters: $\omega = 46$; $n = 4$; $a_0 = -3.4$; $a_1 = 0.242484$; $a_2 = 0.0346714$; $a_3 = -0.00410722$; $a_4 = 0.000131826$ is plotted on Fig. 8.

$Testing\,time\,(day)$	$Failures$	$Cumulative\,failures$
1	2	2
2	1	3
3	1	4
4	1	5
5	2	7
6	2	9
7	2	11
8	1	12
9	7	19
10	2	21
11	1	22
12	2	24
13	2	26
14	4	30
15	1	31
16	6	37
17	1	38
18	3	41
19	1	42
20	3	45
21	1	46

Table 1: On–line IBM entry software package [13]

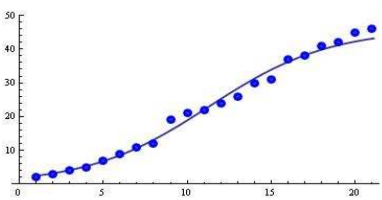

Figure 9: The Verhulst model $s(t)$ for "On–line IBM entry software package data" .

The classical Verhulst model $s(t)$ for "On–line IBM entry software package data" [13] with $\omega = 46$, $k = 0.278078$ and $b = 23.1163$ is plotted on Fig. 9.

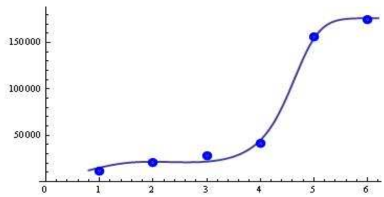

Figure 10: The fitted model $M^*(t)$ for Malicious and High–Risk Android App Volume Growth.

Example 3. We will show how it can be modelled data in [30] for Malicious and High–Risk Android App Volume Growth

$$Malicious\_data := \{\{1, 11000\}, \{2, 21000\}, \{3, 28000\},$$
$$\{4, 41000\}, \{5, 156000\}, \{6, 175000\}\}$$

The fitted model $M^*(t) = \frac{\omega}{1+e^{-F(t)}}$ for $n = 3$, $\omega = 176161$, $a_0 = -4.41$, $a_1 = 3.27714$, $a_2 = -1.45185$, $a_3 = 0.210138$ is depicted on Fig. 10.

Example 4. We will explore how it can be modelled the cumulative data in [31] for the number of users attacked by crypto-ransomware (November 2016 - October 2017)

$$Number\_of\_users\_attacked\_by\_crypto - ransomware$$
$$\_(November\_2016 - October\_2017)\_data := \{\{1, 192729\}, \{2, 398928\},$$
$$\{3, 497471\}, \{4, 585681\}, \{5, 660043\}, \{6, 726802\}, \{7, 824067\},$$
$$\{8, 925574\}, \{9, 980112\}, \{10, 1048504\}, \{11, 1128389\}, \{12, 1237842\}\}$$

The fitted model $M^* = \omega M(t)$ for $n = 3$, $\omega = 1237842$, $a_0 = -3.001$, $a_1 = 1.35015$, $a_2 = -0.19472$, $a_3 = 0.010854$ is depicted on Fig. 11. The fitted model $M^*(t) = \omega M(t)$ for $n = 4$ and $\omega = 1237842$, $a_0 = -3.001$, $a_1 = 1.58441$, $a_2 = -0.315681$, $a_3 = 0.0295429$, $a_4 = -0.000888084$ is given on Fig. 12.

Comparing the graphs in Figures 11–12 shows that approximation of this data with the model at fixed $n = 4$ is preferable.

2. The logistic model is generated by the following reaction network:

$$S + X \xrightarrow{k} 2X \tag{5}$$

where $k = const$. Here we consider the reaction network

$$S + X \xrightarrow{k(t)} 2X \tag{6}$$

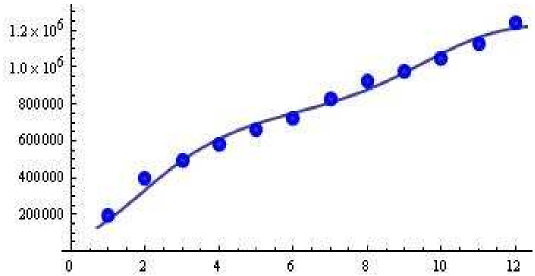

Figure 11: The fitted model $M^*(t)$ for $n = 34$.

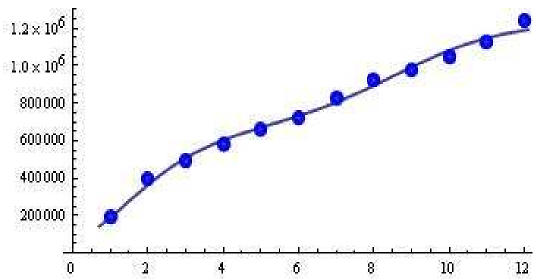

Figure 12: The fitted model $M^*(t)$ for $n = 4$.

where $k(t) = \sum_{i=0}^{n} b_i t^i$.

For example, let $n = 2$. Reaction network (6) induces the following differential system

$$\begin{cases} \dfrac{ds}{dt} & = & -(b_0 + b_1 t + b_2 t^2) s x \\ \dfrac{dx}{dt} & = & (b_0 + b_1 t + b_2 t^2) x s. \end{cases} \tag{7}$$

with $s(0) = s_0$; $x(0) = x_0$.

For $s(0) = s_0 = \frac{1}{2}$; $x(0) = x_0 = \frac{1}{2}$ from $s' + x' = 0$ we get $s + x = C$, $C = s_0 + x_0 = 1$ and

$$\begin{cases} \dfrac{ds}{dt} & = & -(b_0 + b_1 t + b_2 t^2) s(1 - s) \\ \dfrac{dx}{dt} & = & (b_0 + b_1 t + b_2 t^2) x(1 - x). \end{cases} \tag{8}$$

Hence, the new model can be written in the form:

$$s(t) = \frac{e^{-(b_0 t + \frac{b_1}{2} t^2 + \frac{b_2}{3} t^3)}}{1 + e^{-(b_0 t + \frac{b_1}{2} t^2 + \frac{b_2}{3} t^3)}}. \tag{9}$$

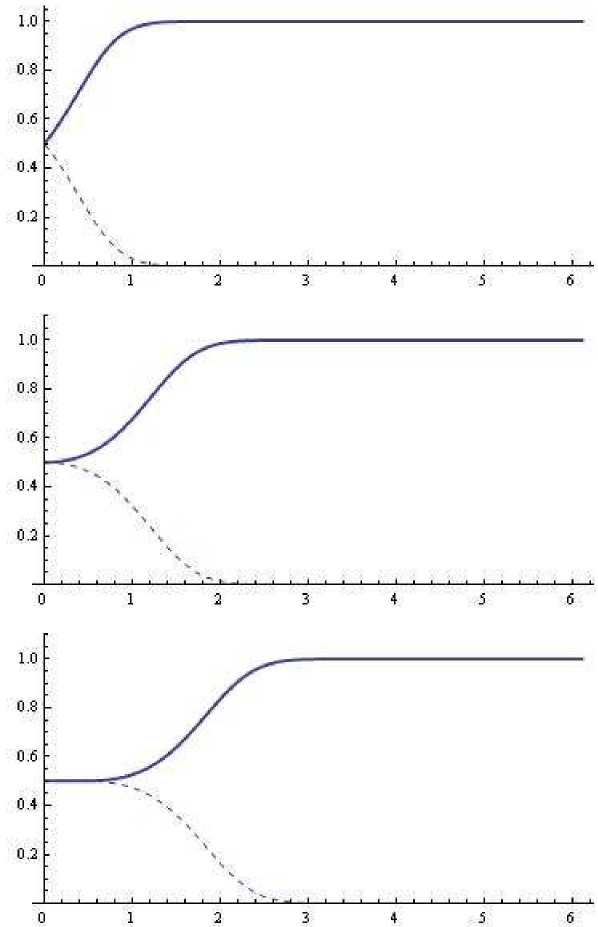

Figure 13: $n = 2$; a) $b_0 = 1.7$, $b_1 = 2.9$, $b_2 = -0.9$; b) $b_0 = 0.01$, $b_1 = 0.8$, $b_2 = -0.3$; c) $b_0 = 0.064$, $b_1 = -0.58$, $b_2 = 0.18$.

$$x(t) = \frac{1}{1 + e^{-(b_0 t + \frac{b_1}{2} t^2 + \frac{b_2}{3} t^3)}}. \tag{10}$$

Apparently, for arbitrary n we have

$$s(t) = \frac{e^{-(b_0 t + \frac{b_1}{2} t^2 + \frac{b_2}{3} t^3 + \ldots + \frac{b_n}{n+1} t^{n+1})}}{1 + e^{-(b_0 t + \frac{b_1}{2} t^2 + \frac{b_2}{3} t^3 + \ldots + \frac{b_n}{n+1} t^{n+1})}}. \tag{11}$$

$$x(t) = \frac{1}{1 + e^{-(b_0 t + \frac{b_1}{2} t^2 + \frac{b_2}{3} t^3 + \ldots + \frac{b_n}{n+1} t^{n+1})}}. \tag{12}$$

We will call this new class of growth functions $x(t)$ generated by reaction network (6) the **”Verhulst curve of growth with polynomial variable transfer” (VCG-PVT)**.

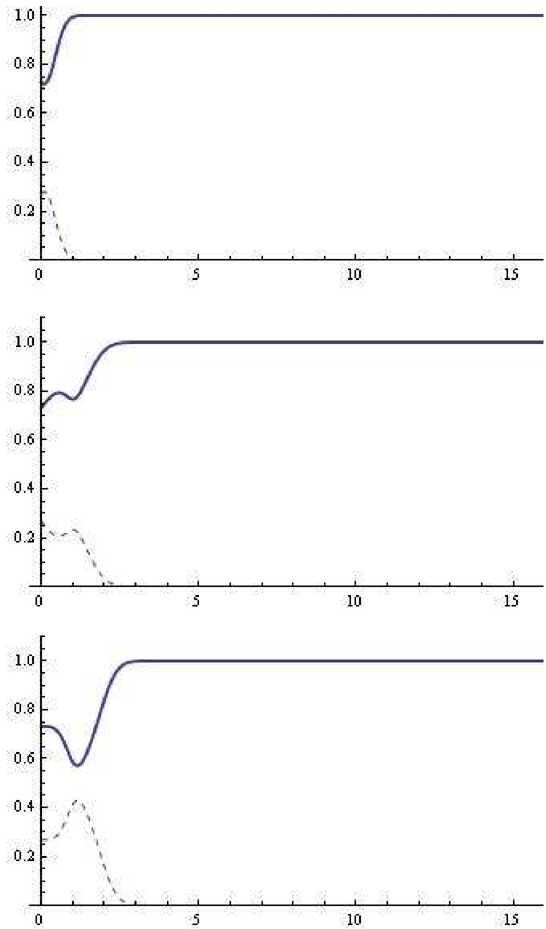

Figure 14: $n = 3$; a) $b_0 = -1.1$, $b_1 = 11.05$, $b_2 = -5.18$, $b_3 = 10.03$; b) $b_0 = 0.9$, $b_1 = -0.58$, $b_2 = 0.18$, $b_3 = -0.5$; c) $b_0 = 0.064$, $b_1 = -0.58$, $b_2 = 0.18$, $b_3 = -0.9$.

Numerical examples, illustrating our results (for $s(t)$–dashed and $x(t)$–thick) are given using *CAS Mathematica* (see, Fig. 13–Fig. 14.).

Example 5. We will analyze the following data in the field of antenna engineering:

$$Data :=$$
$$\{\{0.11, 5.19\}, \{0.55, 4.6\}, \{1, 4.35\}, \{1.5, 3.9\}, \{2, 3.92\},$$
$$\{3, 3.91\}, \{4, 5\}, \{5, 8.39\}, \{6, 10.2\}, \{8, 10.27\}, \{10, 10.27\}\}.$$

The fitted models $x^*(t) = Ax(t)$ for $n = 2$; $A = 10.1$ and $b_0 = 0.01$; $b_1 = -0.58$; $b_2 = 0.22$ is presented on Fig. 15.

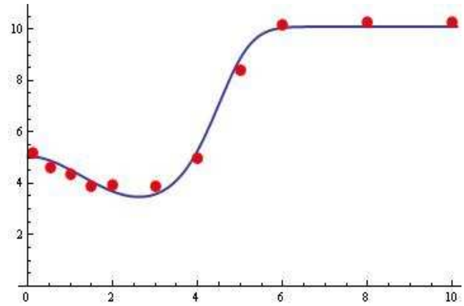

Figure 15: The fitted model $x^*(t) = Ax(t)$.

3. Consider the ”Verhulst curve of growth with polynomial variable transfer” for $n = 1$, i.e.

$$x(t) = \frac{1}{1 + e^{-(b_0 t + \frac{b_1}{2} t^2)}}. \tag{13}$$

The one–sided Hausdorff distance d between the Heaviside function $h_0(t)$ and the function $x(t)$ satisfies the relation

$$x(d) = 1 - d. \tag{14}$$

The following theorem gives upper and lower bounds for d

Theorem 1. Let

$$-2b_0/b_1 > 0, \;\; K = \max\{1 + b_0, \frac{1 + b_1}{2}\} \geq 3.$$

For the one–sided Hausdorff distance d between $h_0(t)$ and the growth function (13) the following inequalities hold:

$$d_l = \frac{1}{K} < d < \frac{\ln K}{K} = d_r. \tag{15}$$

Proof. From (14) we have

$$e^{-d(b_0 + \frac{b_1}{2} d)} = \frac{d}{1 - d}.$$

Let us examine the function:

$$F(d) = d(b_0 + \frac{b_1}{2} d) + \ln d - \ln(1 - d). \tag{16}$$

Consider the function

$$H(d) = Kd + \ln d. \tag{17}$$

From Taylor expansion we obtain $H(d) - F(d) = O(d^2)$. Hence $H(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see, Fig. 16–Fig. 17). Further, for $K \geq 3$ we have

$$H(d_l) = 1 + \ln \frac{1}{K} = 1 - \ln K < 0;$$

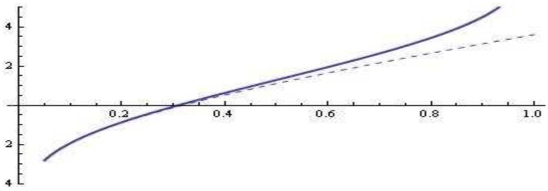

Figure 16: F –(thick) and H –(dashed) for $b_0 = 2.6$; $b_1 = -0.1$; H–distance $d = 0.309853$; $d_l = 0.277778$; $d_r = 0.355815$.

$$H(d_r) = \ln K + \ln \frac{\ln K}{K} = \ln K + \ln \ln K - \ln K = \ln \ln K > 0.$$

This completes the proof of the theorem.

Some computational examples using relations (15) are presented in Table 2. The last column of Table 2 contains the values of d for prescribed values of b_0, $b_1 k$ computed by solving the nonlinear equation (14).

b_0	b_1	d_l	d_r	d by (14)
2.6	-0.1	0.277778	0.355815	0.309852
20.6	-6.9	0.0462963	0.142254	0.105587
30.6	-8.9	0.0316456	0.109277	0.0805256
40.6	-9.9	0.0240385	0.0896178	0.0658538
60	-10	0.0163934	0.0673914	0.0494662
100	-10	0.00990099	0.0450943	0.033636
200	-10	0.00497512	0.0263846	0.0195776
300	-10	0.0033226	0.0189605	0.0141494

Table 2: Bounds for $d(k)$ computed by (15)

The reader may formulate the corresponding approximation problem for arbitrary n following the ideas given in this note, and will be omitted.

4. A practically important class of sigmoid functions is the class of cut functions including the step function as a limiting case.

We will define a **"cut function of the second class $C^{II}(t)$"** as follows:

$$C^{II}(t) = \begin{cases} 0, & \text{if} \quad t \leq t_1, \\ \frac{b_1}{8}t^2 + \frac{b_0}{4}t + \frac{1}{2}, & \text{if} \quad t \in (t_1, z_1), \\ 1, & \text{if} \quad t \geq z_1 \end{cases} \tag{18}$$

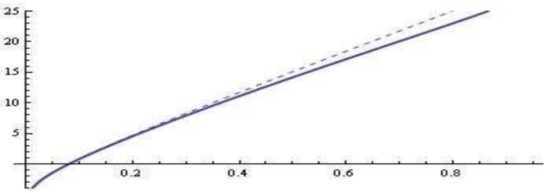

Figure 17: F –(thick) and H –(dashed) for $b_0 = 30.6$; $b_1 = -8.9$; H–distance $d = 0.0805256$; $d_l = 0.0316456$; $d_r = 0.109277$.

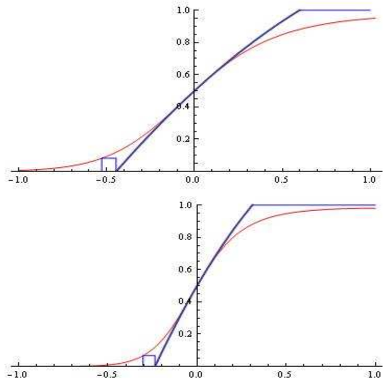

Figure 18: The cut and "growth" functions for : a) $b_0 = 4$, $b_1 = -2.2$, $t_1 = -0.445436$, $z_1 = 0.598508$; H-distance $d = 0.0819851$; b) $b_0 = 7.6$, $b_1 = -7.4$, $t_1 = -0.236035$, $z_1 = 0.309919$; H-distance $d = 0.0666475$.

where

$$t_1 = \frac{-b_0 + \sqrt{b_0^2 - 4b_1}}{b_1}; \ z_1 = \frac{-b_0 + \sqrt{b_0^2 + 4b_1}}{b_1}; \ (b_0^2 - 4b_1 > 0.$$

Let's look again the "Verhulst curve of growth with polynomial variable transfer" for $n = 1$, i.e.

$$x(t) = \frac{1}{1 + e^{-(b_0 t + \frac{b_1}{2} t^2)}}. \tag{19}$$

We study the uniform and Hausdorff approximation of the cut function $C^{II}(t)$ by function $x(t)$. We find an expression for the error of the best uniform approximation. Curiously, the uniform distance between a cut function and the function $x(t)$ of best uniform approximation is an absolute constant (not depending on the slope parameters b_0, b_1). By contrast, it turns out that the Hausdorff distance (H-distance) depends on the slope parameters. We will formulate the following result

Theorem 1. *The function $x(t)$ defined by (19): i) is the growth function of* *best uniform one-sided approximation to function $C^{II}(t)$ in the considered interval; ii) approximates the cut function $C^{II}(t)$ in uniform metric with an error*

$$\rho = \rho(C, x) = \frac{1}{1+e^2} = 0.11920292.... \tag{20}$$

Proof. Consider functions (18) and (19) with same centers. In addition chose c and v to have same slopes at their coinciding centers, cf. Figure 18. Then, for the uniform distance between the cut and logistic functions, we have:

$$\rho = x(t_1) - C^{II}(t_1) = \frac{1}{1+e^2}. \tag{21}$$

This completes the proof of the theorem.

Theorem 2. *The function $x(t)$ is the logistic function of best Hausdorff one-sided approximation to function $C^{II}(t)$ in the considered interval. The function $x(t)$, approximates function $C^{II}(t)$ with an error $d = d(x, C^{II})$ that satisfies the relation:*

$$\ln\frac{d}{1-d} = b_0(t_1 - d) + \frac{1}{2}b_1(t_1 - d)^2. \tag{22}$$

We note that H-distance d satisfies the relation $x(t_1 - d) = d$, which implies (22). Relation (22) shows that the Hausdorff distance d depends on the slope parameters b_0 and b_1.

Concluding Remarks. 1. We consider the following delayed Lotka–Volterra reaction network:

$$\begin{array}{rcl} X_1 & \xrightarrow{k_1,\,\tau_1} & 2X_1 \\ X_1 + X_2 & \xrightarrow{k_2,\,\tau_2} & 2X_2 \\ X_2 & \xrightarrow{k_3} & O \end{array} \tag{23}$$

The dynamic equation of delayed kinetic system is:

$$x'(t) = \left[\begin{array}{c} -k_1x_1(t) - k_2x_1(t)x_2(t) + 2k_1x_1(t-\tau_1) \\ -k_2x_1(t)x_2(t) - k_3x_2(t) + 2k_2x_1(t-\tau_2)x_2(t-\tau_1) \end{array} \right] \tag{24}$$

For the canonical realization of the kinetic system see, for example [32]. It is of interest to observe model by Lotka–Volterra based on the following new reaction networks

$$\begin{array}{rcl} X_1 & \xrightarrow{k_1(t),\,\tau_1} & 2X_1 \\ X_1 + X_2 & \xrightarrow{k_2(t),\,\tau_2} & 2X_2 \\ X_2 & \xrightarrow{k_3} & O \end{array} \tag{25}$$

in the case where, for example, the reaction constants $k_1 = k_1(t)$ and $k_2 = k_2(t)$ are functions of time. Research on this topic is not covered in this paper.

Total Deaths of Novel Coronavirus (2019-nCoV)

Date	Total Deaths	Change in Total	Change in Total (%)
Feb. 18	**2,009**	136	7%
Feb. 17	**1,873**	98	6%
Feb. 16	**1,775**	106	6%
Feb. 15	**1,669**	143	9%
Feb. 14	**1,526**	143	10%
Feb. 13	**1,383**	122	10%
Feb. 12	**1,261**	146	13%
Feb. 11	**1,115**	97	10%
Feb. 10	**1,018**	108	12%
Feb. 9	**910**	97	12%
Feb. 8	**813**	89	12%
Feb. 7	**724**	86	13%
Feb. 6	**638**	73	13%
Feb. 5	**565**	73	15%
Feb. 4	**492**	66	15%
Feb. 3	**426**	64	18%
Feb. 2	**362**	58	19%
Feb. 1	**304**	45	17%
Jan. 31	**259**	46	22%
Jan. 30	**213**	43	25%
Jan. 29	**170**	38	29%
Jan. 28	**132**	26	25%
Jan. 27	**106**	26	33%
Jan. 26	**80**	24	43%
Jan. 25	**56**	15	37%
Jan. 24	**41**	16	64%

Figure 19: The data "Corona virus".

2. Approximation of the data "Corona virus" [28].

The new coronavirus [29], SARS-CoV-2, is the reason for a new disease, Covid-19. As of 22th February, there have been more than 77,977 cases, including 54,298 laboratory confirmed cases. There have been 2,366 deaths [28].

Up to now it is not exactly sure how SARS-CoV-2 is spread. The new virus is part of the coronavirus family which includes the common cold and SARS.

These viruses are spread by cough and sneeze droplets.

Consider the "VCGPVT model": $F(t) = \frac{A}{1+ae^{-(b_0 t+\frac{b_1}{2}t^2+\frac{b_2}{3}t^3)}}$ and classical Verhulst model: $V(t) = \frac{A_1}{1+a_1 e^{-kt}}$. For the data "Corona virus" (see, Fig. 19), comparisons between the fitted model $F(t)$ for $A = 2461.73$, $a = 65.591$, $b_0 = 0.321$, $b_1 = -0.0209628$, $b_2 = 0.000751263$ and the fitted model $V(t)$ for $A_1 = 2836.99$, $a_1 = 43.0795$, $k = 0.17788$ are depicted on Fig. 20

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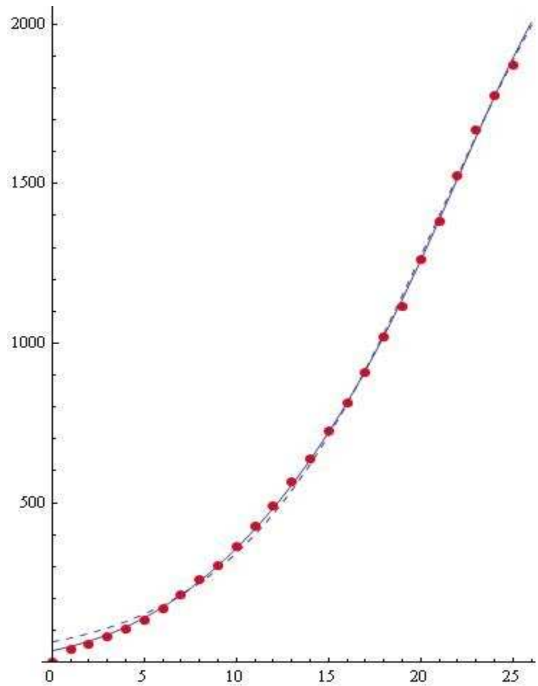

Figure 20: Comparisons between the $F(t)$ – (blue) and $V(t)$ – (dashed) for the data "Corona virus".

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