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EXPONENTIAL STABILITY OF HOPFIELD-TYPE DELAY IMPULSIVE DISCRETE NEURAL NETWORKS AND COMPUTER SIMULATION

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ABSTRACT: A discrete Hopfield-type neural network with constant delays, instantaneous switching topologies at certain times and time variable connection weights is studied. Some criteria for exponential stability are derived. The obtained results are illustrated on an example with different activation functions such as tanh, Swish, and the error function. The example is computer realized by coding the corresponding algorithms for calculating the values of the solution for each step.

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1. INTRODUCTION

Neural networks have received extensive interests in the recent years in connection with their potential applications in signal processing, content addressable memory, pattern recognition, combinatorial optimization. In practice, for computation convenience, more adequate modes are discrete-time neural networks ([10], [12], [13]).

One of the most important problems in the behavior of the equilibrium of the models is stability (see, [2], [5], [9]). At the same time impulses are a very useful mathematical apparatus to model some instantaneous perturbations in the process (see, for example [3], [8]). Difference equations, being a discrete version of differential equations, could have also impulses (see, for example, [4]).

In this paper, we study a class of discrete-time neural networks with constant delays subject to impulsive switching of the topology. We consider the general case when the connection weights between two neurons are changeable in time. Some criteria for exponential stability of the equilibrium are obtained.

The rest of this paper is organized as follows. In Section 2, discrete-time neural networks with constant delays and impulsive perturbations is presented. It is studied the model with a time varying connection weights. Equilibrium point of the studied model is defined. It is deeply connected not only with the difference equations but with the impulsive conditions. Some sufficient conditions for exponential stability of zero equilibrium point are derived. Moreover, a numerical example is presented in Section 3 and the results are discussed. In the particular model of neural networks we use the activation function recently discovered by Google Brain and called Swish (see [11]). As it is shown in [11] by experiments on ImageNet with identical models, the Swish improves top-1 classification accuracy in the networks like Inception-ResNet-v2 by 0.6% and in Mobile NASNet-A by 0.9%. The example is computer realized by the help of Wolfram Mathematica. Following the theoretical schemes for solving the problems, the corresponding algorithms are coded to calculate the values of the solution for each step. The graphs are generated by Wolfram Mathematica.

2. STATEMENT OF THE PROBLEM AND DEFINITION OF SOLUTION

We will introduce basic notation used in this paper. Most of them are well known and used in the literature. Let $\mathbb{Z}_+$ be the set of all nonegative integers; the increasing sequence $\{n_i\}_{i=0}^{\infty}:\ n_0 = 0$ be given; $\mathbb{Z}[a,b] = \{z \in \mathbb{Z}_+ : a \leq z \leq b\},\ a, b \in \mathbb{Z}_+,\ a < b,$

$\mathbb{Z}_a = \{z \in \mathbb{Z}_+ : z \geq a\}$ and

$$I_k = \mathbb{Z}[n_k + 1, n_{k+1} - 1], \ k \in \mathbb{Z}_+.$$

In the paper, we investigate the exponential stability of (1) by finite-dimensional Lyapunov functions.

We will study the exponential stability of discrete neural networks with a delay. We will consider the case when the connections of the network are subject to a long lasting impulses at certain moments which will be modeled by the so called non-instantaneous impulses. Also, we will study the general case of time variable connection weights.

Consider the discrete-time cellular neural network consisting of N interconnected neurons described by a system of delay difference equations of the form

$$\begin{aligned} u_i(n) = a_i u_i(n-1) + \sum_{j=1}^{n} b_{ij}(n) f_j(u_j(n-1)) + \sum_{j=1}^{n} c_{ij}(n) f_j(u_j(n-m_{ij})) + G_i \\ \text{for } n \in \bigcup_{k=0}^{\infty} I_k, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

subject to impulses

$$\begin{aligned} u_i(n_k) = M_{ik} u_i(n_k - 1) + \sum_{j=1}^{n} \psi_{ij}^k S_j^k(u_j(n_k - 1)) \\ + \sum_{j=1}^{n} \phi_{ij}^k S_j^k(u_j(n_k - m_{ij})) + Q_{ik}, \qquad \text{for } k \in \mathbb{Z}_1, \end{aligned} \tag{2}$$

with initial conditions

$$u_i(n) = u_i^0(n), \quad n \in \mathbb{Z}[1-m, 0], \tag{3}$$

where $u_i(n)$ denotes the potential (or voltage) of the i-th neuron at discrete time n, a_i represents rate with which the i-th neuron resets its potential to the resting state when isolated from other neurons and inputs, $m_{ij} \geq 2$ corresponds to the time delay required in processing and transmitting a signal from j-th neuron to i-th neuron, $m = \max_{i,j \in \mathbb{Z}[1,N]} m_{ij}$, b_{ij}, c_{ij} represents the synaptic connection weight between the j-th neuron and the i-th neuron at discrete time n and discrete time $n - m_{ij}$ respectively, f_j is the neuron activation function, G_j denotes the i-th component of an external input source introduced from outside the network to the i-th neuron, M_{ik} represents rate with which the i-th neuron resets its potential to the resting state when isolated from other neurons and inputs at the time of the k-th impulse, ψ_{ij}^k, ϕ_{ij}^k represents the synaptic connection weight between the j-th neuron and the i-th neuron at discrete time n_k and discrete time $n_k - m_{ij}$ respectively, S_j^k is the neuron activation function of the j-th neuron at the time of the k-th impulse, Q_j denotes the i-th component of an external input source introduced from outside the network to the i-th neuron at the time of the k-th impulse, $\varphi_i(n), n \in \mathbb{Z}[-m, 0]$ is the initial function for the i-th neuron.

We will define an equilibrium of the model (1), (2):

Definition 1. A vector $u^* \in \mathbb{R}^n : \ u^* = (u_1^*, u_2^*, \dots, u_n^*)$ is said to be an equilibrium point of the impulsive discrete-time neural network (1), (2) if it satisfies the equalities

$$\begin{aligned} u_i^* &= a_i u_i^* + \sum_{j=1}^{n} b_{ij}(n) f_j(u_j^*) + \sum_{j=1}^{n} c_{ij}(n) f_j(u_j^*) + G_i, \quad n \in \bigcup_{k=0}^{\infty} I_k, i = 1, 2, \dots, N, \\ u_i^* &= M_{ik} u_i^*(n_k - 1) + \sum_{j=1}^{n} (\psi_{ij}^k + \phi_{ij}^k) S_j^k(u_j^*) + Q_{ik}, \quad k \in \mathbb{Z}_1, \end{aligned} \tag{4}$$

Remark 1. Note that if there are impulses in the discrete model then it changes the definition of the equilibrium point. Also, if a point is an equilibrium point of a discrete model, then it is not necessarily to be an equilibrium point of the corresponding discrete model with impulses. But the opposite is true.

Remark 2. If the impulsive discrete-time neural network (1), (2) has an equilibrium point then by a substitution $x = u - u^* \in \mathbb{R}^n$ the neural network (1), (2) is changing to a neural network with no external input source introduced from outside the network and activation functions vanish at 0.

Definition 2. ([9]) The trivial solution of the model (1), (2) is called exponentially stable if there exist constants $\beta > 0$ and $\alpha \in (0, 1)$ such that for any initial value $u^0 : \mathbb{Z}[1 - m, 0] \to \mathbb{R}, \ u^0(n) = (u_1^0(n), u_2^0(n), \dots, u_N^0(n))$ the inequality

$$\sum_{i=1}^{N} |u_i(n)| \leq \beta \alpha^n ||u^0||_0, \quad n = 1, 2, \dots$$

holds, where $||u^0||_0 = \sum\limits_{i=1}^{N} \sup_{n \in \mathbb{Z}[1-m,0]} |u_i^0(n)|$.

Remark 3. As pointed out in [7], $f_j(u)$ can be regarded as a smooth input-output function because the biological information in neurons often lies in a short-time average of the firing rate.

Based on biological meanings and Remark 2, it is assumed that the following conditions are satisfied.

We will introduce the following assumptions:

A1. The activation functions $f_i \in C(\mathbb{R}, \mathbb{R}) : \ f_i(0) = 0$ and there exist positive constants L_i, $i = 1, 2, \dots, N$, such that $|f_i(u)| \le L_i |u|, \ u \in \mathbb{R}, \ i = 1, 2, \dots, N$.

A2. The activation functions $S_i^k \in C(\mathbb{R}, \mathbb{R}) : \ S_i(0) = 0$, and there exist positive constants K_i, $i = 1, 2, \dots, N$, such that $|S_i^k(u)| \le K_i |u|, \ u \in \mathbb{R}, \ \ i = 1, 2, \dots, N$, $k = 1, 2, \dots$.

A3. The connection functions $b_{ij}, c_{ij} : \bigcup_{k=0}^{\infty} I_k \to \mathbb{R}$ are bounded, i.e. there exists constants $\beta_{ij} > 0, \gamma_{ij} > 0$ such that $|b_{ij}(n)| \le \beta_{ij}$, $|c_{ij}(n)| \le \gamma_{ij}$ for $n \in \bigcup_{k=0}^{\infty} I_k$, $k = 1, 2, \dots$ and $i, j = 1, 2 \dots, N$.

A4. For any $i = 1, 2, \dots, N$ the sequence $\{M_{ik}\}_k$ is bounded, i.e. there exists a positive number μ_i such that $M_{ik} \le \mu_i$ for all $k \in \mathbb{Z}_1$.

A5. For any $i, j = 1, 2, \dots, N$ the sequences $\{|\psi_{ji}^k| K_i^k\}_k$ and $\{|\phi_{ji}^k| K_i^k\}_k$ are bounded, i.e. there exist positive numbers η_{ij}, ν_{ij} such that $|\psi_{ji}^k| K_i^k \le \eta_{ij}$ and $|\phi_{ji}^k| K_i^k \le \nu_{ij}$ for all $k \in \mathbb{Z}_1$.

A6. There is no external input source introduced from outside the network, i.e. $G_i = 0, \ Q_{ik} = 0$ for $i = 1, 2, \dots, N$ and $k = 1, 2, \dots$.

Remark 4. Note that in [6] discrete Hopfield-type neural networks with time delays are studied but the activation functions are continuously differentiable and approaches $-1/1$ at infinity. These conditions are stronger than (A1).

Theorem 1. *Let the conditions (A1)-(A6) be satisfied and the inequalities*

$$A_i + \sum_{j=1}^{N} (B_{ji} + C_{ji}) < 1 \quad \textit{for all } i = 1, 2, \dots, N \tag{5}$$

hold, where $A_i = \max\{|a_i|, \mu_i\}$, $B_{ji} = \max\{L_i \beta_{ji}, \eta_{ji}\}$ *and* $C_{ji} = \max\{L_i \gamma_{ji}, \nu_{ij}\}$.

Then the zero equilibrium point of the difference neural network with impulses (1), (2) is exponentially stable.

Proof. Let $u(n) = (u_1(n), u_2(n), \dots, u_N(n))$ be an arbitrary solution of (1), (2).

From conditions (A1), (A3), (A6) we have

$$\begin{aligned} |u_i(n)| \le |a_i| \ |u_i(n-1)| + \sum_{j=1}^{n} \beta_{ij} L_j |u_j(n-1)| + \sum_{j=1}^{n} \gamma_{ij} L_j |u_j(n - m_{ij})|, \\ n \in \bigcup_{k=0}^{\infty} I_k, \quad i = 1, 2, \dots, N, \end{aligned} \tag{6}$$

and from conditions (A2), (A4), (A5), (A6) we obtain

$$\begin{aligned}|u_i(n)| \le |M_{ik}|\ |u_i(n_k-1)| + \sum_{j=1}^{n} |\psi_{ij}^k| K_j^k |u_j(n_k-1)| \\ + \sum_{j=1}^{n} |\phi_{ij}^k| K_j^k |u_j(n_k-m_{ij})|, \quad \text{for } \ k \in \mathbb{Z}_1.\end{aligned} \tag{7}$$

From (6) we get

$$|u_i(1)| \le \Big(|a_i| + L_i \sum_{j=1}^{n} (\beta_{ji} + \gamma_{ji})\Big) ||u_i||_0. \tag{8}$$

Consider the functions $F_i \in C([1,\infty), \mathbb{R})$ defined by

$$F_i(\lambda) = 1 - \lambda A_i - \lambda \sum_{j=1}^{N} B_{ji} - \sum_{j=1}^{N} C_{ji} \lambda^{m_{ji}-1}, \quad i = 1,2,\dots,N. \tag{9}$$

According to inequality (5) we obtain $F_i(1) = 1 - |a_i| - \sum\limits_{j=1}^{N} (B_{ji} + C_{ji}) > 0$ for $i = 1,2,\dots,N$. Since the function F_i is continuous there exists a number $\lambda > 1$ such that for any $i = 1,2,\dots,N$ the inequalities $F_i(\lambda) > 0$ hold.

Define $z_i(n) = \lambda^n |u_i(n)| \ge 0,\ n \ge 0$.

Then we get

$$\begin{aligned} z_i(n) &\le \lambda^n \Big(|a_i|\ |u_i(n-1)| + \sum_{j=1}^{N} \beta_{ij} L_j |u_j(n-1)| + \sum_{j=1}^{N} \gamma_{ij} L_j |u_j(n-m_{ij})|\Big) \\ &= \lambda \Big(|a_i|\ z_i(n-1) + \sum_{j=1}^{N} \beta_{ij} L_j z_j(n-1) + \sum_{j=1}^{N} \gamma_{ij} L_j \lambda^{m_{ij}-1} z_j(n-m_{ij})\Big) \\ &\le \lambda \Big(A_i\ z_i(n-1) + \sum_{j=1}^{N} B_{ij} z_j(n-1) + \sum_{j=1}^{N} C_{ij} \lambda^{m_{ij}-1} z_j(n-m_{ij})\Big) \\ &\qquad n \in I_k, \quad k = 0,1,2,\dots, \ i = 1,2,\dots,N, \end{aligned} \tag{10}$$

and

$$\begin{aligned} z_i(n_k) &= \lambda \Big(|M_{ik}|\ z_i(n_k-1) + \sum_{j=1}^{N} |\psi_{ij}^k| K_j^k z_j(n_k-1) \\ &\qquad + \sum_{j=1}^{N} |\phi_{ij}^k| K_j^k \lambda^{m_{ij}-1} z_j(n_k-m_{ij})\Big) \\ &\le \lambda A_i\ z_i(n_k-1) + \lambda \sum_{j=1}^{N} B_{ij} z_j(n_k-1) + \sum_{j=1}^{N} C_{ij} \lambda^{m_{ij}} z_j(n_k-m_{ij}) \\ &\qquad \text{for } \ k = 1,2,\dots. \end{aligned} \tag{11}$$

Consider the Lyapunov function $V:\ \mathbb{Z}_1 \to \mathbb{R}_+$ defined by

$$V(n) = \sum_{i=1}^{N} \Big\{ z_i(n) + \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} \sum_{l=n-m_{ij}}^{n-1} z_j(l) \Big\}, \quad n \in \mathbb{Z}_1. \tag{12}$$

Then for all $n \in \mathbb{Z}[n_k+1, n_{k+1}-1], \quad k = 0,1,2,\ldots,$ we have

$$\begin{aligned}
V(n) - V(n-1) \leq \sum_{i=1}^{N} \Big\{ &\lambda A_i\ z_i(n-1) + \lambda \sum_{j=1}^{N} B_{ij} z_j(n-1) \\
&+ \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} z_j(n-m_{ij}) + \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} \sum_{l=n-m_{ij}+1}^{n-1} z_j(l) \Big\} \\
&- \sum_{i=1}^{N} \Big\{ z_i(n-1) + \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} \sum_{l=n-m_{ij}}^{n-2} z_j(l) \Big\} \\
\leq -\sum_{i=1}^{N} \Big\{ &(1-\lambda A_i) z_i(n-1) - \lambda \sum_{j=1}^{N} B_{ij} z_j(n-1) \\
&- \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} z_j(n-m_{ij}) \\
&- \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}-1} \sum_{l=n-m_{ij}+1}^{n-1} z_j(l) + \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} \sum_{l=n-m_{ij}}^{n-2} z_j(l) \Big\} \\
= -\sum_{i=1}^{N} \Big\{ &(1-\lambda A_i) z_i(n-1) - \lambda \sum_{j=1}^{N} B_{ij} z_j(n-1) \\
&- \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} z_j(n-m_{ij}) \\
&- \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} z_j(n-1) + \sum_{j=1}^{N} C_{ij}\lambda^{m_{ij}} z_j(n-m_{ij}) \Big\} \\
\leq -\sum_{i=1}^{N} \Big\{ &(1-\lambda A_i) - \lambda \sum_{j=1}^{N} B_{ji} - \sum_{j=1}^{N} C_{ji}\lambda^{m_{ji}} \Big\} z_i(n-1) \\
= -\sum_{i=1}^{N} &F_i(\lambda) z_i(n-1) < 0.
\end{aligned}$$

Therefore, the inequality

$$V(n) \leq V(n_k+1), \quad n \in I_k, \quad k = 0,1,2,\ldots. \tag{13}$$

holds.

For any fixed natural number k we have

$$\begin{aligned}
&V(n_k) - V(n_k - 1) \\
&\le \sum_{i=1}^{N} \Big\{ \lambda A_i \; z_i(n_k - 1) + \lambda \sum_{j=1}^{N} B_{ij} z_j(n_k - 1) \\
&\quad + \sum_{j=1}^{N} C_{ij} \lambda^{m_{ij}} z_j(n_k - m_{ij}) + \sum_{j=1}^{N} C_{ij} \lambda^{m_{ij}} \sum_{l=n_k-m_{ij}+1}^{n_k-1} z_j(l) \Big\} \\
&\quad - \sum_{i=1}^{N} \Big\{ z_i(n_k - 1) + \sum_{j=1}^{N} C_{ij} \lambda^{m_{ij}} \sum_{l=n_k-m_{ij}}^{n_k-2} z_j(l) \Big\} \\
&\le - \sum_{i=1}^{N} \Big\{ (1 - \lambda A_i) - \lambda \sum_{j=1}^{N} B_{ji} - \sum_{j=1}^{N} C_{ji} \lambda^{m_{ji}} \Big\} z_i(n_k - 1) \\
&= - \sum_{i=1}^{N} F_i(\lambda) z_i(n_k - 1) < 0.
\end{aligned}$$

Therefore, the inequality

$$V(n_k) < V(n_k - 1) \quad \text{for} \quad k = 1, 2, \dots \tag{14}$$

holds.

From inequalities (8), (13), (14) and $\sum\limits_{l=-m_{ij}}^{-1} \lambda^l \le \lambda^{-m_{ij}} \frac{-1+\lambda^{m_{ij}}}{-1+\lambda}$ we get

$$\begin{aligned}
V(n) \le V(1) &= \sum_{i=1}^{N} \Big\{ |u_i(1)| + \sum_{j=1}^{N} C_{ij} \lambda^{m_{ij}} \sum_{l=1-m_{ij}}^{0} \lambda^l |u_j(l)| \Big\} \\
&\le \sum_{i=1}^{N} \Big(|a_i| + L_i \sum_{j=1}^{n} (\beta_{ji} + \gamma_{ji}) + \sum_{j=1}^{N} C_{ji} \frac{\lambda^{m_{ji}} - 1}{\lambda - 1} \Big) \sup_{n \in \mathbb{Z}[1-m,0]} |u_i(n)| \\
&\le \beta ||u||_0, \quad n \in \mathbb{Z}_1
\end{aligned} \tag{15}$$

where $\beta = \max\limits_{i \in \mathbb{Z}[1,N]} \Big(|a_i| + L_i \sum_{j=1}^{n} (\beta_{ji} + \gamma_{ji}) + \sum_{j=1}^{N} C_{ji} \frac{\lambda^{m_{ji}}-1}{\lambda-1} \Big)$.

From the definition (13) and inequality (15) we obtain

$$\sum_{i=1}^{N} |u_i(n)| = \frac{1}{\lambda^n} \sum_{i=1}^{N} z_i(n) \le \frac{1}{\lambda^n} V(n) \le \frac{1}{\lambda^n} \beta ||u||_0, \quad n \in \mathbb{Z}_1. \tag{16}$$

□

3. APPLICATION

In this section we will consider a particular neural network to illustrate the obtained results in the paper.

Example 1. Let $u_k = 6k,\ k = 1, 2, \dots$. Consider the discrete model of neural network with three agents, impulses caused by a switched topology at some times and time variable bounded connection weights between neurons:

$$
\begin{aligned}
u_1(n) &= -0.3u_1(n-1) - 0.15\mathrm{erf}(u_1(n-6)) - 0.2\frac{u_2(n-8)}{1+e^{-u_2(n-8)}} \\
&\quad + 0.15\tanh(u_3(n-6)) - 0.05\frac{u_2(n-1)}{1+e^{-u_2(n-1)}} + 0.2\tanh(u_3(n-1)) \\
u_2(n) &= -0.2u_2(n-1) - 0.06\mathrm{erf}(u_1(n-10)) - 0.01\frac{u_2(n-4)}{1+e^{-u_2(n-4)}} \\
&\quad + 0.1\mathrm{erf}(u_1(n-1)) - 0.1\frac{u_2(n-1)}{1+e^{-u_2(n-1)}} + 0.1\tanh(u_3(n-1)) \\
u_3(n) &= -0.1u_3(n-1) + 0.05\mathrm{erf}(u_1(n-7)) - 0.1\frac{u_2(n-10)}{1+e^{-u_2(n-10)}} \\
&\quad - 0.02\tanh(u_3(n-6)) - 0.1\mathrm{erf}(u_1(n-1)) - 0.05\frac{u_2(n-1)}{1+e^{-u_2(n-1)}} \\
&\quad + 0.01\tanh(u_3(n-1)) \\
&\quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,
\end{aligned}
\tag{17}
$$

with impulses at times n_k for $k \in \mathbb{Z}_1$:

$$
\begin{aligned}
u_1(n_k) &= 0.2u_1(n_k-1) - 0.11\tanh(u_1(n_k-6)) + 0.01\mathrm{erf}(u_3(n_k-6)) \\
&\quad + 0.05\tanh(u_1(n_k-1)) - 0.09\frac{u_2(n_k-1)}{1+e^{-u_2(n_k-1)}} + 0.07\mathrm{erf}(u_3(n_k-1)) \\
u_2(n_k) &= 0.3u_2(n_k-1) - \frac{1}{30}\tanh(u_1(n_k-10)) - 0.05\frac{u_2(n_k-4)}{1+e^{-u_2(n_k-4)}} \\
&\quad + 0.1\mathrm{erf}(u_3(n_k-5)) + \frac{1}{30}\tanh(u_1(n_k-1)) \\
&\quad + 0.1\frac{u_2(n_k-1)}{1+e^{-u_2(n_k-1)}} - 0.09\mathrm{erf}(u_3(n_k-1)) \\
u_3(n_k) &= 0.08u_3(n_k-1) - 0.04\tanh(u_1(n_k-7)) - 0.1\frac{u_2(n_k-10)}{1+e^{-u_2(n_k-10)}} \\
&\quad + \frac{1}{30}\tanh(u_1(n_k-1)) - 0.02\mathrm{erf}(u_3(n_k-1)).
\end{aligned}
\tag{18}
$$

In this case $L_1 = 1.4,\ L_2 = 1.1,\ L_3 = 1.2,\ K_1 = 1.2,\ K_2 = 1.1,\ K_3 = 1.4,$
Then the inequalities

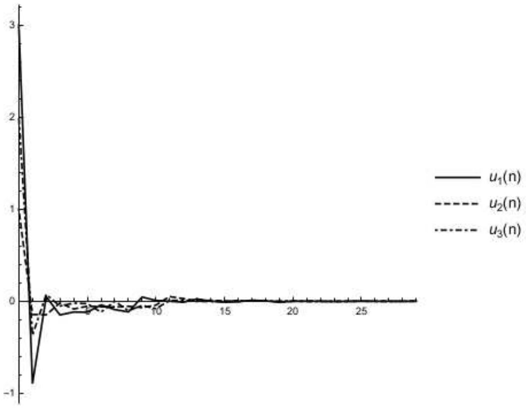

Figure 1. Graph of the solution of (17), (18) on the interval $[0, 29]$ with initial values $u^0 = (3, 1, 2)$.

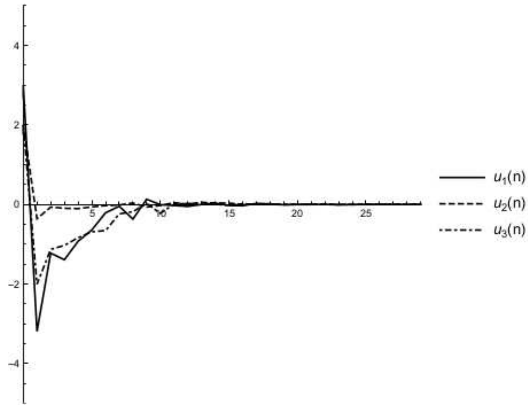

Figure 2. Graph of the solution of (17), (18) on the interval $[0, 29]$ with initial values $u^0 = (3(n+1), 2|n+1|, 2(n+1))$.

$0.3 + \max(1.4*0.15, 1.2*0.11) + \max(1.4*0.06, 1.2*1/30) + \max(1.40.05, 1.2*0.04) + \max(1.4*0, 1.2*0.05) + \max(1.4*0.1, 1.2*1/30) + \max(1.4*0.05, 1.2*1/30) < 1$,

$0.3 + \max(1.1*0.2, 1.1*0) + \max(1.1*0.01, 1.1*0.05) + \max(1.1*0.1, 1.1*0.1) + \max(1.1*0.05, 1.1*0.09) + \max(1.1*0.1, 1.1*0.1) + \max(1.1*0.05, 1.1*0) < 1$,

and

$0.1 + \max(1.2*0.15, 1.4*0.01) + \max(1.2*0, 1.4*0.1) + \max(1.2*0.02, 1.4*0) + \max(1.2*0.2, 1.4*0.07) + \max(1.2*0.1, 1.4*0.09) + \max(1.2*0.01, 1.4*0.02) < 1$

hold.

Therefore, the conditions of Theorem are satisfied and the zero equilibrium of the neural network (17), (18) is exponentially stable.

The graph of the solution of the neural network (17), (18) with constant initial values $u^0 = (3, 1, 2)$ and $u^0(n) = (3(n+1), 2|n+1|, 2(n+1))$, $n \in \mathbb{Z}[-9, 0]$ is given on Figure 1 and Figure 2, respectively. It could be seen from the graphs the zero is exponentially stable.

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