

COMPUTER SIMULATIONS FOR DELAY IMPULSIVE DISCRETE NEURAL NETWORKS

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ABSTRACT: A discrete Hopfield-type neural network with constant delays, instantaneous switching topologies at certain times and time variable connection weights is studied. Several models for the neural network are considered to illustrate the influence of different factors on the behavior of the solutions. These factors include the initial conditions, the impulsive conditions, the activation functions and the number of agents in the network. Each model is computer realized by coding the corresponding algorithms for calculating the values of the solution for each step.

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1.. INTRODUCTION

In the past years great attention has been paid to neural networks and they are broadly used with application in signal processing, content addressable memory, pattern recognition, combinatorial optimization. From a practical point of view, for computation

convenience, more adequate models are discrete-time neural networks (see [12], [14], [15]). One of the most important problems in the behavior of the equilibrium of the models is stability (see, [2], [5], [11], [7]). At the same time, impulses are a very useful mathematical apparatus to model some instantaneous perturbations in the process. Difference equations, being a discrete version of differential equations (see, [1], [6], [10]), could have also impulses (see, for example, [4]).

In this paper, we consider a class of discrete-time neural networks with constant delays, instantaneous switching topologies at certain times and time variable connection weights. The equilibrium point of the studied model is defined which is deeply connected not only with the difference equations but also with the impulsive conditions. In the main section several models for discrete delay neural network are presented and simulation study is conducted about the impact of different factors on the behavior of the solutions with respect to their convergence to the equilibrium point of the systems. These factors cover not only the influence of the initial and the impulsive conditions, but also the influence of the activation functions and the number of the agents included in the network. Following the appropriate theoretical schemes for solving the problems, the corresponding algorithms are coded to calculate the values of the solutions. The graphs are generated by the help of Wolfram Mathematica.

2.. STATEMENT OF THE PROBLEM AND DEFINITION OF SOLUTION

We will introduce basic notation used in this paper. Most of them are well known and used in the literature. Let $\mathbb{Z}_+$ be the set of all nonnegative integers; the increasing sequence $\{n_i\}_{i=0}^\infty : n_0 = 0$ be given; $\mathbb{Z}[a, b] = \{z \in \mathbb{Z}_+ : a \leq z \leq b\}$, $a, b \in \mathbb{Z}_+$, $a < b$, $\mathbb{Z}_a = \{z \in \mathbb{Z}_+ : z \geq a\}$ and

$$I_k = \mathbb{Z}[n_k + 1, n_{k+1} - 1], \quad k \in \mathbb{Z}_+.$$

We will study the behavior of the solutions of discrete neural networks with a delay. We will consider the case when the connections of the network are subject to a long lasting impulses at certain moments which will be modeled by the so-called non-instantaneous impulses. Also, we will study the general case of time variable connection weights.

Consider the discrete-time cellular neural network consisting of N interconnected

neurons described by a system of delay difference equations of the form

$$u_i(n) = a_i u_i(n-1) + \sum_{j=1}^n b_{ij}(n) f_j(u_j(n-1)) + \sum_{j=1}^n c_{ij}(n) f_j(u_j(n-m_{ij})) + G_i$$

$$\text{for } n \in \bigcup_{k=0}^{\infty} I_k, \quad i = 1, 2, \dots, N, \quad (1)$$

subject to impulses

$$u_i(n_k) = M_{ik} u_i(n_k - 1) + \sum_{j=1}^n \psi_{ij}^k S_j^k(u_j(n_k - 1))$$

$$+ \sum_{j=1}^n \phi_{ij}^k S_j^k(u_j(n_k - m_{ij})) + Q_{ik}, \quad \text{for } k \in \mathbb{Z}_1, \quad (2)$$

with initial conditions

$$u_i(n) = u_i^0(n), \quad n \in \mathbb{Z}[1-m, 0], \quad (3)$$

where $u_i(n)$ denotes the potential (or voltage) of the i -th neuron at discrete time n , a_i represents rate with which the i -th neuron resets its potential to the resting state when isolated from other neurons and inputs, $m_{ij} \geq 2$ corresponds to the time delay required in processing and transmitting a signal from j -th neuron to i -th neuron, $m = \max_{i,j \in \mathbb{Z}[1,N]} m_{ij}$, b_{ij}, c_{ij} represents the synaptic connection weight between the j -th neuron and the i -th neuron at discrete time n and discrete time $n - m_{ij}$, respectively, f_j is the neuron activation function, G_j denotes the i -th component of an external input source introduced from outside the network to the i -th neuron, M_{ik} represents rate with which the i -th neuron resets its potential to the resting state when isolated from other neurons and inputs at the time of the k -th impulse, ψ_{ij}^k, ϕ_{ij}^k represents the synaptic connection weight between the j -th neuron and the i -th neuron at discrete time n_k and discrete time $n_k - m_{ij}$, respectively, S_j^k is the neuron activation function of the j -th neuron at the time of the k -th impulse, Q_j denotes the i -th component of an external input source introduced from outside the network to the i -th neuron at the time of the k -th impulse, $\varphi_i(n), n \in \mathbb{Z}[-m, 0]$ is the initial function for the i -th neuron.

Let us define an equilibrium of the model (1), (2):

Definition 1. A vector $u^* \in \mathbb{R}^n$: $u^* = (u_1^*, u_2^*, \dots, u_n^*)$ is said to be an equilibrium

point of the impulsive discrete-time neural network (1), (2), if it satisfies the equalities

$$\begin{aligned} u_i^* &= a_i u_i^* + \sum_{j=1}^n b_{ij}(n) f_j(u_j^*) + \sum_{j=1}^n c_{ij}(n) f_j(u_j^*) + G_i, \quad n \in \bigcup_{k=0}^{\infty} I_k, i = 1, 2, \dots, N, \\ u_i^* &= M_{ik} u_i^*(n_k - 1) + \sum_{j=1}^n (\psi_{ij}^k + \phi_{ij}^k) S_j^k(u_j^*) + Q_{ik}, \quad k \in \mathbb{Z}_1. \end{aligned} \quad (4)$$

In connection with the given above definition let us make the following remark:

Remark 1. Note that if there are impulses in the discrete model, then it changes the definition of the equilibrium point. Also, if a point is an equilibrium point of a discrete model, then it is not necessarily to be an equilibrium point of the corresponding discrete model with impulses. But the opposite is true.

□

3.. SIMULATION STUDY OF THE IMPACTS ON THE DISCRETE DELAY NEURAL NETWORK

In this section we will consider several models for discrete delay neural network. The aim of these models is to conduct a simulation study about the impact of different factors on the behavior of the solutions with respect to their convergence to the equilibrium point of the system.

In the first model we will illustrate the influence of the initial values to the behavior of the solution.

Model 1. (*Impact of the initial conditions*) Consider the discrete model of neural network with two agents, impulses caused by a switched topology at initially given times and time variable bounded connection weights between neurons which is modeled by the impulsive discrete system:

$$\begin{aligned} u_1(n) &= -0.1u_1(n-1) + 0.2 \tanh(u_1(n-1)) - 0.1 \tanh(u_2(n-1)) \\ &\quad + 0.5 \tanh(u_1(n-3)) - 0.1 \tanh(u_2(n-3)) \\ u_2(n) &= -0.1u_2(n-1) + 0.1 \tanh(u_1(n-1)) + 0.2 \tanh(u_2(n-1)) \\ &\quad + 0.2 \tanh(u_1(n-3)) - 0.5 \tanh(u_2(n-3)) \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k, \end{aligned} \quad (5)$$

with impulsive conditions

$$u_1(n_k) = -0.2u_1(n_k - 1) \quad \text{and} \quad u_2(n_k) = 0.3u_2(n_k - 1) \quad \text{for } n_k = 6k, \quad k = 1, 2, \dots \quad (6)$$

and constant initial conditions

$$u_1(n) = 3 \quad \text{and} \quad u_2(n) = 1 \quad \text{for } n \in \mathbb{Z}[-3, 0]. \quad (7)$$

We will study if the behavior of the solutions of (5), (6) will be affected by a change in the initial conditions. For this purpose, we introduce three additional initial conditions

$$u_1(n) = -1 \quad \text{and} \quad u_2(n) = 2 \quad \text{for } n \in \mathbb{Z}[-3, 0], \quad (8)$$

$$u_1(n) = 4 \quad \text{and} \quad u_2(n) = -3 \quad \text{for } n \in \mathbb{Z}[-3, 0], \quad (9)$$

$$u_1(n) = 1 \quad \text{and} \quad u_2(n) = 0.5 \quad \text{for } n \in \mathbb{Z}[-3, 0]. \quad (10)$$

By the help of the computer realization the values of the solutions are calculated and they are given in Table 1, where $u_i^p(n)$ ($i = 1, 2$, $p = 1, 2, 3, 4$) denotes the solution of the IVP (5), (6) with the initial values (7), (8), (9) and (10), respectively.

n	$u_1^1(n)$	$u_2^1(n)$	$u_1^2(n)$	$u_2^2(n)$	$u_1^3(n)$	$u_2^3(n)$	$u_1^4(n)$	$u_2^4(n)$
1	0.244219	-0.625921	0.498541	0.340692	-0.029962	-0.717687	0.898315	0.039843
2	0.447837	-0.464108	0.569962	0.362153	-0.160833	-0.741183	0.796754	-0.041950
3	0.476554	-0.454488	0.579018	0.371984	-0.155583	-0.729529	0.801695	-0.048219
4	0.179208	-0.193461	0.138927	0.198830	0.091910	0.102529	-0.160522	0.076466
5	0.234382	-0.182815	0.221075	0.189482	0.190625	0.219317	-0.230018	0.117627
6	-0.046877	0.036563	-0.044215	-0.037897	0.057187	0.065795	-0.069005	0.035288
7	0.069099	-0.108676	0.087410	0.083180	-0.009341	-0.079087	0.095890	0.000829
8	0.104078	-0.114879	0.130513	0.090112	-0.042173	-0.162795	0.174780	-0.012712
9	-0.014586	0.016453	-0.019602	-0.012233	-0.031773	-0.052974	0.055715	-0.017497
10	0.037146	-0.039295	0.026509	0.041938	0.013835	0.014169	-0.026761	0.013211
11	0.058395	-0.046393	0.052911	0.049073	0.046911	0.055300	-0.060580	0.029842
12	-0.011679	0.009279	-0.010582	-0.009815	0.014073	0.016590	-0.018174	0.008953
13	0.014606	-0.021785	0.016686	0.017759	0.000748	-0.012353	0.015803	0.001692
14	0.025862	-0.029647	0.032570	0.023140	-0.010237	-0.040307	0.044074	-0.003165
15	-0.003638	0.004046	-0.004624	-0.003173	-0.007810	-0.013429	0.014627	-0.004442
16	0.007645	-0.007908	0.004837	0.008836	0.001402	0.000881	-0.003564	0.001944
17	0.014576	-0.011670	0.012715	0.012574	0.011195	0.013512	-0.015384	0.007288
18	-0.002915	0.002334	-0.002543	-0.002515	0.003358	0.004054	-0.004615	0.002186
19	0.003055	-0.004214	0.002982	0.003754	0.000872	-0.001384	0.002033	0.000762
20	0.006386	-0.007469	0.007990	0.005857	-0.002289	-0.009649	0.010736	-0.000678
21	-0.000926	0.000980	-0.001085	-0.000823	-0.001853	-0.003272	0.003671	-0.001078
22	0.001533	-0.001544	0.000812	0.001826	-0.000103	-0.000380	-0.000162	0.000179
23	0.003586	-0.002886	0.003019	0.003161	0.002565	0.003138	-0.003705	0.001711
24	-0.000717	0.000577	-0.000604	-0.000632	0.000769	0.000942	-0.001111	0.000513
25	0.000628	-0.000770	0.000473	0.000781	0.000363	0.000033	0.000072	0.000264
26	0.001563	-0.001837	0.001920	0.001461	-0.000466	-0.002220	0.002510	-0.000119
27	-0.000233	0.000233	-0.000250	-0.000209	-0.000418	-0.000761	0.000878	-0.000249
28	0.000296	-0.000289	0.000117	0.000368	-0.000121	-0.000224	0.000122	-0.000021
29	0.000870	-0.000703	0.000708	0.000781	0.000563	0.000691	-0.000847	0.000386

Table 1. Values of the solutions $u_i^p(n)$ ($i = 1, 2$; $p = 1, 2, 3, 4$; $n \in \mathbb{Z}[1, 29]$) of (5), (6)

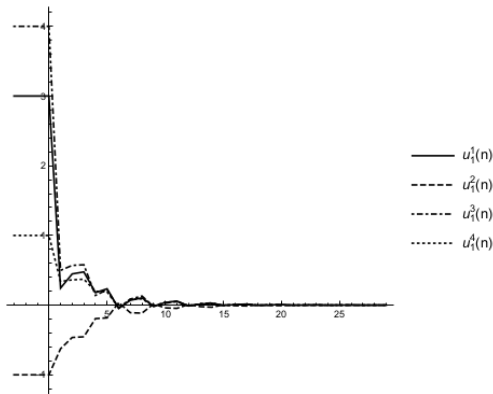


Figure 1. Graphs of the solutions $u_1^p(n)$ ($p = 1, 2, 3, 4$) of (5), (6) on the interval $[-3, 29]$.

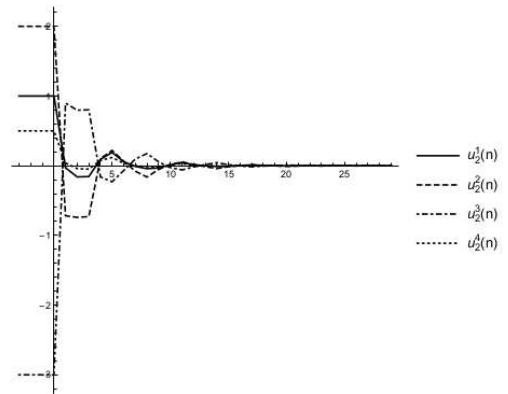


Figure 2. Graphs of the solutions $u_2^p(n)$ ($p = 1, 2, 3, 4$) of (5), (6) on the interval $[-3, 29]$.

The obtained values in Table 1 are used to graph the solutions (see Figures 1 and 2).

From Table 1, Figures 1 and 2, it could be seen that the zero solution of the neural network (5), (6) is exponentially stable and the change of the initial conditions has impact on the speed of convergence of the solutions. We can conclude that if the initial value is closer to 0, then it leads to faster approaching the zero.

□

In the next model we will study the influence of the impulsive conditions on the behavior of the solution.

Model 2. (*Impact of the impulsive conditions*) Consider the discrete model of neural network with two agents (5) with impulsive conditions:

$$u_1(n_k) = -2u_1(n_k - 1) \quad \text{and} \quad u_2(n_k) = 3u_2(n_k - 1) \quad \text{for} \quad n_k = 6k, \quad k = 1, 2, \dots \quad (11)$$

and with constant initial values (7).

Following the concept of this model, we will represent additional impulsive conditions to study if the behavior of the solutions of (5), (7) will be affected by the change in the impulsive conditions:

$$u_1(n_k) = 2u_1(n_k - 1), \quad u_2(n_k) = -3u_2(n_k - 1), \quad (12)$$

$$u_1(n_k) = n_k u_1(n_k - 1), \quad u_2(n_k) = n_k u_2(n_k - 1), \quad (13)$$

$$u_1(n_k) = (-2)^{n_k} u_1(n_k - 1), \quad u_2(n_k) = 3^{n_k} u_2(n_k - 1), \quad (14)$$

$$u_1(n_k) = 2^{n_k} u_1(n_k - 1), \quad u_2(n_k) = -3^{n_k} u_2(n_k - 1). \quad (15)$$

The problems are coded and the calculated values of the solutions are given in Tables 2 and 3, where by $u_i^p(n)$ ($i = 1, 2$; $p = 1, 2, \dots, 5$) we have denoted the corresponding solution of the IVP (5), (7) with the impulsive conditions (11), (12), (13), (14), and (15), respectively.

n	$u_1^1(n)$	$u_1^2(n)$	$u_1^3(n)$	$u_1^4(n)$	$u_1^5(n)$
1	0.244219	0.244219	0.244219	0.244219	0.244219
2	0.447837	0.447837	0.447837	0.447837	0.447837
3	0.476554	0.476554	0.476554	0.476554	0.476554
4	0.179208	0.179208	0.179208	0.179208	0.179208
5	0.234382	0.234382	0.234382	0.234382	0.234382
6	-0.468765	0.468765	1.406290	15.0005	15.000500
7	-0.012745	0.171729	0.034636	-1.320560	-1.120560
8	0.095775	0.114376	0.087079	0.154928	-0.053240
9	-0.255738	0.284852	0.373658	0.338023	0.681051
10	0.002794	0.079315	0.060874	-0.279256	-0.510894
11	0.058553	0.058039	0.056940	-0.059348	0.067000
12	-0.117105	0.116078	0.683275	-243.0890	274.434000
13	-0.016323	0.057945	0.045060	24.040300	-27.328400
14	0.020913	0.033038	0.016459	-2.299910	2.434540
15	-0.067168	0.079485	0.257386	-0.266028	0.653497
16	-0.009675	0.029907	0.042011	0.274618	-0.650546
17	0.013343	0.015992	0.015148	-0.263925	0.643025
18	-0.026685	0.031983	0.272671	-69186.3	168565
19	-0.005304	0.019755	0.054220	6918.5700	-16856.4
20	0.004572	0.010008	0.008105	-691.7820	1685.5300
21	-0.014049	0.024417	0.135854	68.2782	-167.6530
22	-0.004326	0.010599	0.028179	-5.927820	15.8653
23	0.002644	0.004717	0.004800	-0.307215	-0.686526
24	-0.005287	0.009434	0.115200	-5154210	-11518000
25	-0.001328	0.006867	0.030094	515421	1.1518×10^6
26	0.001021	0.003168	0.004746	-51542	-115180
27	-0.002277	0.008222	0.062439	5153.7	11517.5
28	-0.001312	0.003736	0.015389	-514.87	-1151.25
29	0.000431	0.001462	0.002033	50.987	114.625

Table 2. Values of the solutions $u_1^p(n)$ ($p = 1, 2, \dots, 5$) of (5), (7) with the impulsive conditions (11), (12), (13), (14), and (15), respectively

n	$u_2^1(n)$	$u_2^2(n)$	$u_2^3(n)$	$u_2^4(n)$	$u_2^5(n)$
1	-0.029962	-0.029962	-0.029962	-0.029962	-0.029962
2	-0.160833	-0.160833	-0.160833	-0.160833	-0.160833
3	-0.155583	-0.155583	-0.155583	-0.155583	-0.155583
4	0.091910	0.091910	0.091910	0.091910	0.091910
5	0.190625	0.190625	0.190625	0.190625	0.190625
6	0.571874	-0.571874	1.143750	138.965000	-138.965
7	-0.007924	-0.012803	0.127067	-13.606900	13.7862
8	-0.050205	-0.032412	-0.032105	1.025860	-1.307530
9	-0.341271	0.353956	-0.225018	-0.232768	0.652704
10	-0.055213	0.100732	-0.042308	0.336733	-0.552905
11	0.038947	0.056897	0.035273	-0.351365	0.328874
12	0.116841	-0.170690	0.423275	-186730	-174777
13	0.028058	-0.039552	0.130275	18672.5	17477.8
14	-0.006594	-0.014986	0.011131	-1866.79	-1747.82
15	-0.080039	0.109442	-0.078355	186.681	175.381
16	-0.031966	0.050131	-0.038384	-18.7941	-17.9807
17	0.003317	0.022092	-0.001910	2.01018	2.23782
18	0.009952	-0.066276	-0.034372	7.78785×10^8	-866976000
19	0.012370	-0.022476	0.050756	-77878500	8.66976×10^7
20	0.001716	-0.008118	0.014468	7.78785×10^6	-8669760
21	-0.009683	0.039673	0.072658	-778786	866976
22	-0.009618	0.021591	0.006220	77879.2	-86698.2
23	-0.001338	0.009279	-0.002173	-7788.52	8670.42
24	-0.004014	-0.027836	-0.052163	-2.19971E+15	-2.44878E+15
25	0.003014	-0.010513	0.008787	2.19971×10^{14}	2.44878×10^{14}
26	0.001366	-0.004060	0.005934	-2.19971E+13	-2.44878E+13
27	0.001188	0.015712	0.050064	2.19971×10^{12}	2.44878×10^{12}
28	-0.001881	0.009023	0.012858	-2.19971E+11	-2.44878E+11
29	-0.000798	0.003940	0.000807	2.19971×10^{10}	2.44878×10^{10}

Table 3. Values of the solutions $u_2^p(n)$ ($p = 1, 2, \dots, 5$) of (5), (7) with the impulsive conditions (11), (12), (13), (14), and (15), respectively

From Tables 2 and 3, Figures 3 and 4, it can be seen that the zero solution of the neural network (5), (7) is exponentially stable and the change of the impulsive conditions has impact on the speed of convergence of the solutions. We can conclude that if the impulsive condition is closer to 0, then it leads to faster approaching the zero.

□

In the next model we will study the influence of the activation functions on the behavior of the solution.

Model 3. (*Impact of the activation functions*) Consider the discrete model of neural network with two agents, impulses caused by a switched topology at some times and time variable bounded connection weights between neurons which is the impulsive

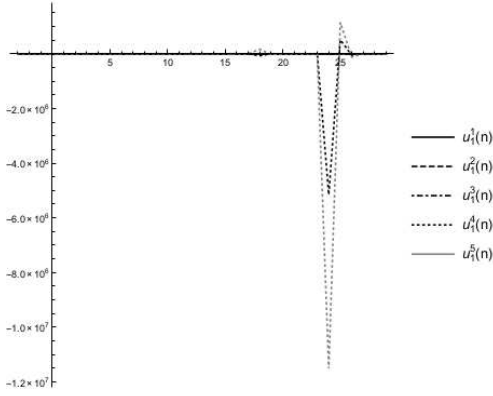


Figure 3. Graphs of the solutions $u_1^p(n)$ ($p = 1, 2, \dots, 5$) of (5), (7) on the interval $[-3, 29]$ with the impulsive conditions (11), (12), (13), (14), and (15), respectively.

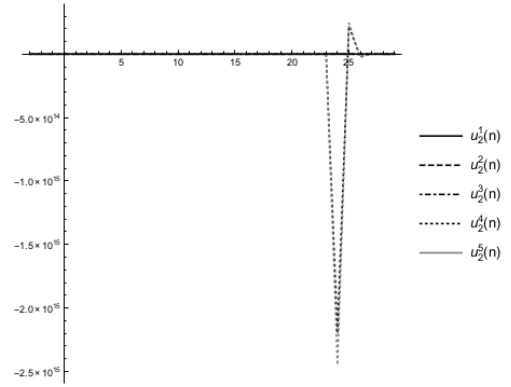


Figure 4. Graphs of the solutions $u_2^p(n)$ ($p = 1, 2, \dots, 5$) of (5), (7) on the interval $[-3, 29]$ with the impulsive conditions (11), (12), (13), (14), and (15), respectively.

discrete modification of the example given in [13]:

$$\begin{aligned}
 u_1(n) &= -0.1u_1(n-1) + 0.2 \tanh(u_1(n-1)) - 0.1 \sin(u_2(n-1)) \\
 &\quad + 0.5 \sin(u_1(n-3)) - 0.1 \sin(u_2(n-3)) \\
 u_2(n) &= -0.1u_2(n-1) + 0.1 \sin(u_1(n-1)) + 0.2 \sin(u_2(n-1)) \\
 &\quad + 0.2 \sin(u_1(n-3)) - 0.5 \sin(u_2(n-3)) \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,
 \end{aligned} \tag{16}$$

with impulsive conditions

$$u_1(n_k) = -0.2u_1(n_k - 1) \quad \text{and} \quad u_2(n_k) = 0.3u_2(n_k - 1) \quad \text{for } n_k = 6k, \quad k = 1, 2, \dots \tag{17}$$

and constant initial values (7). The graph of the solution of the neural network (16), (17), (7) is given on Figure 5.

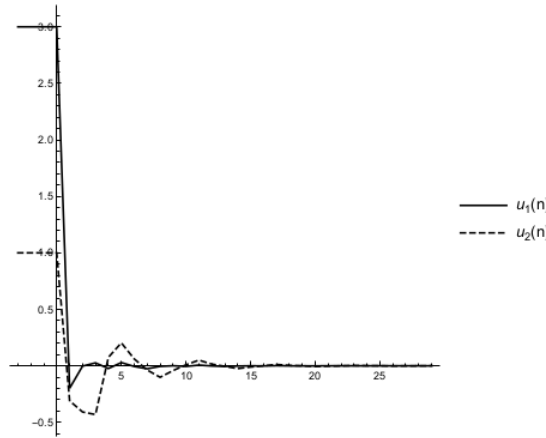


Figure 5. Graph of the solution of (16), (17), (7) on the interval $[-3, 29]$.

From Table 4 and Figure 5, it could be seen that the zero solution of the neural network (16), (17), (7) is exponentially stable.

Now, we will compare the obtained results from the computer simulations of the problems in Model 3 and Model 1, to make some conclusions if the included activation functions in the right sides of the equations have impact on the behavior of the solutions. For convenience, let us denote by $u_i^p(n)$ ($i = 1, 2$; $p = 1, 2$) the corresponding solutions of (16), (17), (7) and (5), (6), (7).

n	$u_1^1(n)$	$u_1^2(n)$	$u_2^1(n)$	$u_2^2(n)$
1	-0.198723	0.244219	-0.310105	-0.029962
2	-0.002428	0.447837	-0.408421	-0.160833
3	0.025886	0.476554	-0.431344	-0.155583
4	-0.023796	0.179208	0.075200	0.091910
5	0.028610	0.234382	0.203221	0.190625
6	-0.005722	-0.046877	0.060966	0.057187
7	-0.026075	0.069099	-0.036807	-0.009341
8	-0.004806	0.104078	-0.101477	-0.042173
9	0.000696	-0.014586	-0.042202	-0.031773
10	-0.005068	0.037146	0.009037	0.013835
11	0.006317	0.058395	0.050087	0.046911
12	-0.001263	-0.011679	0.015026	0.014073
13	-0.005066	0.014606	-0.004156	0.000748
14	-0.001939	0.025862	-0.024692	-0.010237
15	0.000141	-0.003638	-0.010428	-0.007810
16	-0.001061	0.007645	0.000036	0.001402
17	0.001390	0.014576	0.011854	0.011195
18	-0.000278	-0.002915	0.003556	0.003358
19	-0.000917	0.003055	0.000098	0.000872
20	-0.000592	0.006386	-0.005731	-0.002289
21	0.000019	-0.000926	-0.002466	-0.001853
22	-0.000220	0.001533	-0.000477	-0.000103
23	0.000303	0.003586	0.002677	0.002565
24	-0.000061	-0.000717	0.000803	0.000769
25	-0.000149	0.000628	0.000269	0.000363
26	-0.000158	0.001563	-0.001266	-0.000466
27	2.0397×10^{-7}	-0.000233	-0.000556	-0.000418
28	-0.000046	0.000296	-0.000220	-0.000121
29	0.000065	0.000870	0.000575	0.000563

Table 4. Values of the solutions $u_i^p(n)$ ($i = 1, 2; p = 1, 2$) of IVP (16), (17), (7) and (5), (6), (7), respectively

The calculated values for the solutions are used to graph them (see Figures 6 and 7).

From Table 4, Figures 6 and 7 it can be seen that the change in the activation functions impacts on the speed of convergence of the solutions.

□

In the next model we will investigate the impact of the number of agents included in the neural network.

Model 4. (*Impact of the number of agents*) Consider the discrete model of neural network with three agents, impulses caused by a switched topology at some times and time variable bounded connection weights between neurons (the continuous case is

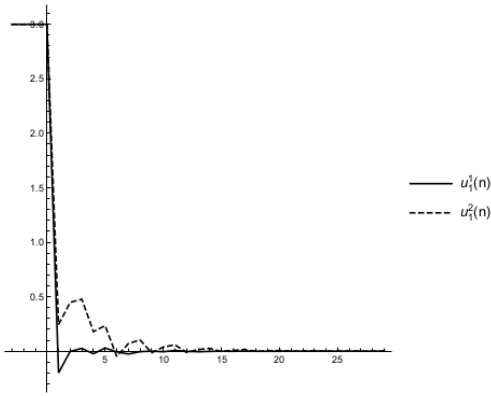


Figure 6. Graphs of the solutions $u_1^p(n)$ ($p = 1, 2$) on the interval $[-3, 29]$ of (16), (17), (7) and (5), (6), (7), respectively.

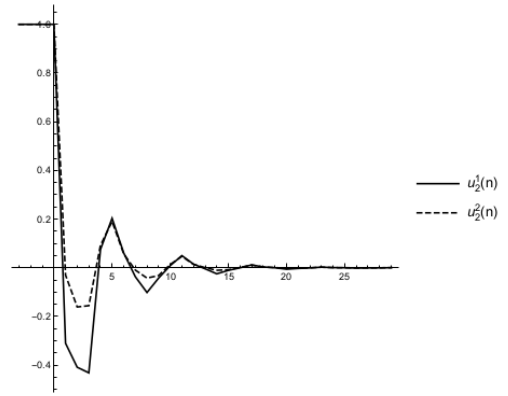


Figure 7. Graphs of the solutions $u_2^p(n)$ ($p = 1, 2$) on the interval $[-3, 29]$ of (16), (17), (7) and (5), (6), (7), respectively.

studied in [3]):

$$\begin{aligned}
 u_1(n) &= -0.4u_1(n-1) - 0.04\sin(u_1(n-1)) + 0.06\sin(u_2(n-1)) \\
 &\quad + 0.02\sin(u_3(n-1)) + 0.03\tanh(u_1(n-2)) \\
 &\quad - 0.04\tanh(u_2(n-2)) + 0.05\tanh(u_3(n-2)) \\
 u_2(n) &= -0.55u_2(n-1) + 0.03\sin(u_1(n-1)) - 0.08\sin(u_2(n-1)) \\
 &\quad + 0.04\sin(u_3(n-1)) - 0.05\tanh(u_1(n-2)) \\
 &\quad - 0.02\tanh(u_2(n-2)) - 0.03\tanh(u_3(n-2)) \\
 u_3(n) &= -0.5u_3(n-1) - 0.05\sin(u_1(n-1)) - 0.02\sin(u_2(n-1)) \\
 &\quad + 0.01\sin(u_3(n-1)) + 0.06\tanh(u_1(n-2)) \\
 &\quad + 0.04\tanh(u_2(n-2)) - 0.04\tanh(u_3(n-2)) \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,
 \end{aligned} \tag{18}$$

with impulses at times $n_k = 6k$, $k = 1, 2, \dots$

$$u_1(n_k) = 0.9u_1(n_k - 1), \quad u_2(n_k) = 1.3u_2(n_k - 1), \quad u_3(n_k) = 0.8u_3(n_k - 1), \tag{19}$$

and constant initial values

$$u_1(n) = 0.1, \quad u_2(n) = 0.2 \quad \text{and} \quad u_3(n) = 0.3 \quad \text{for } n \in \mathbb{Z}[-1, 0]. \tag{20}$$

The calculated values of the solutions of (18), (19), (20) are given in Table 5 and they are used for the graphs in Figure 8. From Table 5 and Figure 8 it can be seen that the zero solution of the neural network (18), (19) is exponentially stable.

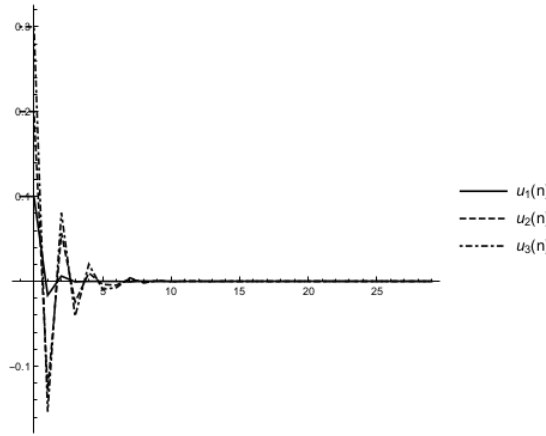


Figure 8. Graph of the solution of (18), (19), (20) on the interval $[-1, 29]$.

Now, we will compare the obtained results from the computer simulations of the problems in Model 4 and Model 3, to make some conclusions if the number of the agents included in the network system has impact on the behavior of the solutions. For convenience, let us denote by $u_i^p(n)$ ($i = 1, 2, 3$; $p = 1, 2$) the corresponding solutions of (18), (19), (20) and (16), (17), (7), respectively.

n	$u_1^1(n)$	$u_2^1(n)$	$u_3^1(n)$	$u_1^2(n)$	$u_2^2(n)$
1	-0.016502	-0.128748	-0.153787	-0.198723	-0.310105
2	0.006154	0.056790	0.080977	-0.002428	-0.408421
3	-0.000687	-0.024392	-0.041131	0.025886	-0.431344
4	-0.000028	0.009835	0.020083	-0.023796	0.075200
5	-0.000097	-0.003638	-0.009408	0.028610	0.203221
6	-0.000087	-0.004730	-0.007527	-0.005722	0.060966
7	-0.000724	0.003036	0.004012	-0.026075	-0.036807
8	0.000391	-0.001449	-0.001884	-0.004806	-0.101477
9	-0.000239	0.000704	0.000850	0.000696	-0.042202
10	0.000140	-0.000351	-0.000378	-0.005068	0.009037
11	-0.000083	0.000183	0.000165	0.006317	0.050087
12	-0.000075	0.000237	0.000132	-0.001263	0.015026
13	0.000048	-0.000151	-0.000070	-0.005066	-0.004156
14	-0.000037	0.000089	0.000035	-0.001939	-0.024692
15	0.000026	-0.000053	-0.000017	0.000141	-0.010428
16	-0.000018	0.000032	8.15457×10^{-6}	-0.001061	0.000036
17	0.000012	-0.000020	-0.000004	0.001390	0.011854
18	0.000011	-0.000027	-0.000003	-0.000278	0.003556
19	-0.000005	0.000017	1.45139×10^{-6}	-0.000917	0.000098
20	4.67196×10^{-6}	-0.000011	-0.000001	-0.000592	-0.005731
21	-3.4796×10^{-6}	6.69249×10^{-6}	7.93957×10^{-7}	0.000019	-0.002466
22	2.46071×10^{-6}	-4.27753×10^{-6}	-4.514×10^{-7}	-0.000220	-0.000477
23	-1.68078×10^{-6}	2.76692×10^{-6}	2.10867×10^{-7}	0.000303	0.002677
24	-1.5127×10^{-6}	3.597×10^{-6}	1.68693×10^{-7}	-0.000061	0.000803
25	7.34226×10^{-7}	-2.28237×10^{-6}	-7.75692×10^{-8}	-0.000149	0.000269
26	-6.42379×10^{-7}	1.45545×10^{-6}	9.33151×10^{-8}	-0.000158	-0.001266
27	4.81283×10^{-7}	-9.21209×10^{-7}	-8.68528×10^{-8}	2.0397×10^{-7}	-0.000556
28	-3.41598×10^{-7}	5.91537×10^{-7}	5.28606×10^{-8}	-0.000046	-0.00022
29	2.33797×10^{-7}	-3.83836×10^{-7}	-2.51498×10^{-8}	0.000065	0.000575

Table 5. Values of the solutions $u_i^p(n)$, $i = 1, 2, 3$, $p = 1, 2$, $n = 1, 2, \dots, 29$, of (18), (19), (20) and (16), (17), (7), respectively

From Table 5, Figures 9 and 10 it can be seen that the number of the agents included in the network system impacts on the speed of convergence of the solutions.

□

In the next model we will study the case when different activation functions and external perturbations are included in the neural network.

Model 5. (*Impact of the initial conditions in case of nonzero equilibrium point*) Let $n_k = 6k$ for $k = 1, 2, \dots$. Consider the discrete model of neural network with three agents, impulses caused by a switched topology at some times and time variable

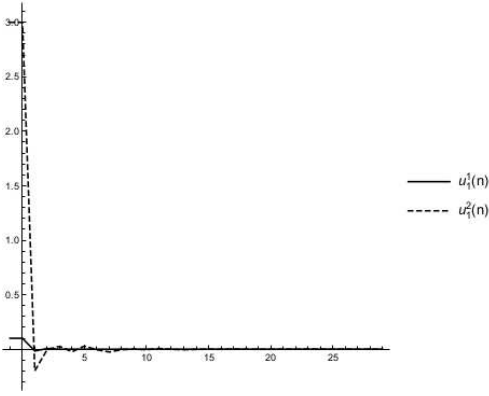


Figure 9. Graphs of the solutions $u_1^p(n)$ ($p = 1, 2$) on the interval $[-1, 29]$ of (18), (19), (20) and (16), (17), (7), respectively.

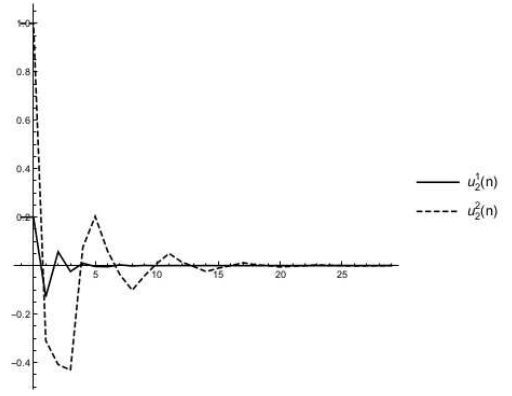


Figure 10. Graphs of the solutions $u_2^p(n)$ ($p = 1, 2$) on the interval $[-1, 29]$ of (18), (19), (20) and (16), (17), (7), respectively.

bounded connection weights between neurons (the continuous case is studied in [3]):

$$\begin{aligned}
 u_1(n) &= -0.4u_1(n-1) - 0.04 \tanh(u_1(n-1)) + 0.06 \tanh(u_2(n-1)) \\
 &\quad + 0.02 \tanh(u_3(n-1)) + 0.03 \tanh(u_1(n-3)) \\
 &\quad - 0.04 \tanh(u_2(n-3)) + 0.05 \tanh(u_3(n-3)) + 0.45 \\
 u_2(n) &= -0.55u_2(n-1) + 0.03 \tanh(u_1(n-1)) - 0.08 \tanh(u_2(n-1)) \\
 &\quad + 0.04 \tanh(u_3(n-1)) - 0.05 \tanh(u_1(n-3)) \\
 &\quad + 0.02 \tanh(u_2(n-3)) - 0.03 \tanh(u_3(n-3)) - 0.75 \\
 u_3(n) &= -0.5u_3(n-1) - 0.05 \tanh(u_1(n-1)) - 0.02 \tanh(u_2(n-1)) \\
 &\quad + 0.01 \tanh(u_3(n-1)) + 0.06 \tanh(u_1(n-3)) \\
 &\quad + 0.04 \tanh(u_2(n-3)) - 0.04 \tanh(u_3(n-3)) + 0.8 \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,
 \end{aligned} \tag{21}$$

with impulses at times n_k for $k \in \mathbb{Z}_1$:

$$\begin{aligned}
 u_1(n_k) &= -0.4u_1(n_k-1) - 0.01 \tanh(u_1(n_k-1)) + 0.02 \tanh(u_2(n_k-1)) \\
 &\quad + 0.07 \tanh(u_3(n_k-1)) + 0.45 \\
 u_2(n_k) &= -0.55u_2(n_k-1) - 0.02 \tanh(u_1(n_k-1)) - 0.06 \tanh(u_2(n_k-1)) \\
 &\quad + 0.01 \tanh(u_3(n_k-1)) - 0.75 \\
 u_3(n_k) &= -0.5u_3(n_k-1) + 0.01 \tanh(u_1(n_k-1)) + 0.02 \tanh(u_2(n_k-1)) \\
 &\quad - 0.03 \tanh(u_3(n_k-1)) + 0.8
 \end{aligned} \tag{22}$$

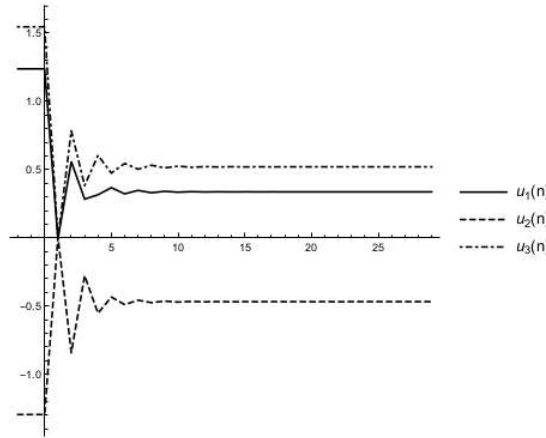


Figure 11. Graph of the solution of (21), (22), (23) on the interval $[-2, 29]$.

and constant initial values

$$u_1(n) = 1.238, \quad u_2(n) = -1.295 \quad \text{and} \quad u_3(n) = 1.546 \quad \text{for } n \in \mathbb{Z}[-2, 0]. \quad (23)$$

The graph of the solution of the neural network (21), (22) (23) is given on Figure 11.

We will study if a change in the initial conditions will affect the behavior of the solutions of (21), (22) with respect to their convergence to the equilibrium point. Let us consider the following constant initial conditions

$$u_1(n) = -1.2, \quad u_2(n) = 1.3 \quad \text{and} \quad u_3(n) = -1.5 \quad \text{for } n \in \mathbb{Z}[-2, 0]. \quad (24)$$

Now, we will compare the obtained results from the computer simulations of the problems. For convenience, let us denote by $u_i^p(n)$ ($i = 1, 2, 3$; $p = 1, 2$) the corresponding solutions of (21), (22), (23) and (21), (22), (24), respectively. The calculated values of the solutions are given in Table 6.

n	$u_1^1(n)$	$u_2^1(n)$	$u_3^1(n)$	$u_1^2(n)$	$u_2^2(n)$	$u_3^2(n)$
1	-0.00693883	0.00610937	-0.00915346	0.892211	-1.50908	1.58605
2	0.558656	-0.84127	0.784461	-0.0761353	0.296765	0.0193361
3	0.283594	-0.27781	0.382454	0.396445	-0.851723	0.809213
4	0.315647	-0.55191	0.604222	0.351833	-0.269467	0.366775
5	0.367634	-0.434261	0.474799	0.274135	-0.54757	0.614688
6	0.322194	-0.489242	0.54468	0.366017	-0.41879	0.468934
7	0.347624	-0.456941	0.502777	0.312789	-0.492682	0.556007
8	0.330227	-0.474911	0.53217	0.350828	-0.452998	0.495228
9	0.340687	-0.463925	0.512376	0.328108	-0.477018	0.536301
10	0.334387	-0.470658	0.525171	0.34185	-0.462446	0.509511
11	0.338257	-0.466489	0.516838	0.333567	-0.471526	0.527043
12	0.335999	-0.469075	0.521875	0.338385	-0.465898	0.516414
13	0.337287	-0.467477	0.518939	0.335645	-0.469372	0.52262
14	0.336406	-0.468429	0.520901	0.337516	-0.467302	0.518463
15	0.336992	-0.467865	0.519629	0.336275	-0.468532	0.521161
16	0.336614	-0.468199	0.520431	0.337076	-0.467803	0.519457
17	0.336857	-0.468	0.51992	0.336559	-0.468236	0.520544
18	0.336713	-0.468122	0.520226	0.336865	-0.467971	0.519893
19	0.336795	-0.468046	0.520048	0.336693	-0.468137	0.520271
20	0.336741	-0.468092	0.520166	0.336808	-0.468037	0.52002
21	0.336776	-0.468064	0.52009	0.336733	-0.468097	0.520182
22	0.336753	-0.468081	0.520138	0.33678	-0.468061	0.52008
23	0.336768	-0.468071	0.520108	0.33675	-0.468083	0.520145
24	0.336759	-0.468077	0.520126	0.336768	-0.468069	0.520106
25	0.336764	-0.468073	0.520115	0.336758	-0.468078	0.520128
26	0.336761	-0.468075	0.520122	0.336765	-0.468072	0.520114
27	0.336763	-0.468074	0.520118	0.33676	-0.468076	0.520123
28	0.336761	-0.468075	0.520121	0.336763	-0.468074	0.520117
29	0.336762	-0.468074	0.520119	0.336761	-0.468075	0.520121

Table 6. Values of the solutions $u_i^p(n)$ ($i = 1, 2, 3$; $p = 1, 2$), $n = 1, 2, \dots, 29$, of (21), (22), (23) and (21), (22), (24), respectively

From Table 6, Figure 12, Figure 13 and Figure 14 we can see that the stated conclusion in Model 1 is also valid. The change of the initial condition affects the speed of the convergence to the equilibrium point which in this case is not the zero solution. From Table 6, Figure 12, Figure 13 and Figure 14 we can see that the stated conclusion in Model 1 is also valid. The change of the initial condition affects the speed of the convergence to the equilibrium point which in this case is not the zero solution.

4.. CONCLUSIONS

In this paper by computer simulations the influence of various factors on the behavior of the solution of discrete models of neural networks is illustrated. It is shown the influence of the initial condition (see Model 1 and Model 5), the influence of the impulsive

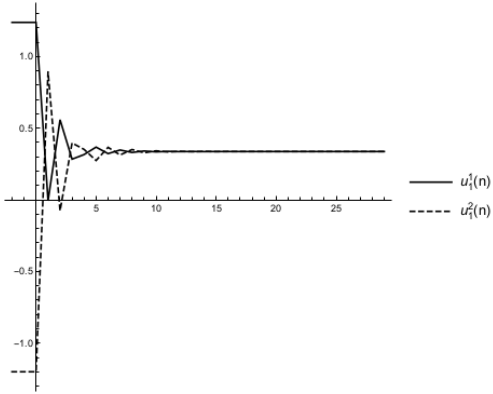


Figure 12. Graphs of the solutions of $u_1^p(n)$ ($p = 1, 2$) on the interval $[-2, 29]$ of (21), (22), (23) and (21), (22), (24), respectively

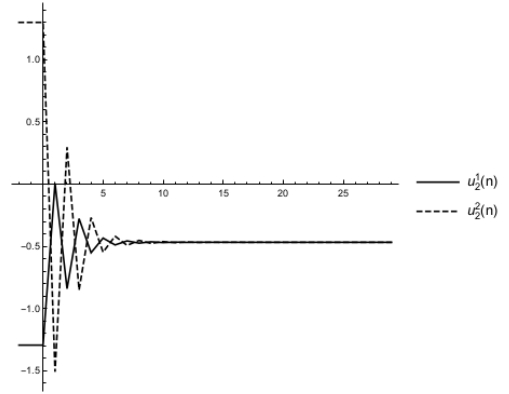


Figure 13. Graphs of the solutions of $u_2^p(n)$ ($p = 1, 2$) on the interval $[-2, 29]$ of (21), (22), (23) and (21), (22), (24), respectively

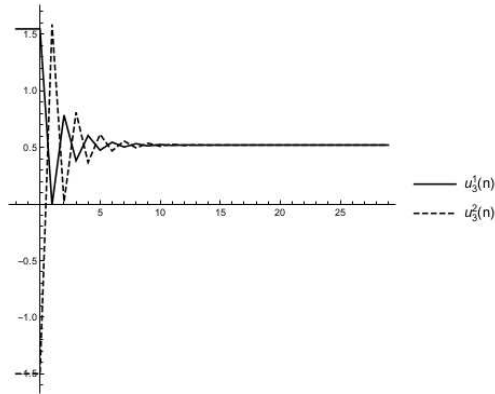


Figure 14. Graphs of the solutions of $u_3^p(n)$ ($p = 1, 2$) on the interval $[-2, 29]$ of (21), (22), (23) and (21), (22), (24), respectively

conditions (see Model 2), the influence of the activation function (see Model 3) and the influence of the number of the agents included in the system (see Model 4).

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