International Journal of Differential Equations and Applications
Volume **19**, No. 1 (2020), pages: 97-107
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu

ijdea.eu

SEVEN-POINT FINITE ELEMENT OF INTEGRAL TYPE APPLIED TO CONFORMING – NONCONFORMING METHOD

M.R. Racheva

Department of Mathematics and Informatics
Technical University of Gabrovo
Gabrovo, BULGARIA

ABSTRACT: The paper deals with integral variants of a seven-point triangular finite element. This is an enriched quadratic element constructed by adding a non-negative basis bulb-function which vanishes on the element boundary. The polynomial space which seven-point element uses is $P_2 + span\{x^2y + xy^2\}$.

The main goal here is to determine two-sided bounds of eigenvalues for a second-order elliptic operator. The method consists of finite element solving of the problem by means of seven-point conforming element and then constructing an nonconforming interpolant of the approximate conforming eigendunctions. Thus, solving only once the eigenvalue problem, we get upper and lower bounds for the exact eigenvalues. Furthermore, the fact that the nonconforming interpolants uses the nodal values of the conforming approximate eigenfunctions gives an obvious computational advantage.

Finally, numerical experiments confirming the efficiency of the proposed algorithm are also provided.

AMS Subject Classification: 65N30, 65N25, 65N12, 65N15

Key Words: seven-point finite element, conforming/nonconforming FEM, eigenvalue problem

Received: December 10, 2019 **Revised:** September 1, 2020
Published: November 20, 2020 **doi:** 10.12732/ijdea.v19i1.7
Academic Publications, Ltd. https://acadpubl.eu

1. INTRODUCTION AND PRELIMINARIES

Finite element application using combinations of conforming and nonconforming approaches are subject to intensive research in recent years (see e.q. [2, 3, 7]). Here, we consider a special triangular element in the plane. The 7-point 2-simplex finite element of reference $(K, \mathcal{P}, \Sigma)$ is defined as [4]:

- $K = \{(x, y) :\ x \geq 0, y \geq 0, x + y \leq 1\}$ is the unit 2-simplex;
- $\mathcal{P}$ is a set of polynomials which satisfies $P_2 \subset \mathcal{P} \subset P_3$, where P_s is the space of all polynomials of degree, not exceeding s on K;
- $\Sigma = \Big\{(x, y) :\ x = \frac{i}{2}; y = \frac{j}{2}; i + j \leq 2; i, j \in \{0, 1, 2\} \cup (\frac{1}{3}, \frac{1}{3})\Big\}$ is a set of the Lagrangian interpolation nodes for the 6-node element adding the barycenter.

The diagonalization of the mass matrix (lumped mass matrix) is an advantageous practical step when parabolic- or eigenvalue problems are solved by finite element method. In the family of isoparametric triangular elements this is a unique finite element giving lumped mass approximation (see [4, 5]). This fact is due also to the presence of a suitable quadrature formula with positive coefficients exact for polynomials of P_3 (see [8], p. 184). On the other hand, a key role of some postprocessing methods play by certain locally supported, nonnegative functions (the seventh-point basic function in our case) that are commonly referred to as bubble functions [1].

The main goal of this paper is to present an interesting and useful application. It is well-known that the eigenvalues obtained by conforming FE method approximate the exact ones from above. This fact comes from the min-max characterization of the eigenvalues (see [6], p. 699). Therefore, the question is to find a relevant approximation of the exact eigenvalues from below. Here, we propose an efficient method for obtaining eigenvalues estimates from below. For this purpose, we modify the 7-point conforming element by replacing the nodal values except for the values in the vertices with the integral values on the sides and in the element itself. So, an integral variant of 7-point triangular element is introduced. The basis functions of the reference finite element are (see Fig. 1):

$$\begin{aligned}
\varphi_1(x, y) &= z^2 + 10xyz - 2(x + y)z, & \varphi_4(x, y) &= 6xy(1 - 5z),\\
\varphi_2(x, y) &= 3x^2 + 10xyz - 2x, & \varphi_5(x, y) &= 6yz(1 - 5x),\\
\varphi_3(x, y) &= 3y^2 + 10xyz - 2y, & \varphi_6(x, y) &= 6xz(1 - 5y),
\end{aligned}$$

$$\varphi_7(x, y) = 120xyz,$$

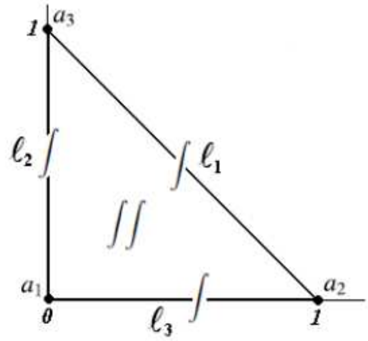

Figure 1: Reference finite element

where $0 \leq x, y \geq 1, z = 1 - x - y$.

Now, we introduce a model eigenvalue problem. Let Ω be a bounded polygonal domain in R^2 with boundary $\Gamma = \partial\Omega$. Let also $H^m(\Omega)$ be the usual $m-$th order $(m \geq 0)$ Sobolev space on Ω with a norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$ and $(\cdot,\cdot)$ denote the $L_2(\Omega)-$inner product.

The weak form of eigenvalue problem is: Find a number $\lambda \in R$ and a function $u \in V \equiv H_0^1(\Omega)$, $\|u\|_{0,\Omega} = 1$ such that

$$a(u, v) = \lambda(u, v), \quad \forall v \in V, \tag{1}$$

where

$$a(u, v) = \int\int_{\Omega} \nabla u \cdot \nabla v \, dx \, dy, \quad \forall u, v \in V.$$

The problem (1) has a countable sequence of real eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and the corresponding eigenfunctions $u_1, u_2, \ldots$ can be assumed to satisfy $\|u_j\|_{0,\Omega} = 1$; $(u_i, u_j) = \delta_{ij}, i, j \geq 1$.

Let $V_h \subset V$ be a conforming finite element space. If $\{\tau_h\}$ is a family of triangulations of Ω (h is the mesh parameter), then V_h consists of integral type 7-point rightangular triangles.

For any test function v and $K \in \tau_h$, the degree of freedom are (see Fig. 1): $v(a_j)$, $\frac{1}{|l_j|}\int_{l_j} v(s)\, ds$ and $\frac{1}{|K|}\int\int_K v(x, y)\, dx\, dy$, where $a_j, j = 1, 2, 3$ are the vertices, $l_j, j = 1, 2, 3$ are the edges of K and $|l_j| = \int_{l_j} ds$; $|K| = \int\int_K dx\, dy$.

Thus, the corresponding approximation of (1) is: Find a number $\lambda_h \in R$ and a function $u_h \in V_h, \|u_h\|_{0,\Omega} = 1$ such that

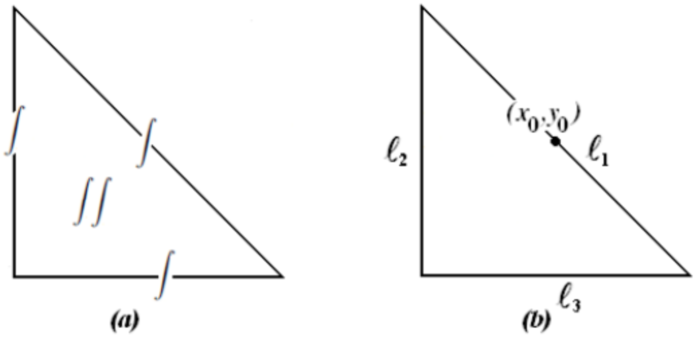

Figure 2: (a) EC-R element; (b) Triangular element

$$a(u_h, v_h) = \lambda_h(u_h, v_h), \quad \forall v_h \in V_h. \tag{2}$$

2. MAIN RESULTS

Supposing that (2) is already solved, the main question is how to get a good approximation from below of λ in an easy and effective way.

Let us consider nonconforming finite element space of the extension of Crouzeix-Raviart element (denoted by EC-R). Its degrees of freedom could ne obtained from those of the integral type 7-point 2-simplex element by removing the degrees of freedom in the element vertices (see Fig. 2a).

So, the degrees of freedom of EC-R element are: $\dfrac{1}{|l_j|}\displaystyle\int_{l_j} v(s)\,ds, j = 1, 2, 3$ and $\dfrac{1}{|K|}\displaystyle\int\int_K v(x, y)\,dx\,dy.$

This nonconforming finite element space we denote by $\widetilde{V}_h$ and define by:

$$\widetilde{V}_h = \big\{v \in L_2(\Omega) : v_{|K} = span\{1, x, y, x^2 + y^2\},$$

$$v \text{ is integrally continuous on } \Omega\big\}.$$

In $\widetilde{V}_h$ we define the following interpolation operator: $\widetilde{i}_h : V \to \widetilde{V}_h$. For any $v \in V$:

$$\int_{l_j} \widetilde{i}_h v\,ds = \int_{l_j} v\,ds, \ \ j = 1, 2, 3;$$

$$\int\int_K \widetilde{i}_h v\,dx\,dy = \int\int_K v\,dx\,dy, \tag{3}$$

for $K \in \tau_h$ with edges $l_j, j = 1,2,3$.

First, a mesh-dependent bilinear form is defined, which approximates $a(\cdot,\cdot)$ on $\widetilde{V}_h + H_0^1(\Omega)$:

$$a_h(u,v) = \sum_{K\in\tau_h} \int\int_K \nabla u \cdot \nabla v \, dx\, dy.$$

Lemma 1. *Let τ_h be uniform partition of Ω generated from rightangular triangular elements. Then for any $v \in V$ and $\widetilde{v}_h \in \widetilde{V}_h$*

$$a_h(\widetilde{i}_h v - v, \widetilde{v}_h) = 0. \tag{4}$$

Proof. If K_0 is a fixed reference triangle, then the result on arbitrary $K \in \tau_h$ will be transfered from K_0 using an affine transformation.

The equations of the edges of K_0 are respectively:

$$l_1 : x - x_0 + y - y_0 = 0; \quad l_2 : x - x_0 = -\frac{h_1}{2}; \quad l_3 : y - y_0 = -\frac{h_2}{2},$$

where (x_0, y_0) is the midpoint of the hypotenuse (Fig. 2b) and $h = \sqrt{h_1^2 + h_2^2}$.

The following notations for partial derivatives are adopted:
$\partial_x \cdot = \dfrac{\partial \cdot}{\partial x}$, $\partial_y \cdot = \dfrac{\partial \cdot}{\partial y}$ and so on.

Thus, for any $v \in V$, $\widetilde{v}_h \in \widetilde{V}_h$

$$\begin{aligned} a_h(\widetilde{i}_h v - v, \widetilde{v}_h) &= \textstyle\sum_{K\in\tau_h} \int\int_K \nabla(\widetilde{i}_h v - v)\cdot \nabla \widetilde{v}_h \, dx\, dy \\ &= \textstyle\sum_{K\in\tau_h} \int\int_K \Big(\partial_x(\widetilde{i}_h v - v)\partial_x \widetilde{v}_h + \partial_y(\widetilde{i}_h v - v)\partial_y \widetilde{v}_h\Big)\, dx\, dy. \end{aligned} \tag{5}$$

Since $\widetilde{v}_h$ is an incomplete quadratic polynomial on K_0, we have

$$\begin{aligned} \widetilde{v}_h(x,y) &= \widetilde{v}_h(x_0,y_0) + (x-x_0)\partial_x \widetilde{v}_h(x_0,y_0) + (y-y_0)\partial_y \widetilde{v}_h(x_0,y_0) \\ &+ \frac{1}{2}(x-x_0)^2 \partial_{xx}\widetilde{v}_h(x_0,y_0) + \frac{1}{2}(y-y_0)\partial_{yy}\widetilde{v}_h(x_0,y_0). \end{aligned}$$

Then,

$$\partial_x \widetilde{v}_h(x,y) = \partial_x \widetilde{v}_h(x_0,y_0) + (x-x_0)\partial_{xx}\widetilde{v}_h,$$

$$\partial_y \widetilde{v}_h(x,y) = \partial_y \widetilde{v}_h(x_0,y_0) + (y-y_0)\partial_{yy}\widetilde{v}_h,$$

and in addition $\partial_{xx}\widetilde{v}_h = \partial_{yy}\widetilde{v}_h = const.$

Thus, we obtain:

$$\int\int_K \partial_x(\widetilde{i}_h v - v)\partial_x \widetilde{v}_h\,dx\,dy = \int\int_K \partial_x(\widetilde{i}_h v - v)\partial_x \widetilde{v}_h(x_0,y_0)\,dx\,dy$$

$$+\int\int_K \partial_x(\widetilde{i}_h v - v)(x-x_0)\partial_{xx}\widetilde{v}_h\,dx\,dy$$

$$= \partial_x\widetilde{v}_h(x_0,y_0) + \left(\int_{l_1} - \int_{l_2}\right)(\widetilde{i}_h v - v)\,dy$$

$$+\partial_{xx}\widetilde{v}_h\left(\int_{l_1}-\int_{l_2}\right)(\widetilde{i}_h v - v)(x_0,y_0)\,dy - \int\int_K (\widetilde{i}_h v - v)\partial_{xx}\widetilde{v}_h\,dx\,dy.$$

Using the properties (3), the first and the third terms disappear and then

$$\int\int_K \partial_x(\widetilde{i}_h v - v)\partial_x\widetilde{v}_h\,dx\,dy = \partial_{xx}\widetilde{v}_h\int_{l_1}(x-x_0)(\widetilde{i}_h v - v)\,dy$$

$$+\frac{h_1}{2}\partial_{xx}\widetilde{v}_h\int_{l_2}(\widetilde{i}_h v - v)\,dy = \partial_{xx}\widetilde{v}_h\int_{l_1}(x-x_0)(\widetilde{i}_h v - v)\,dy.$$

For the second term of (5), by analogy

$$\int\int_K \partial_y(\widetilde{i}_h v - v)\partial_y\widetilde{v}_h\,dx\,dy = \partial_{yy}\widetilde{v}_h\int_{l_1}(y-y_0)(\widetilde{i}_h v - v)\,dx.$$

Having in mind that $\partial_{xx}\widetilde{v}_h = \partial_{yy}\widetilde{v}_h$, we easily get the equality (4). □

Now, let us introduce for any $v_h \in V + \widetilde{V}_h$ the notation

$$|v_h|^2_{a_h} = a_h(\widetilde{i}_h v_h, \widetilde{i}_h v_h).$$

For our main result, it is enough to assume the following interpolation inequality for $v \in V$:

$$|\widetilde{i}_h v - v|_{a_h} \geq Ch^{2-\delta}, \tag{6}$$

where δ is a small positive number.

Theorem 1. *Let (λ_h, u_h) be an approximation of the exact eigenpair (λ, u) obtained from (2) using integral type 7-point 2-simplex finite elements. If the inequality (6) is fulfilled, the number*

$$\widetilde{\Lambda}_h = a_h(\widetilde{i}_h u_h, \widetilde{i}_h u_h)$$

approximates λ from below when h is small enough, so two-sided bounds of λ are obtained:

$$\widetilde{\Lambda}_h \leq \lambda \leq \lambda_h. \tag{7}$$

Proof. Using that $\|u_h\|_{0,\Omega} = 1$, from (4) it follows:

$$\begin{aligned} &a_h(\widetilde{i}_h u_h - u_h, \widetilde{i}_h u_h - u_h) \\ &= a_h(\widetilde{i}_h u_h, \widetilde{i}_h u_h) - 2a_h(\widetilde{i}_h u_h, u_h) + a_h(u_h, u_h) \\ &= a(u_h, u_h) - a_h(\widetilde{i}_h u_h, \widetilde{i}_h u_h) = \lambda_h - \widetilde{\Lambda}_h, \end{aligned}$$

consequently

$$\lambda_h - \widetilde{\Lambda}_h = |\widetilde{i}_h u_h - u_h|^2_{a_h} \geq 0.$$

Since λ_h is a conforming approximation of λ by quadratic (incomplete cubic) triangular elements, we have the estimate (see [6]):

$$0 \leq \lambda_h - \lambda \leq C_1 h^4.$$

On the other hand, we obtain asymptotically the inequality:

$$\begin{aligned} \lambda - \widetilde{\Lambda}_h = \lambda - \lambda_h + \lambda_h - \widetilde{\Lambda}_h \quad &= -(\lambda_h - \lambda) + |\widetilde{i}_h u_h - u_h|^2_{a_h} \\ &\geq -C_1 h^4 + C_2 h^{4-2\delta} \geq 0. \end{aligned}$$

□

We can present the following

Algorithm

1. Find eigenpairs (λ_h, u_h) from (2) by means of conforming integral type 7-point finite element space V_h;
2. Construct $\widetilde{V}_h$ from V_h by eliminating the vertex degrees of freedom and then find the interpolant $\widetilde{i}_h u_h \in \widetilde{V}_h$;
3. Calculate the number $\widetilde{\Lambda}_h = a_h(\widetilde{i}_h u_h, \widetilde{i}_h u_h)$ which approximates λ asymptotically from below.

As a result of this algorithm implementation, two-sided bounds (7) of eigenvalues λ are obtained.

3. COMPUTATIONAL ASPECTS AND NUMERICAL EXAMPLE

The results from numerical experiments given in this section serve as a verification and a confirmation of the validity, reliability and effectiveness of the proposed algorithm for obtaining two-sided bounds for eigenvalues.

We will demonstrate the proposed algorithm for problem (1) with $\Omega = (0,\pi)\times(0,\pi)$.

The reason for this choice is that the exact eigenvalues are known: $\lambda = s_1^2+s_2^2$, $s_{1;2} = 1,2,\ldots$, so that $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 5$, $\lambda_4 = 8$, ...

We use triangular meshes, so that the square Ω is uniformly divided into $2n^2$ triangle elements. Thus the mesh parameter h is equal to $\dfrac{\pi\sqrt{2}}{n}$ and the numerical experiment is implemented for $n = 4; 8; 12; 16; 20$.

We solve the variational discrete problem (2) using conforming triangular 7-point finite elements for V_h. As a result, we obtain the approximate eigenvalues λ_h , which give upper bounds for the corresponding exact eigenvalues and the approximate eigenfunctions u_h which we interpolate. Interpolation is done by means of EC-R nonconforming finite element space (Fig. 2). Obtaining the intepolants $\widetilde{i}_h u_h \in \widetilde{V}_h$ and calculating the numbers $\widetilde{\Lambda}_h = a_h(\widetilde{i}_h u_h, \widetilde{i}_h u_h)$ we get lower bounds for the exact eigenvalues. The results from our numerical experiment for the first three eigenvalues are given in Table 1.

As expected, due to the use of conforming finite element method, the approximate values $\lambda_{j,h},\ j = 1,2,3$ are greater than the exact ones and the sequences $\{\lambda_{j,h}\},\ j = 1,2,3$ obtained when the mesh parameter h decreases are decreasing. On the part of the nonconforming interpolation implementation, as seen in Table 1, the resulting approximation of the exact eigenvalues is from below. When the mesh parameter h decreases, the sequences $\{\widetilde{\Lambda}_{j,h}\},\ j = 1,2,3$ are increasing and go to the corresponding exact eigenvalue. So that, the statement of Theorem 1 is confirmed.

As it is seen from the numerical results, the proposed algorithm works not only for simple eigenvalues, but also in case of multiple eigenvalues.

Instead of applying the proposed algorithm for obtaining two-sided bounds of the exact eigenvalues, one may obtain similar results solving numerically the problem (1) twice – by 7-point conforming finite elements and by EC-R nonconforming elements. The approximations $\widetilde{\lambda}_{j,h}$ obtained sovling the EVP (1) on $\widetilde{V}_h$ have error of the same order as the numbers $\{\widetilde{\Lambda}_{j,h}\}$. For sake of comparison, in Table 1 we also give the values of $\widetilde{\lambda}_{j,h},\ j = 1,2,3$. Let us observe, that for the numerical example under consideration the approximations $\widetilde{\Lambda}_{j,h}$ are even better than $\widetilde{\lambda}_{j,h}$.

One of the advantages of the proposed algorithm is those, that we do not solve the EVP twice – on conforming and nonconforming finite element space; we solve the EVP just once and thus construct nonconforming intepolants on the basis of the conforming finite element eigenfunctions.

An additional advantage comes from the use of integral-type degrees of freedom for the 7-point conforming finite elements during the equation (2) numerical solving . In

n		$j=1$	$j=2$	$j=3$
4	$\lambda_{j,h}$	2.005892076	5.036142495	5.087584349
	$\widetilde{\Lambda}_{j,h}$	1.952135604	4.677483488	4.705826984
	$\widetilde{\lambda}_{j,h}$	1.903764937	4.236635509	4.258846597
8	$\lambda_{j,h}$	2.000419980	5.003449620	5.007049395
	$\widetilde{\Lambda}_{j,h}$	1.987315226	4.912752722	4.913833395
	$\widetilde{\lambda}_{j,h}$	1.974736916	4.789503228	4.791205288
12	$\lambda_{j,h}$	2.000085596	5.000651616	5.001453225
	$\widetilde{\Lambda}_{j,h}$	1.994314936	4.960446764	4.960650368
	$\widetilde{\lambda}_{j,h}$	1.988664673	4.904549262	4.904896264
16	$\lambda_{j,h}$	2.000027409	5.000210644	5.000538034
	$\widetilde{\Lambda}_{j,h}$	1. 996794489	4.977635149	4. 977760359
	$\widetilde{\lambda}_{j,h}$	1.993602390	4.945929497	4.946041332
20	$\lambda_{j,h}$	2.000011259	5.000139752	5.000307228
	$\widetilde{\Lambda}_{j,h}$	1.997562790	4.973734155	4.973734805
	$\widetilde{\lambda}_{j,h}$	1.995698360	4.965702041	4.965729041

Table 1: Approximations of the first three eigenvalues computed by 7-node triangular conforming FE ($\lambda_{j,h}$), by EC-R nonconforming interpolation ($\widetilde{\Lambda}_{j,h}$) and by EC-R nonconforming FE ($\widetilde{\lambda}_{j,h}$)

actual fact, solving (2) we get the approximate eigenfunction $u_h \in V_h$ into the form

$$u_h = \sum_{i=1}^{N} A_i \Phi_i(x, y),$$

where N is the number of the degrees of freedom for V_h, $\Phi_i(x,y), i = 1, 2, \ldots, N$ are the shape functions and $A_i, i = 1, 2, \ldots, N$ are either the values of u_h at the vertices of the elements from τ_h, or the integral values of u_h over the edges of the elements from τ_h and on the elements themselves.

On its part, the interpolant $\widetilde{i}_h u_h \in \widetilde{V}_h$ has a representation

$$\widetilde{i}_h u_h = \sum_{i=1}^{N_1} B_i \Psi_i(x,y),$$

where N_1 is the number of the degrees of freedom for $\widetilde{V}_h$, $\Psi_i(x,y), i = 1,2,\ldots,N_1$ are the shape functions and $B_i, i = 1,2,\ldots,N_1$ are the integral values of $\widetilde{i}_h u_h$ over the edges of the elements from τ_h and on these elements.

Due to use of integral type degrees of freedom for the 7-point conforming finite elements, once we have obtained u_h, the values $B_i, i = 1,2,\ldots,N_1$ are already known, because they are among $A_i, i = 1,2,\ldots,N$. Thus all we have to calculate for obtaining lower bounds of eigenvalues is $a_h(\widetilde{i}_h u_h, \widetilde{i}_h u_h)$. To do this, we use the structure of the stiffeness matrix from (2) by eliminating those columns and rows which correspond to the vertices of the finite element from the mesh.

In conclusion, the proposed algorithm could be also applicable for more general second-order and for fourth-order eigenvalue problem. It should be considered in a separate work.

REFERENCES

[1] M. Ainsworth, J.T. Oden, *A-posteriori Error Estimation in Finite Element Analysis*, Wiley, New York, 2000.

[2] A. Andreev, M. Racheva, Two-sided bounds of eigenvalues of second- and fourth-order elliptic operators, *Appl. Math.*, **59** No. 4 (2014), 371-390.

[3] A. B. Andreev, M. R. Racheva, A new algorithm for two-sided eigenvalue approximation, *Comp. rend. Acad. bulg. Sci.*, **70** No 9 (2017), 1207-1214.

[4] A. B. Andreev, T.D. Todorov, Lumped mass approximation for an isoparametric finite element eigenvalue problem, . *Sib. JNM*, **2**(4) (1999), 295-308.

[5] A. B. Andreev, T.D. Todorov, Lumped mass error estimates for an isoparametric finite element eigenvalue problem. *Sib. JNM*, **3**(3) (2000), 215-228.

[6] I. Babuška, J. Osborn, Eigenvalue Problems, In: *Handbook of Numerical Analysis*, Vol. II, (Eds. P. G. Lions and Ciarlet P.G.), Finite Element Methods (Part 1), North-Holland, Amsterdam (1991), 641-787.

[7] S. Brenner, L.R. Scott, *The Mathematical Theory for Finite Element Methods*, Springer-Verlag, New York, 1994.

[8] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, New York, Oxford, 1978.