

ANOTHER INTERPRETATION FOR THE WEISSLER'S CRITICAL EXPONENT IN THE THEORY OF SEMILINEAR PARABOLIC PDE

Yoshifumi Mimura¹, Yoshihiko Yamaura²

^{1,2}Department of Mathematics

College of Humanities and Sciences

Nihon University

25-40 Sakurajosui 3-chome, Setagaya-ku, Tokyo 156-8550, JAPAN

ABSTRACT: The semilinear partial differential equation which may have either a time-global solution or a blow-up one according to the choice of the initial datum is treated. In particular, a time-local curve of maximal slope for the Lyapunov functional is constructed through the energy minimizing scheme established in 1980 by Marino, Saccon and Tosques. The main objective of this paper is to give another natural interpretation for the Weissler's critical exponent known in the investigation of the existence of the time-local solutions.

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1. INTRODUCTION

In this paper, we investigate the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u.$$

We obtain in Theorem 2 a condition on the exponent p such that the functional whose

first variation is the equation above is *locally* bounded from below by simple interpolation inequality. As a result, the condition exactly coincides with the Weissler's sufficient one for the existence of time-local L^2 -valued solutions with any L^2 -initial data (see Remark 4). In fact, under this condition, we actually construct a time-local curves of maximal slope ("CMS" henceforth. See (16) for the definition) by exploiting the minimizing scheme by [6] (see Theorem 7). Additionally, we can make sure the obtained CMS is nothing but a *strong evolution curve* (see (17) for the definition), that is, a solution of PDE. In this sense, we are able to give another natural interpretation of the Weissler's critical exponent.

The construction of CMS itself have already been established under general conditions in the paper [9] owing to "the restriction trick". Nevertheless, we make sure in this paper the result can be completed without such a special technique under the Weissler's condition. Furthermore, our argument does not use the H -class theory developed in [6], but straightforward computations. As a Corollary, we get a slight superiority of our solutions only in the restricted p by (6) (see Remark 10).

Let us explain the background of this investigation. In order to construct a solution to the gradient flow equation with an energy functional decreasing with respect to the time variable, the *minimizing movement scheme* is one of the useful approaches. In case the energy functional is globally bounded from below, coercive, lower semicontinuous and convex in some sense, the existence and the uniqueness of the time-global solution has been established, and its asymptotic behavior can be also investigated. In the reference [3], the study of functionals defined on a general metric space is developed. As an important example, a curve of maximal slope associated to the equation with the mass conservation law and the energy dissipation is constructed in terms of the Wasserstein distance. It is remarkable that this example have never been built as a gradient flow on the usual functional space with the integral norm. On the other hand, in the case where the partial differential equation does not necessarily have a time-global solution or its Lyapunov functional is non-convex, only a few actual examples are known. Therefore, it seems to be an important subject to investigate an example under such settings.

A semilinear heat equation has a variational structure, but the Lyapunov functional is non-convex and the solution may not necessarily exist globally in time, that is, there can exist a solution blowing up in finite time. Trying to apply the minimizing movement method to this problem, it turns out that this scheme does not work well. Such a difficulty is caused not only by the formal reason that the variational problem has no global minimizers, but also by the essential reason that the solution constructed through

the minimizing movement method is inevitably the time-global solution. However, recalling the classical result of the construction of local solutions by semi-group theory, we find out that the construction method itself does not depend on the asymptotic behavior of solutions. Observing such a background, we raise naturally the following problem: *Is it possible to construct a time-local CMS in the time period before the blow up time is attained ?* In other words, we would like to apply to a certain minimizing scheme the following usual argument in the initial value problem: *construct first a time-local solution, and then investigate whether it can be extended to a time-global one.* In this paper, we make sure that the problem is realized in the case of a semilinear heat equation as an example, by adopting the minimizing scheme established by Marino, Saccon and Tosques ([6]).

We comment here that the research of this article is a trial for investigation of the existence of the time-local solutions to the Keller-Segel system on the Wasserstein metric space ([7]). For this purpose, we proceed our argument by relying on the interpolation inequality. In fact, it is significant that we try to study the most simple method for the aim of investigating the analysis on the Wasserstein metric space.

The notion of curves of maximal slope is first introduced by DeGiorgi, Marino and Tosques in 1980 [4] and the actual construction scheme is first proposed in [6]. The aim of the investigation of CMS is to construct a gradient flow to functionals which are non-convex or are not smooth. In fact, Saccon applied in [10] the theory of CMS to a one dimensional free boundary problem whose energy functional is neither convex nor smooth. Another construction method of CMS is established by Ambrosio et al., which is called minimizing movement method ([2], [3] and refer to [11], [12] for a free boundary problem). The both construction methods by [6] and by [3] are based on similar minimizing schemes. The latter one is more constructive way than the former one, because the latter one gives directly an approximation to the desired CMS. In fact, for this reason, the scheme by Ambrosio et al. is also applied to the theory of numerical analysis (see [8] for instance). However, once a minimizing movement is constructed it must be inevitably a time-global curve. In contrast, in order to get the CMS by the method of [6], a certain reparametrization process with respect to the time variable is indispensable. The reparametrization is expressed by an integration of the “ slope ” (see (12) for the definition) along the curve defined by a constructive way. The slope is a generalized notion of the modulus of the gradient to the functional, and it is in general difficult to be computed directly. Although the method is not constructive for this reason, the reparametrization process, indeed, makes us possible to capture also the blow-up solution. Moreover, since in the variational routine of this method we have

only to find minimizers in balls of finite radius, the functional is required to be only locally bounded from below, not globally bounded in the whole of the space.

Notation We list up the notation used throughout this paper. We denote by $\mathbb{N}$ the set of all natural numbers. Let Ω be a domain in $\mathbb{R}^m$, where m is a positive integer. $|\Omega|$ is a m -dimensional Lebesgue measure of Ω . The Lebesgue space $L^p(\Omega)$ ($1 \leq p < \infty$) is defined to be the sets of p -summable functions u on Ω equipped with the norm $\|u\|_{L^p(\Omega)} := \left(\int_{\Omega} |u|^p dx \right)^{1/p}$. $L^\infty(\Omega)$ is the set of essentially bounded measurable functions on Ω , and $\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{\Omega} |u|$, where “ess sup” means the essential supremum. Denote by $C_c^\infty(\Omega)$ the set of infinitely differentiable functions on Ω whose supports are compact and are contained in Ω . We use the notation $W^{1,p}(\Omega)$, $p \geq 1$, to represent the space of p -summable functions on Ω whose distributional derivatives are also p -summable on Ω . $W_0^{1,p}(\Omega)$, $p \geq 1$ is the closure of $C_c^\infty(\Omega)$ with respect to the $W^{1,p}(\Omega)$ -norm

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_{\Omega} (|u|^p + |\nabla u|^p) dx \right)^{1/p}.$$

In case $p = 2$, we also use the notation $H_0^1(\Omega)$ instead of $W_0^{1,2}(\Omega)$. In particular, as the norm of $H_0^1(\Omega)$ we adopt

$$\|u\|_{H_0^1(\Omega)} := \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}$$

which is equivalent to the norm $\|\cdot\|_{W^{1,2}(\Omega)}$. The inner product of the space $L^2(\Omega)$ is denoted by $\langle u, v \rangle_{L^2(\Omega)} := \int_{\Omega} uv dx$ for $u, v \in L^2(\Omega)$. For the set of real number of the form $A = \{F(x) : x \in X\}$, X being a set and F a $(-\infty, +\infty]$ -valued functional defined on X , the notation “argmin A ” means the set of all elements of X which give a minimal value of F if it exists. The letter $C = C(*)$ denotes a positive constant possibly dependent on the quantities $*$ in the parenthesis. Subscripts are sometimes attached to C in order to distinguish some of the constants from the others.

2. THE LOCAL BOUNDEDNESS FROM BELOW OF THE FUNCTIONAL

Let m be a positive integer larger than or equal to 2, and let Ω be a bounded domain in $\mathbb{R}^m$. For a positive constant p larger than 1, we consider the functional defined on $H_0^1(\Omega)$ as follows:

$$J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx \quad \text{for } u \in H_0^1(\Omega). \quad (1)$$

By using the Sobolev inequality for $m \geq 2$, or the fundamental theory of calculus for $m = 1$, $J(u)$ is shown to take a finite value if

$$p < \begin{cases} \frac{2m}{m-2} - 1 & (m \geq 3), \\ +\infty & (m = 2) \end{cases} \quad (2)$$

and the following inequality holds:

$$\|u\|_{L^{p+1}(\Omega)} \leq C(m, p, \Omega) \|u\|_{H_0^1(\Omega)}. \quad (3)$$

The formal calculation tells us that the stationary point of J is closely related to the semilinear partial differential equation $\Delta u + |u|^{p-1}u = 0$. On the corresponding time evolutionary problem, the following fact is known :

Theorem 1 (Tsutsumi, 1972). *For a given $u_0 \in H_0^1(\Omega)$, we consider the initial boundary value problem*

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & x \in \Omega, \ t > 0, \\ u = 0 & x \in \partial\Omega, \ t > 0, \\ u(\cdot, 0) = u_0 & x \in \Omega. \end{cases}$$

If $J(u_0) < 0$, then the solution blows up in finite time.

Our goal of this paper is to construct a time-local curve of maximal slope to $J(u)$ (Theorem 7). We adopt L^2 -distance as a metric in the definition of a curve of maximal slope. For this reason, we need to extend the domain of definition $H_0^1(\Omega)$ of J to the whole of the space $L^2(\Omega)$ in the following way:

$$J(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx & \text{for } u \in H_0^1(\Omega), \\ +\infty & \text{for } u \in L^2(\Omega) \setminus H_0^1(\Omega). \end{cases} \quad (4)$$

We first try to apply the scheme by minimizing movement method. For this, as a first step, we seek a minimizer of the functional

$$u \in L^2(\Omega) \quad \longmapsto \quad \Phi(u) := J(u) + \frac{1}{2h} \|u - u_0\|_{L^2(\Omega)}^2,$$

where h is a positive constant and u_0 is a given initial datum. However, we cannot find a minimizer, because $\Phi(u)$ is unbounded from below on the whole of $L^2(\Omega)$. In fact,

for an arbitrarily fixed $u_* \in H_0^1(\Omega)$, $u_* \not\equiv 0$ and $\gamma > 0$, we have

$$\begin{aligned}
 \Phi(\gamma u_*) &= \frac{1}{2} \int_{\Omega} |\nabla(\gamma u_*)|^2 dx - \frac{1}{p+1} \int_{\Omega} |\gamma u_*|^{p+1} dx \\
 &\quad + \frac{1}{2h} \|\gamma u_* - u_0\|_{L^2(\Omega)}^2 \\
 &\leq \frac{\gamma^2}{2} \int_{\Omega} |\nabla u_*|^2 dx - \frac{\gamma^{p+1}}{p+1} \int_{\Omega} |u_*|^{p+1} dx \\
 &\quad + \frac{\gamma^2}{h} \|u_*\|_{L^2(\Omega)}^2 + \frac{1}{h} \|u_0\|_{L^2(\Omega)}^2 \\
 &=: C_1 \gamma^2 - C_2 \gamma^{p+1} + C_3 \gamma^2 + C_4.
 \end{aligned} \tag{5}$$

Since $p+1 > 2$ and the constants C_i ($i = 1, 2, 3, 4$) are independent of γ and $p+1 > 2$, the last side diverges to $-\infty$ as $\gamma \rightarrow +\infty$. This reveals us that $\Phi(u)$ is not bounded from below. Thus, we cannot apply the scheme through the minimizing movement method.

We now turn to the scheme due to Marino et.al [6]. By neglecting terms containing the constant h in (5), we observe that $J(\gamma u_*)$ also diverges to $-\infty$ as $\gamma \rightarrow +\infty$. However, this does not mean the scheme does not work to this problem. Because, in the scheme of [6], the variational problem is investigated only in closed balls of $L^2(\Omega)$, and therefore to passage to the limit $\gamma \rightarrow +\infty$ in (5) is not allowed. In fact, we establish the following theorem:

Theorem 2 (local boundedness from below). *Assume*

$$p < 1 + \frac{4}{m}. \tag{6}$$

Then, $J(u)$ is bounded from below on any closed ball $\overline{B}_\rho(u_0) \equiv \{u \in L^2(\Omega) \mid \|u - u_0\|_{L^2(\Omega)} \leq \rho\}$, where $\rho > 0$ and $u_0 \in H_0^1(\Omega)$.

Remark 3. The upper bound for p in (6) is less than that in (2) for any dimension m .

Remark 4. The number $1 + \frac{4}{m}$ ($m \geq 2$) in (6) is the Weissler's critical exponent. Namely, the existence of a time-local solution of the partial differential equation $u_t = \Delta u + |u|^{p-1}u$ with the initial datum $u_0 \in L^q(\Omega)$ is proved under the condition $q > \frac{m(p-1)}{2}$ in the case $m \geq 2$ by Weissler in the 1980s. In particular, in the case of $q = 2$, the condition is equivalent to $p < 1 + \frac{4}{m}$, which is nothing but the condition (6) of Theorem 2. We here in particular notice that the Weissler's critical exponent is obtained such that the mapping considered in the semi-group theory is a contracting one. Thus, although the background is quite different from ours the condition of p is exactly same.

The proof of Theorem 2

It is sufficient to show that $\inf_{u \in \overline{B}_\rho(u_0) \cap H_0^1(\Omega)} J(u) > -\infty$ for arbitrary $u_0 \in H_0^1(\Omega)$ and $\rho > 0$. In the sequel of this proof, let u be a function in $H_0^1(\Omega)$ such that $\|u - u_0\|_{L^2(\Omega)} \leq \rho$. We divide our argument to the case $m \geq 3$, $m = 2$ and $m = 1$.

Case 1. The case $m \geq 3$. Since from (6) the inequalities $2 < p + 1 < 1 + \frac{4}{m} < \frac{2m}{m-2}$ hold, we can apply the interpolation inequality [5, (7.9)], so that we have $\|u\|_{L^{p+1}(\Omega)} \leq \|u\|_{L^2(\Omega)}^\lambda \|u\|_{L^{\frac{2m}{m-2}}(\Omega)}^{1-\lambda}$, where $\lambda \in (0, 1)$ is a number satisfying $\frac{1}{p+1} = \lambda \cdot \frac{1}{2} + (1-\lambda) \cdot \frac{m-2}{2m}$, and hence, $\lambda = \frac{m}{p+1} - \frac{m-2}{2}$. Applying the Sobolev inequality to the last norm, we obtain

$$\|u\|_{L^{p+1}(\Omega)} \leq C_5(m) \|u\|_{L^2(\Omega)}^\lambda \|\nabla u\|_{L^2(\Omega)}^{1-\lambda}.$$

Using this inequality and the Young inequality, we have for any positive number ε

$$\begin{aligned} J(u) &\geq \\ &\frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \varepsilon C_6(m, p) \|\nabla u\|_{L^2(\Omega)}^{(1-\lambda)(p+1)P} - C_7(\varepsilon, p) \|u\|_{L^2(\Omega)}^{\lambda(p+1)Q}, \end{aligned}$$

where $P = \frac{2}{(1-\lambda)(p+1)}$ and Q is the Hölder conjugate exponent of P . positive number satisfying $\frac{1}{P} + \frac{1}{Q} = 1$. We here notice that the assumption $p < 1 + \frac{4}{m}$ is equivalent to the condition $(1-\lambda)(p+1) < 2$, and so $P > 1$. In this way, we find

$$J(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \varepsilon C_6 \|\nabla u\|_{L^2(\Omega)}^2 - C_7 \|u\|_{L^2(\Omega)}^{\lambda Q(p+1)}.$$

We here choose ε such that $\varepsilon C_6 = \frac{1}{4}$. Then, the sum of the first two terms is positive, and hence dropping it, we see

$$J(u) \geq -C_8(p, m) \|u\|_{L^2(\Omega)}^q \geq -C_9(p, m) \left(\rho^q + \|u_0\|_{L^2(\Omega)}^q \right) > -\infty,$$

where $q = \lambda Q(p+1)$.

Case 2. The case $m = 2$. Let $1 < p < 1 + \frac{4}{2} = 3$. Then we can choose r larger than $p+1$ such that

$$p < 3 - \frac{4}{r}. \quad (7)$$

Furthermore, for this r , we can choose $\sigma \in (1, 2)$ such that $\frac{2\sigma}{2-\sigma} = r$. Since $2 < p+1 < r$, we can apply Interpolation inequality to get

$$\|u\|_{L^{p+1}(\Omega)} \leq \|u\|_{L^2(\Omega)}^\lambda \|u\|_{L^r(\Omega)}^{1-\lambda}, \quad (8)$$

where $\lambda \in (0, 1)$ is a number satisfying $\frac{1}{p+1} = \lambda \cdot \frac{1}{2} + (1-\lambda) \cdot \frac{1}{r}$, and hence, $\lambda = \frac{2(r-p-1)}{(p+1)(r-2)}$. We next apply the Sobolev imbedding $H_0^1(\Omega) \subset W_0^{1,\sigma}(\Omega) \hookrightarrow L^{\frac{2\sigma}{2-\sigma}}(\Omega)$, so that

$$\|u\|_{L^r(\Omega)} = \|u\|_{L^{\frac{2\sigma}{2-\sigma}}(\Omega)} \leq C_{10}(p) \|\nabla u\|_{L^2(\Omega)}. \quad (9)$$

Combining (8) with (9), we have

$$\|u\|_{L^{p+1}(\Omega)} \leq C_{11}(p) \|\nabla u\|_{L^2(\Omega)}^{1-\lambda} \|u\|_{L^2(\Omega)}^\lambda.$$

Therefore, we obtain

$$\int_{\Omega} |u|^{p+1} dx \leq C_{12}(p) \|\nabla u\|_{L^2(\Omega)}^{(p+1)(1-\lambda)} \|u\|_{L^2(\Omega)}^{(p+1)\lambda}. \quad (10)$$

Now the condition (7): $p < 3 - \frac{4}{r}$ is equivalent to the condition $(p+1)(1-\lambda) < 2$. So we can use the Young inequality in (10): For any $\varepsilon > 0$,

$$\int_{\Omega} |u|^{p+1} dx \leq \varepsilon C_{13}(p) \|\nabla u\|_{L^2(\Omega)}^2 + C_{14}(\varepsilon, p) \|u\|_{L^2(\Omega)}^q,$$

where $q = Q(p+1)\lambda$, $Q = \frac{P}{P-1}$ and $P = \frac{2}{(p+1)(1-\lambda)}$. By making use of the last inequality, we arrive at the conclusion in the same way as *Case 1*. $\square$

Remark 5. In the proof above, we show that for any dimension m and $p > 1$ satisfying (6),

$$J(u) \geq \frac{1}{4} \int_{\Omega} |\nabla u|^2 dx - C_{15} \|u\|_{L^2(\Omega)}^q \quad \text{for } u \in H_0^1(\Omega), \quad (11)$$

where C_{15} and q are positive constants depending only on m and p .

In light of (11), we can show the following existence result whose proof is elementary in the calculus of variation:

Corollary 6 (The existence of minimizers). *Suppose that p satisfies (6). For any $\rho > 0$ and $u_0 \in H_0^1(\Omega)$, there exists a minimizer of the functional J on a closed ball $\overline{B}_\rho(u_0) = \{u \in L^2(\Omega) : \|u - u_0\|_{L^2(\Omega)} \leq \rho\}$.*

3. A CONSTRUCTION OF A CURVE OF MAXIMAL SLOPE

The aim of this section is to show the following main theorem of this paper:

Theorem 7 (Existence of a time-local CMS). *Suppose that p satisfies (6). Then, there exists a time-local curve of maximal slope for the functional J as in (4) at the initial datum $u_0 \in H_0^1(\Omega)$ with $J(u_0) < +\infty$.*

Before showing Theorem 7, we sum up the definitions related to the curve of maximal slope. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\| \cdot \|_H$, and $\phi : H \rightarrow (-\infty, +\infty]$. Let $v_0 \in H$ be an arbitrary element satisfying $\phi(v_0) < +\infty$. Then the *slope* of $\phi(v)$ at $v = v_0$ is defined by the following real value or $+\infty$:

$$|\nabla\phi|(v_0) = \overline{\lim}_{v \rightarrow v_0} \frac{(\phi(v_0) - \phi(v))^+}{\|v - v_0\|_H}, \quad (12)$$

where $\sigma^+ = \max\{\sigma, 0\}$ for a real number σ . In the reference [6], the slope is defined as $-\lim_{\rho \rightarrow 0} \frac{\chi_{v_0}(\rho) - \chi_{v_0}(0)}{\rho}$, where $\chi_{v_0}(\rho) = \inf_{\overline{B}_\rho(v_0)} \phi$. This is checked to be same as our definition. $\phi(v)$ is said to be *Fréchet subdifferentiable* at $v = v_0$ if there exists $\alpha \in H$ such that

$$\lim_{v \rightarrow v_0} \frac{\phi(v) - \phi(v_0) - \langle \alpha, v - v_0 \rangle_H}{\|v - v_0\|_H} \geq 0. \quad (13)$$

Such an $\alpha \in H$ is called a *Fréchet subdifferential* of ϕ at $v_0 \in H$. We define $\partial^-\phi(v_0)$ the set of all Fréchet subdifferentials of ϕ at $v_0 \in H$. We denote by $\text{grad}^-\phi(v_0)$ the element of $\partial^-\phi(v_0)$ which is the norm-minimizer in this set. Here we remark that the norm-minimizer is uniquely determined because the set $\partial^-\phi(v_0)$ is directly checked to be closed and convex set in H .

Now, the slope is immediately shown to be expressed by

$$\begin{aligned} |\nabla\phi|(v_0) = \min \left\{ C \geq 0 : \right. \\ \left. \begin{aligned} \phi(v) &\geq \phi(v_0) - C\|v - v_0\|_H - o(\|v - v_0\|_H) \\ &\text{as } \|v - v_0\|_H \rightarrow 0 \end{aligned} \right\}. \end{aligned} \quad (14)$$

On the other hand, by definition $\alpha \in \partial^-\phi(v_0)$ satisfies

$$\phi(v) \geq \phi(v_0) - \|\alpha\|_H \|v - v_0\|_H - o(\|v - v_0\|_H) \quad \text{as } \|v - v_0\|_H \rightarrow 0.$$

So in particular we immediately have

$$|\nabla\phi|(v_0) \leq \|\alpha\|_H \quad \text{for } \alpha \in \partial^-\phi(v_0). \quad (15)$$

The notion of curve of maximal slope is defined in the reference [6, Definition 1.2] as follows: Let T be a positive constant. Let (X, d) be a metric space. We say that $u : [0, T] \rightarrow X$ is a *curve of maximal slope almost everywhere* for ϕ if there exists a negligible set $N \subset [0, T]$ such that $u(t)$ is continuous in $[0, T]$, $\phi(u(t)) < +\infty$ for all

$t \in [0, T] \setminus N$ and the following two inequalities with respect to the slope hold:

$$\begin{aligned}
 d(u(t_2), u(t_1)) &\leq \int_{t_1}^{t_2} |\nabla \phi|(u(t)) dt \\
 &\quad \text{for all } t_1, t_2 \in I \text{ with } t_1 \leq t_2, \\
 \phi(u(t_2)) - \phi(u(t_1)) &\leq - \int_{t_1}^{t_2} |\nabla \phi|^2(u(t)) dt \\
 &\quad \text{for all } t_1, t_2 \in I \setminus N \text{ with } t_1 \leq t_2.
 \end{aligned} \tag{16}$$

If N can be taken as the empty set, $u(t)$ is said to be a curve of maximal slope everywhere for ϕ . The notion of CMS is checked to be equivalent to the one introduced in [3, Definition 1.3.2].

Furthermore, in case the metric space X is given by a Hilbert space H , we say that $u = u(t)$ ($t \in [0, T]$) is a *strong evolution curve almost everywhere* for ϕ if there exists a negligible set $N \subset [0, T]$ such that $u(t)$ is continuous in $[0, T]$ and absolutely continuous on $(0, T]$, $\phi(u(t)) < +\infty$ for all $t \in [0, T] \setminus N$, $\phi(u(t)) \leq \phi(u(0))$ for all $t \in [0, T] \setminus N$, $\phi(u(t))$ is non-increasing on $[0, T] \setminus N$ and the following differential inclusion holds:

$$\partial^- \phi(u(t)) \neq \emptyset \quad \text{and} \quad -u'(t) \in \partial^- \phi(u(t)) \quad \text{for all } t \in [0, T] \setminus N. \tag{17}$$

If N can be taken as the empty set, $u(t)$ is said to be a strong evolution curve everywhere for ϕ .

The relation between the curve of maximal slope and the strong evolution curve is shown in [6, Theorem 1.11]:

Theorem 8 ([6, Theorem 1.11]). *Suppose that $u(t)$ is a curve of maximal slope almost everywhere for ϕ . If $\partial^- \phi(u(t)) \neq \emptyset$ and $|\nabla \phi|(u(t)) = \|\text{grad}^- \phi(u(t))\|_H$ for almost everywhere on $[0, T]$, then $u(t)$ is a strong evolution curve almost everywhere for ϕ .*

Let us show Theorem 7. Set

$$D_0 := \overline{B}_S(u_0) \cap \left\{ u \in L^2(\Omega) : J(u) \leq J(u_0) \right\} \tag{18}$$

for a positive number S and for u_0 with $J(u_0) < +\infty$. Having already shown Corollary 6, we have only to check the following three conditions to make sure the existence of a time-local curve of maximal slope (see [6, section 4]). For any $u \in D_0$,

$$J(u) \leq \varliminf_{\substack{v \rightarrow u \\ v \in D_0}} J(v), \tag{19}$$

$$|\nabla J|(u) \leq \varliminf_{\substack{v \rightarrow u \\ v \in D_0}} |\nabla J|(v), \tag{20}$$

$$\overline{\lim_{\substack{v \rightarrow u \\ v \in D_0 \\ |\nabla J|(v) \leq C}}} J(v) \leq J(u) \quad \text{for any fixed positive number } C. \quad (21)$$

First part of this section is the proof of Theorem 7 for the case $m = 1$, while the second part is for the case $m \geq 2$.

Remark 9. The functional $u \in L^2(\Omega) \mapsto \int_{\Omega} |u|^{p+1} dx$ does not belong to the class of C^1 -functionals on $L^2(\Omega)$, and, in fact, it is not even continuous on $L^2(\Omega)$.

Remark 10. In [9], a restriction trick essentially based on the Interpolation inequality is used, and as a result time-local solutions for a general form of the functional are shown to be constructed without any restrictions to $p < 2^* = \frac{m}{m-2}$ ($m \geq 3$). In the theory of [6], CMS is constructed as a curve with the parameter defined by $L^2(\Omega)$ -distance, which we here call this curve “pre-CMS” (cf. the one-dimensional argument in Section 2). Because of the restriction to the set $\{u \in L^2(\Omega) \mid \|u\|_{L^{p+1}(\Omega)} < \|u_0\|_{L^{p+1}(\Omega)}\}$ in [9], we can construct a pre-CMS only for sufficiently small parameter with respect to the value of the $L^{p+1}(\Omega)$ -norm of the initial data u_0 . On the other hand, in this paper, as far as we concern with the construction of pre-CMS we need not restrict the value of parameter, and this result is considered to be useful in the study of the extension to the time-global solution.

The proof of Theorem 7

Define

$$J(u) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx & \text{for } u \in H_0^1(\Omega), \\ +\infty & \text{for } u \in L^2(\Omega) \setminus H_0^1(\Omega). \end{cases} \quad (22)$$

Let us begin with showing some results on the slope and Fréchet subdifferential of J .

Lemma 11. *Let $p < 1 + \frac{4}{m}$. Suppose (u_j) be a sequence in $H_0^1(\Omega)$ such that $\sup_{j \in \mathbb{N}} \|u_j\|_{H_0^1(\Omega)} < +\infty$. Then there exists a subsequence (u_{j_k}) such that*

$$u_{j_k} \rightarrow u \quad \text{strongly in } L^{p+1}(\Omega) \quad \text{as } k \rightarrow \infty.$$

Proof. Let $m \geq 3$. The condition $p < 1 + \frac{4}{m}$ implies $p+1 < m^* (= \frac{2m}{m-2})$. Hence, the Sobolev compact imbedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ holds. This imbedding turns out to hold also in the case $m = 2$ by noticing $H_0^1(\Omega) = W_0^{1,2}(\Omega) \subset W_0^{1,r}(\Omega) \hookrightarrow L^{\frac{2r}{r-2}}(\Omega)$ for any $r < 2$. We thus establish our conclusion by the assumption $\sup_{j \in \mathbb{N}} \|u_j\|_{H_0^1(\Omega)} < +\infty$. $\square$

Lemma 12. For any positive number ε and K , there exists a positive number M such that if $u \in H_0^1(\Omega)$ satisfies $\|\nabla u\|_{L^2(\Omega)} \leq K$, then $\int_{\{u \geq M\}} u^{p+1} dx < \varepsilon$ holds, where $\{u \geq M\} = \{x \in \Omega \mid u(x) \geq M\}$.

Proof. We argue by contradiction. Were the statement false, there would exist positive numbers ε and K such that the following holds: there exists a sequence $(u_j) \subset H_0^1(\Omega)$ such that

$$\begin{cases} \|\nabla u_j\|_{L^2(\Omega)} \leq K \\ \int_{\{u_j \geq j\}} u_j^{p+1} dx \geq \varepsilon \end{cases} \quad (j = 1, 2, \dots). \quad (23)$$

Let $v_j := u_j^{p+1} \chi_{\{u_j \geq j\}}$. Here, $\chi_{\{u_j \geq j\}}$ is the function which takes 1 on the set $\{u_j \geq j\}$, and 0 elsewhere. Then by the second inequality of (23)

$$\int_{\Omega} v_j dx \geq \varepsilon \quad \text{for } j \in \mathbb{N}. \quad (24)$$

Furthermore, by the first inequality of (23), we have from Lemma 11 that there exists a subsequence (u_{j_k}) and $u \in L^{p+1}(\Omega)$ such that

$$\begin{cases} \text{(a) } u_{j_k} \rightarrow u \text{ strongly in } L^{p+1}(\Omega), \\ \text{(b) } u_{j_k} \rightarrow u \text{ almost everywhere in } \Omega. \end{cases} \quad (25)$$

Since $u(x)$ is a real number for almost all $x \in \mathbb{R}$, we have from (25)-(b) that

$$\begin{aligned} v_{j_k}(x) \equiv u_{j_k}(x)^{p+1} \chi_{\{u_{j_k} \geq j_k\}}(x) &\longrightarrow 0 \\ &\text{for a.a. } x \in \Omega \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (26)$$

Furthermore,

$$|v_{j_k}| \leq |u_{j_k}|^{p+1}, \quad (27)$$

$$|u_{j_k}|^{p+1} \rightarrow |u|^{p+1} \quad \text{a.e. in } \Omega, \quad (28)$$

$$\int_{\Omega} |u_{j_k}|^{p+1} dx \rightarrow \int_{\Omega} |u|^{p+1} dx, \quad (29)$$

where the last convergence is followed by (25)-(a). By exploiting the *Extended Lebesgue dominated convergence theorem* (see Lemma 13 below), (26)–(29) enable us to lead

$$\int_{\Omega} v_{j_k} dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This is a contradiction to (24). □

The following assertion is shown by a slight variation of the proof of well-known Lebesgue dominated convergence theorem.

Lemma 13 (Extended Lebesgue convergence theorem). *Let (u_j) be a sequence of measurable functions. If $u_j \rightarrow u$ a.e. in Ω , there exists v_j such that $|u_j| \leq v_j$ in Ω for each $j \in \mathbb{N}$, and there exists $v \in L^1(\Omega)$ such that $v_j \rightarrow v$ a.e. in Ω , $\int_{\Omega} v_j dx \rightarrow \int_{\Omega} v dx$, then*

$$\lim_{j \rightarrow \infty} \int_{\Omega} u_j dx = \int_{\Omega} u dx.$$

Lemma 14. *Let $v_0 \in H_0^1(\Omega)$. Define for $v \in H_0^1(\Omega)$*

$$\phi_{v_0}(v) :=$$

$$J(v) - J(v_0) - \left\{ \int_{\Omega} \nabla v_0 \nabla(v - v_0) dx - \int_{\Omega} |v_0|^{p-1} v_0 (v - v_0) dx \right\}.$$

Then the following estimate holds: for any $v \in H_0^1(\Omega)$ satisfying the hypothesis

$$(P1) \ J(v) \leq J(v_0) + 1,$$

$$(P2) \ \|v_0 - v\|_{L^2(\Omega)} \leq 1.$$

it holds that $\phi_{v_0}(v) \geq -C\|v_0 - v\|_{L^2(\Omega)}^2$, where C is a positive constant.

Proof. Let $v \in H_0^1(\Omega)$ satisfy (P1) and (P2). Then, by make use of (11), we estimate

$$\begin{aligned} \int_{\Omega} |\nabla v|^2 dx &\leq 4C_{15} \|v\|_{L^2(\Omega)}^q + 4J(v) \\ &\leq C_{17} + C_{18} \|v_0\|_{L^2(\Omega)}^q + 4\{J(v_0) + 1\}. \end{aligned} \tag{30}$$

As a result, there exists a positive constant C_{19} independent of $v \in H_0^1(\Omega)$ satisfying (P1) and (P2) such that $\int_{\Omega} |\nabla v|^2 dx \leq C_{19}$. Thus, for $w := |v_0| + |v|$ we have

$$\int_{\Omega} |\nabla w|^2 dx \leq 2\|\nabla v_0\|_{L^2(\Omega)}^2 + 2C_{19}.$$

Since $w \in H_0^1(\Omega)$, we can apply Lemma 12 with $u = w$, $K := \sqrt{2\|\nabla v_0\|_{L^2(\Omega)}^2 + 2C_{19}}$ and $\varepsilon := (4C_{21})^{-\frac{p+1}{p-1}}$, where $C_{21} = C_{21}(m, p)$ is a constant depending only on p and m determined later in (34). In this way, we can find a positive number M such that

$$C_{21} \left\{ \int_{\{|v_0|+|v|\geq M\}} (|v_0| + |v|)^{p+1} dx \right\}^{\frac{p-1}{p+1}} < \frac{1}{4}. \tag{31}$$

Now we rewrite

$$\begin{aligned} \phi_{v_0}(v) &= \frac{1}{2} \int_{\Omega} |\nabla v - \nabla v_0|^2 dx \\ &\quad + \frac{1}{p+1} \int_{\Omega} \left(|v_0|^{p+1} - |v|^{p+1} + (p+1)|v_0|^{p-1} v_0 (v - v_0) \right) dx. \end{aligned} \tag{32}$$

By make use of the Taylor expansion to $t \in \mathbb{R} \mapsto |t|^{p+1}$ at $t = |v_0(x)|$, we obtain that for any $x \in \Omega$, there exists $\theta(x) \in (0, 1)$ such that

$$\begin{aligned} |v(x)|^{p+1} &= |v_0(x)|^{p+1} + (p+1)|v_0(x)|^{p-1}v_0(x)(v(x) - v_0(x)) + \\ &+ \frac{1}{2}(p+1) \times \\ &\times \left\{ (p-1)|u(x)|^{p-3}u(x)^2 + |u(x)|^{p-1} \right\}_{u(x)=\theta(x)v_0(x)+(1-\theta(x))v(x)} \times \\ &\quad \times (v(x) - v_0(x))^2. \end{aligned}$$

Hence, the second integral term of the right side of (32) is estimated from below by $-C_{20} \int_{\Omega} |\theta(x)v_0 + (1-\theta(x))v|^{p-1}(v - v_0)^2 dx$. Therefore, we have

$$\begin{aligned} \phi_{v_0}(v) &\geq \\ \frac{1}{2} \int_{\Omega} |\nabla(v_0 - v)|^2 dx &- C_{20}(p) \int_{\Omega} |\theta v_0 + (1-\theta)v|^{p-1}(v - v_0)^2 dx. \end{aligned} \quad (33)$$

The second integral of (33) is estimated furthermore as follows:

$$\begin{aligned} &C_{20} \int_{\Omega} |\theta v_0 + (1-\theta)v|^{p-1}(v - v_0)^2 dx \\ &\leq C_{20} \int_{\{|v_0|+|v|\geq M\}} (|v_0| + |v|)^{p-1}(v - v_0)^2 dx \\ &\quad + C_{20} \int_{\{|v_0|+|v|<M\}} (|v_0| + |v|)^{p-1}(v - v_0)^2 dx \\ &\leq C_{20} \left\{ \int_{\{|v_0|+|v|\geq M\}} (|v_0| + |v|)^{p+1} dx \right\}^{\frac{p-1}{p+1}} \left(\int_{\Omega} |v_0 - v|^{p+1} dx \right)^{\frac{2}{p+1}} \\ &\quad + M^{p-1} C_{20} \int_{\Omega} |v_0 - v|^2 dx \\ &\leq C_{21}(m, p, |\Omega|) \left\{ \int_{\{|v_0|+|v|\geq M\}} (|v_0| + |v|)^{p+1} dx \right\}^{\frac{p-1}{p+1}} \|v_0 - v\|_{H_0^1(\Omega)}^2 \\ &\quad + M^{p-1} C_{20} \int_{\Omega} |v_0 - v|^2 dx, \end{aligned} \quad (34)$$

where in the last inequality we use the inequality (3). Apply here (31) to the first term o the last side of (34) in order to obtain

$$\begin{aligned} &C_{20} \int_{\Omega} |\theta v_0 + (1-\theta)v|^{p-1}(v - v_0)^2 dx \\ &\leq \frac{1}{4} \|v_0 - v\|_{H_0^1(\Omega)}^2 + M^{p-1} C_{20} \|v_0 - v\|_{L^2(\Omega)}^2. \end{aligned}$$

By combining this with (33), the two terms $\|v_0 - v\|_{H_0^1(\Omega)}^2$ are gathered as $\frac{1}{4} \|v_0 - v\|_{H_0^1(\Omega)}^2$, which yields the conclusion. $\square$

Theorem 15. *Let v_0 be a function in $H_0^1(\Omega)$. Then,*

(i) $\alpha \in L^2(\Omega)$ belongs to $\partial^- J(v_0)$ if and only if

$$\int_{\Omega} \left(\nabla v_0 \nabla w - |v_0|^{p-1} v_0 w \right) dx = \int_{\Omega} \alpha w dx \quad \text{for } w \in H_0^1(\Omega).$$

In particular, if $J(u)$ is Fréchet subdifferentiable at $u = v_0$, then the Fréchet subdifferential is uniquely determined.

(ii) $|\nabla J|(v_0) < +\infty$ holds if and only if $\partial^- J(v_0) \neq \emptyset$.

(iii) If $|\nabla J|(v_0) < +\infty$, then $|\nabla J|(v_0) = \|\text{grad}^- J(v_0)\|_{L^2(\Omega)}$.

Proof. (i) Let $\alpha \in \partial^- J(v_0)$. By definition we have

$$\lim_{\substack{v \in L^2(\Omega) \\ \|v - v_0\|_{L^2(\Omega)} \rightarrow 0}} \frac{J(v) - J(v_0) - \langle \alpha, v - v_0 \rangle_{L^2(\Omega)}}{\|v - v_0\|_{L^2(\Omega)}} \geq 0.$$

Let ε be a positive number and w be a function in $H_0^1(\Omega)$. Substituting $v_0 + \varepsilon w$ as v , we get

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{J(v_0 + \varepsilon w) - J(v_0)}{\varepsilon} \geq \int_{\Omega} \alpha w dx. \quad (35)$$

The fraction of the left side is computed as follows:

$$\begin{aligned} \frac{J(v_0 + \varepsilon w) - J(v_0)}{\varepsilon} &= \frac{1}{2} \int_{\Omega} (2 \nabla v_0 \nabla w + \varepsilon |\nabla w|^2) dx \\ &- \frac{1}{p+1} \int_{\Omega} \frac{p+1}{2} (|v_0|^2 + \theta(x) [2\varepsilon v_0 w + \varepsilon^2 |w|^2])^{\frac{p-1}{2}} \times \\ &\quad \times [2v_0 w + \varepsilon |w|^2] dx \end{aligned}$$

for some $\theta(x) \in (0, 1)$. By the Lebesgue dominated convergence theorem we derive

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{J(v_0 + \varepsilon w) - J(v_0)}{\varepsilon} = \int_{\Omega} \left(\nabla v_0 \nabla w - |v_0|^{p-1} v_0 w \right) dx.$$

So by (35)

$$\int_{\Omega} \left(\nabla v_0 \nabla w - |v_0|^{p-1} v_0 w \right) dx \geq \int_{\Omega} \alpha w dx. \quad (36)$$

In the argument leading to (36), we replace w by $-w$, and multiply the resultant inequality by -1 to have the reverse inequality of (36). So we reach the desired equality.

We next assume that $\alpha \in L^2(\Omega)$ satisfy

$$\int_{\Omega} \left(\nabla v_0 \nabla w - |v_0|^{p-1} v_0 w \right) dx = \int_{\Omega} \alpha w dx \quad \text{for } w \in H_0^1(\Omega). \quad (37)$$

Let $v \in H_0^1(\Omega)$ be a function satisfying the condition (P2) of Lemma 14: $\|v_0 - v\|_{L^2(\Omega)} \leq 1$. If v satisfies also the condition (P1) of Lemma 14: $J(v) \leq J(v_0) + 1$, we have from Lemma 14 that

$$\frac{J(v) - J(v_0) - \langle \alpha, v - v_0 \rangle_{L^2(\Omega)}}{\|v - v_0\|_{L^2(\Omega)}} \geq -C_{21}\|v - v_0\|_{L^2(\Omega)}. \quad (38)$$

If v does not satisfy (P1) of Lemma 14, then we directly have

$$\frac{J(v) - J(v_0) - \langle \alpha, v - v_0 \rangle_{L^2(\Omega)}}{\|v - v_0\|_{L^2(\Omega)}} \geq \frac{1}{\|v - v_0\|_{L^2(\Omega)}} - \|\alpha\|_{L^2(\Omega)}. \quad (39)$$

Gathering (38) and (39), we have

$$\begin{aligned} & \frac{J(v) - J(v_0) - \langle \alpha, v - v_0 \rangle_{L^2(\Omega)}}{\|v - v_0\|_{L^2(\Omega)}} \\ & \geq \min \left\{ -C_{21}\|v - v_0\|_{L^2(\Omega)}, \frac{1}{\|v - v_0\|_{L^2(\Omega)}} - \|\alpha\|_{L^2(\Omega)} \right\}. \end{aligned} \quad (40)$$

We notice here that the argument leading to (40) holds under the condition $v \in H_0^1(\Omega)$. However, even if $v \in L^2(\Omega) \setminus H_0^1(\Omega)$, the inequality (40) turns out to hold because the left side is $+\infty$. Now pass to the limit $v \rightarrow v_0$ in $L^2(\Omega)$ in (40), we arrive at

$$\begin{aligned} & \lim_{v \rightarrow v_0} \frac{J(v) - J(v_0) - \langle \alpha, v - v_0 \rangle_{L^2(\Omega)}}{\|v - v_0\|_{L^2(\Omega)}} \\ & \geq \lim_{v \rightarrow v_0} \min \left\{ -C_{21}\|v - v_0\|_{L^2(\Omega)}, \frac{1}{\|v - v_0\|_{L^2(\Omega)}} - \|\alpha\|_{L^2(\Omega)} \right\} \\ & = 0. \end{aligned}$$

Suppose α_1 and α_2 are both Fréchet subdifferentials. By applying the equality of (i) to α_1 , α_2 and subtracting the former resultant equality from the latter one, we have

$$\int_{\Omega} (\alpha_1 - \alpha_2)w \, dx = 0 \quad \text{for } w \in H_0^1(\Omega).$$

Using the density of $H_0^1(\Omega)$ in $L^2(\Omega)$, we can substitute $(\alpha_1 - \alpha_2)$ as w , so that we have $\alpha_1 = \alpha_2$.

(ii) Let $\alpha \in \partial^- J(v_0)$. By (15) with $H = L^2(\Omega)$,

$$|\nabla J|(v_0) \leq \|\alpha\|_{L^2(\Omega)}, \quad (41)$$

and so in particular we have $|\nabla J|(v_0) < +\infty$.

On the other hand, suppose $|\nabla J|(v_0) < +\infty$, which gives us that $v_0 \in H_0^1(\Omega)$ by the definition of the slope. Then by (14) with $H = L^2(\Omega)$ we in particular have

$$J(v) \geq J(v_0) - |\nabla J|(v_0) \|v_0 - v\|_{L^2(\Omega)} - o(\|v_0 - v\|_{L^2(\Omega)})$$

$$\text{as } \|v_0 - v\|_{L^2(\Omega)} \rightarrow 0.$$

Let $w \in H_0^1(\Omega)$ and substitute $v_0 + \varepsilon w$, $\varepsilon > 0$, as v in the above inequality. Then

$$J(v_0 + \varepsilon w) - J(v_0) \geq -\varepsilon |\nabla J|(v_0) \|w\|_{L^2(\Omega)} - o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

Dividing the both sides by ε and passing to the limit $\varepsilon \rightarrow 0$, we have

$$\int_{\Omega} \left(\nabla v_0 \nabla w - |v_0|^{p-1} v_0 w \right) dx \geq -|\nabla J|(v_0) \|w\|_{L^2(\Omega)}. \quad (42)$$

The argument leading to (42) is valid if we replace w by $-w$. Together with the resultant inequality and (42) we get

$$\left| \int_{\Omega} \left(\nabla v_0 \nabla w - |v_0|^{p-1} v_0 w \right) dx \right|$$

$$\leq |\nabla J|(v_0) \|w\|_{L^2(\Omega)} \quad \text{for } w \in H_0^1(\Omega). \quad (43)$$

If we define $T \in H^{-1}(\Omega)$ as

$$Tw := \int_{\Omega} \left(\nabla v_0 \nabla w - |v_0|^{p-1} v_0 w \right) dx \quad \text{for } w \in H_0^1(\Omega),$$

then we derive from (43)

$$|Tw| \leq |\nabla J|(v_0) \|w\|_{L^2(\Omega)} \quad \text{for } w \in H_0^1(\Omega). \quad (44)$$

By a standard density argument, T is extended to a functional on $L^2(\Omega)$, still denoted by T , and as a result the inequality (44) remains valid for any $w \in L^2(\Omega)$. Therefore, by the Riesz representation theorem, there exists $\alpha \in L^2(\Omega)$ such that

$$Tw = \langle \alpha, w \rangle_{L^2(\Omega)} \quad (w \in L^2(\Omega)). \quad (45)$$

Restricting w as $H_0^1(\Omega)$ -functions, we have the integral identity of (i), which yields that α is a subdifferential of $J(u)$ at $u = v_0$.

(iii) Suppose $|\nabla J|(v_0) < +\infty$. Let α be as in the second half of the proof of (ii). Substituting $w = \alpha$ in (45), we obtain $T\alpha = \|\alpha\|_{L^2(\Omega)}^2$. Furthermore, by substituting $w = \alpha$ in (44), we have the inequality $\|\alpha\|_{L^2(\Omega)}^2 = |T\alpha| \leq |\nabla J|(v_0) \|\alpha\|_{L^2(\Omega)}$. So we have $\|\alpha\|_{L^2(\Omega)} \leq |\nabla J|(v_0)$, and in fact the equality holds because of (41). Since the Fréchet subdifferential of J is known to be unique as shown in (i), α is a unique Fréchet subdifferential. Therefore, in particular, α is the norm minimizing element of $\partial^- J(v_0)$, and therefore it is inevitable that $\alpha = \text{grad}^- J(v_0)$. We thus establish the equality $|\nabla J|(v_0) = \|\text{grad}^- J(v_0)\|_{L^2(\Omega)}$. $\square$

Remark 16. If v_0 satisfies the integration equality in (i) of Theorem 15 we can know that $v_0 \in H^2(\Omega)$ owing to the regularity argument for the weak solutions of the elliptic PDE (see Theorem 1.10 of [10]). However, we proceed our argument without using this regularity result.

Finally, for showing that CMS is able to be constructed, we shall check (19)-(21):

The proof of sufficient conditions (19) – (21)

• (19) : Let us show a stronger result : $J(u) \leq \varliminf_{v \rightarrow u} J(v)$. We may assume $\varliminf_{v \rightarrow u} J(v) < +\infty$, since otherwise the desired inequality is trivial. Let (u_j) be sequence in D_0 such that $u_j \rightarrow u$ in $L^2(\Omega)$ as $j \rightarrow \infty$ and the liminf of (19) is attained.

$$\lim_{j \rightarrow \infty} J(u_j) = \varliminf_{v \rightarrow u} J(v). \quad (46)$$

Since we may assume $J(u_j) < +\infty$, u_j belongs to $H_0^1(\Omega)$. Hence we can apply (11) to have

$$J(u_j) \geq \frac{1}{4} \int_{\Omega} |\nabla u_j|^2 dx - C_{15} \|u_j\|_{L^2(\Omega)}^q.$$

Therefore

$$\sup_{j \in \mathbb{N}} \int_{\Omega} |\nabla u_j|^2 dx \leq 4 \sup_{j \in \mathbb{N}} J(u_j) + 4C_{15} \sup_{j \in \mathbb{N}} \|u_j\|_{L^2(\Omega)}^q. \quad (47)$$

The right side is finite, and as a result, the uniform boundedness of $\|u_j\|_{H_0^1(\Omega)} = \|\nabla u_j\|_{L^2(\Omega)}$ is shown. Henceforth, by taking a subsequence (u_{j_k}) , we have

$$\nabla u_{j_k} \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega). \quad (48)$$

Furthermore, by Lemma 11 there exists a subsequence (u_{j_k}) such that

$$u_{j_k} \rightarrow u \quad \text{strongly in } L^{p+1}(\Omega). \quad (49)$$

With the aid of (48) and (49), the Banach-Steinhaus theorem tells us

$$\begin{aligned} \varliminf_{j \rightarrow \infty} J(u_j) &= \\ &= \lim_{k \rightarrow \infty} J(u_{j_k}) \geq \frac{1}{2} \varliminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_{j_k}|^2 dx - \frac{1}{p+1} \lim_{k \rightarrow \infty} \int_{\Omega} |u_{j_k}|^{p+1} dx \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx = J(u). \end{aligned}$$

□

Before proceeding to the proof of (20), we show the following convergence result.

Lemma 17 (weak-strong closure of $\partial^- J$). *Let $(u_j), u \in H_0^1(\Omega)$ and $(\alpha_j), \alpha \in L^2(\Omega)$ satisfy*

$$\begin{cases} \text{(A1)} & u_j \rightarrow u \text{ strongly in } L^2(\Omega) \text{ as } j \rightarrow \infty, \\ \text{(A2)} & \alpha_j \rightharpoonup \alpha \text{ weakly in } L^2(\Omega) \text{ as } j \rightarrow \infty. \end{cases}$$

If $\alpha_j \in \partial^- J(u_j)$ holds for any $j = 1, 2, \dots$, then $\alpha \in \partial^- J(u)$.

Proof. Choose $v \in H_0^1(\Omega)$ such that

$$\begin{cases} \text{(B1)} & J(v) \leq J(u) + \frac{1}{2}, \\ \text{(B2)} & \|v - u\|_{L^2(\Omega)} \leq \frac{1}{2}. \end{cases}$$

The lower semicontinuity $J(u) \leq \lim_{j \rightarrow \infty} J(u_j)$ holds due to the assumption (A1) (see the proof of (19) above). Therefore together with (B1), we have

$$J(v) \leq J(u) + \frac{1}{2} \leq \varliminf_{j \rightarrow \infty} J(u_j) + \frac{1}{2}.$$

On the other hand, by the definition of liminf, for sufficiently large j it holds that

$$J(u_j) \geq \varliminf_{j \rightarrow \infty} J(u_j) - \frac{1}{2}.$$

So we have for sufficiently large j

$$J(v) \leq J(u_j) + 1. \quad (50)$$

Furthermore, by (A1) and (B2),

$$\|v - u_j\|_{L^2(\Omega)} \leq 1 \quad (51)$$

holds for j large enough. Since $\alpha_j \in \partial^- J(u_j)$, we have from the only if part of (i) in Theorem 15

$$\int_{\Omega} \left(\nabla u_j \nabla w - |u_j|^{p-1} u_j w \right) dx = \int_{\Omega} \alpha_j w dx \quad \text{for } w \in H_0^1(\Omega). \quad (52)$$

For sufficiently large j , owing to (50) and (51) we can apply Lemma 14 to obtain

$$\begin{aligned} J(v) - J(u_j) - \left\{ \int_{\Omega} \nabla u_j \nabla (v - u_j) dx - \int_{\Omega} |u_j|^{p-1} u_j (v - u_j) dx \right\} \\ \geq -C \|u_j - v\|_{L^2(\Omega)}^2 \quad \text{for } v \in H_0^1(\Omega). \end{aligned}$$

Applying (52) to the two integral terms of the left side of the above inequality as $w = v - u_j$, we obtain

$$J(v) - J(u_j) - \langle \alpha_j, v - u_j \rangle_{L^2(\Omega)} \geq -C \|u_j - v\|_{L^2(\Omega)}^2 \quad \text{for } v \in H_0^1(\Omega). \quad (53)$$

With the help of the convergences (A1) and (A2), $\langle \alpha_j, v - u_j \rangle_{L^2(\Omega)}$ converges to $\langle \alpha, v - u \rangle_{L^2(\Omega)}$ as $j \rightarrow \infty$. Again using (19), we here passage to the limit $j \rightarrow \infty$ in (53) to have

$$J(v) - J(u) - \langle \alpha, v - u \rangle_{L^2(\Omega)} \geq -C\|u - v\|_{L^2(\Omega)}^2 \quad (54)$$

for $v \in H_0^1(\Omega)$ with (B1), (B2).

Here, for $v \in H_0^1(\Omega)$ which does not satisfy (B1), it holds that

$$J(v) - J(u) - \langle \alpha, v - u \rangle > \frac{1}{2} - \|\alpha\|_{L^2(\Omega)}\|u - v\|_{L^2(\Omega)}.$$

In addition, for $v \in L^2(\Omega) \setminus H_0^1(\Omega)$ it holds that $J(v) = +\infty$. Consequently, we can remove the condition (B1) and the restriction $v \in H_0^1(\Omega)$ in (54) to obtain

$$\begin{aligned} & \frac{J(v) - J(u) - \langle \alpha, v - u \rangle_{L^2(\Omega)}}{\|u - v\|_{L^2(\Omega)}} \\ & \geq \min \left\{ -C\|u - v\|_{L^2(\Omega)}, \frac{1}{2\|u - v\|_{L^2(\Omega)}} - \|\alpha\|_{L^2(\Omega)} \right\} \\ & \quad \text{for } v \in L^2(\Omega) \text{ with (B2).} \end{aligned}$$

By passing to the limit $v \rightarrow u$, we finally have $\alpha \in \partial^- J(u)$ by the definition of the Fréchet subdifferential. $\square$

Having shown Lemma 17, we continue to prove the sufficient condition to the existence of CMS.

- (20) : We may assume that $\lim_{\substack{v \rightarrow u \\ v \in D_0}} |\nabla J|(v) < +\infty$. Then we choose a sequence (u_j) in D_0 converging to u such that

$$\begin{cases} \text{(a)} \quad \lim_{j \rightarrow \infty} |\nabla J|(u_j) = \lim_{\substack{v \rightarrow u \\ v \in D_0}} |\nabla J|(v), \\ \text{(b)} \quad \sup_{j \in \mathbb{N}} |\nabla J|(u_j) < +\infty. \end{cases} \quad (55)$$

Then by (b) of (55) and (ii) of Theorem 15, $J(u)$ is Fréchet subdifferential at $u = u_j$ and the Fréchet subdifferential is uniquely determined, which we denote by $\alpha_j \in \partial^- J(u_j)$. Then by (iii) of Theorem 15 the equality $\|\alpha_j\|_{L^2(\Omega)} = |\nabla J|(u_j)$ holds for $j \in \mathbb{N}$. Therefore, by (b) of (55) again, by selecting a subsequence if necessary $\alpha_{j_k} \rightharpoonup \exists \alpha$ weakly in $L^2(\Omega)$ for some $\alpha \in L^2(\Omega)$. Moreover, by the assumption u_{j_k} converges to u_0 strongly in $L^2(\Omega)$ as $j \rightarrow \infty$. Hence, we can exploit here Lemma 17 to have $\alpha \in \partial^- J(u)$. By the general property (15), the inequality $|\nabla J|(u) \leq \|\alpha\|_{L^2(\Omega)}$ holds. So, the Banach-Steinhaus theorem to the convergence $\alpha_{j_k} \rightharpoonup \alpha$ as $k \rightarrow \infty$ tells us

$$|\nabla J|(u) \leq \|\alpha\|_{L^2(\Omega)} \leq \liminf_{k \rightarrow \infty} \|\alpha_{j_k}\|_{L^2(\Omega)} = \liminf_{k \rightarrow \infty} |\nabla J|(u_{j_k}).$$

□

We finally check the last sufficient condition.

• (21) : We may assume $J(u) < +\infty$, which in particular leads us that u belongs to $H_0^1(\Omega)$. Now we choose a sequence (u_j) in D_0 such that $u_j \rightarrow u$ in $L^2(\Omega)$ as $n \rightarrow \infty$ and the limsup of (21) is attained. Here we sum up the properties of (u_j) which we shall use later:

$$\begin{cases} \text{(C1)} & u_j \rightarrow u \text{ in } L^2(\Omega) \quad \text{as } j \rightarrow \infty, \\ \text{(C2)} & |\nabla J|(u_j) \leq C \quad \text{for } j \in \mathbb{N}, \\ \text{(C3)} & J(u_j) \leq J(u_0) \quad \text{for } j \in \mathbb{N}, \end{cases}$$

where C is a positive constant in (21). In particular, by (C3), u_j belongs to $H_0^1(\Omega)$. Therefore, due to (11) and (C3) we estimate

$$\int_{\Omega} |\nabla u_j|^2 dx \leq 4 \left(J(u_0) + C_{15} \|u_j\|_{L^2(\Omega)}^q \right).$$

For sufficiently large n , by (C1) it holds that

$$\|u_j\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + 1,$$

and therefore we establish the uniform boundedness of $\|u_j\|_{H_0^1(\Omega)} = \|\nabla u_j\|_{L^2(\Omega)}$:

$$\int_{\Omega} |\nabla u_j|^2 dx \leq C_{22} \left(J(u_0) + \|u_0\|_{L^2(\Omega)}^q \right).$$

Hence by Lemma 11, choosing a subsequence if necessary, we obtain

$$\lim_{j \rightarrow \infty} \int_{\Omega} |u_{j_k}|^{p+1} dx = \int_{\Omega} |u|^{p+1} dx. \quad (56)$$

By (C2) it holds $|\nabla J|(u_j) < +\infty$, and taking (19) into account we also have $|\nabla J|(u_0) < +\infty$. Thus, by (i), (ii) and (iii) of Theorem 15 we claim that there exist $\alpha_{j_k}, \alpha \in L^2(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} \alpha_{j_k} w dx &= \int_{\Omega} \left(\nabla u_{j_k} \nabla w - |u_{j_k}|^{p-1} u_{j_k} w \right) dx, \\ \int_{\Omega} \alpha w dx &= \int_{\Omega} \left(\nabla u \nabla w - |u|^{p-1} u w \right) dx \end{aligned}$$

for any $w \in H_0^1(\Omega)$. Making use of these identity with $w = u - u_{j_k} \in H_0^1(\Omega)$, we observe

$$\begin{aligned} \int_{\Omega} |\nabla u_{j_k}|^2 dx &= \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} \nabla(u - u_{j_k}) \cdot \nabla(u + u_{j_k}) dx, \\ &\leq \int_{\Omega} |\nabla u|^2 dx + \\ &\quad + \|\alpha_{j_k}\|_{L^2(\Omega)} \|u_{j_k} - u\|_{L^2(\Omega)} + \|\alpha\|_{L^2(\Omega)} \|u_{j_k} - u\|_{L^2(\Omega)} \\ &\quad + \int_{\Omega} |u_{j_k}|^p |u_{j_k} - u| dx + \int_{\Omega} |u|^p |u_{j_k} - u| dx. \end{aligned} \quad (57)$$

By recalling $\|\alpha_{j_k}\|_{L^2(\Omega)} = |\nabla J|(u_{j_k}) \leq C$ as well as $\|u_{j_k} - u\|_{L^2(\Omega)} \rightarrow 0$, the sum of the second and the third term of the right side of (57) converge to zero as $j \rightarrow \infty$. For treating the fourth term we set

$$r := \frac{2m^2}{m^2 - 2m + 8}.$$

Then we can easily check that

$$\begin{aligned} \text{(a)} \quad & 1 < r < \frac{2m}{m-2}, \\ \text{(b)} \quad & p \frac{r}{r-1} < \frac{2m}{m-2}. \end{aligned} \tag{58}$$

Since the sequence $(u_{j_k}) \subset W_0^{1,2}(\Omega)$ is bounded by (a) of (58) and the Sobolev compact imbedding theorem $W_0^{1,2}(\Omega) \hookrightarrow L^r(\Omega)$. Here in case $m = 2$, we consider $\frac{2m}{m-2} = +\infty$. Recalling here $u_{j_k} \rightarrow u$ in $L^2(\Omega)$ we have from the uniqueness of the limit that

$$(u_{j_k} - u) \rightarrow 0 \quad \text{in } L^r(\Omega). \tag{59}$$

By the Hölder inequality we have

$$\begin{aligned} \int_{\Omega} |u_{j_k}|^p |u_{j_k} - u| dx &\leq \left(\int_{\Omega} |u_{j_k} - u|^r dx \right)^{\frac{1}{r}} \left(\int_{\Omega} |u_{j_k}|^{p \frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} \\ &= \|u_{j_k} - u\|_{L^r(\Omega)} \left(\int_{\Omega} |u_{j_k}|^{p \frac{r}{r-1}} dx \right)^{\frac{r-1}{r}}. \end{aligned} \tag{60}$$

In case $m = 2$, we may assume that $u_{j_k} \rightarrow u$ in $L^\sigma(\Omega)$ for any positive number $\sigma > 1$, and therefore the last side turns out to converge to zero. In case $m \geq 3$, the last side of (60) is estimated furthermore by the Hölder inequality in the following way :

$$\begin{aligned} & \|u_{j_k} - u\|_{L^r(\Omega)} \left(\int_{\Omega} |u_{j_k}|^{p \frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} \\ & \leq C(\Omega, m, p) \|u_{j_k} - u\|_{L^r(\Omega)} \|u_{j_k}\|_{L^{\frac{2m}{m-2}}(\Omega)}^p \\ & \leq C \|u_{j_k} - u\|_{L^r(\Omega)} \|\nabla u_{j_k}\|_{L^2(\Omega)}^p, \end{aligned}$$

where we use the Sobolev inequality in the last inequality. Since the sequence $\|\nabla u_{j_k}\|_{L^2(\Omega)}$ is uniformly bounded, the last side converges to zero by passing to the limit $k \rightarrow \infty$ owing to (59). In the same way, the last term of the right side of (57) also converges to zero by passing to the limit $k \rightarrow \infty$. We thus obtain

$$\overline{\lim}_{k \rightarrow \infty} \int_{\Omega} |\nabla u_{j_k}|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx. \tag{61}$$

Finally we arrive at our conclusion by (56) and (61). $\square$

Corollary 18 (the existence of a strong evolution curve). *Let p as in (6). Then for any $u_0 \in H_0^1(\Omega)$, there exists a time-local strong evolution curve to the functional J .*

Proof. Let $u(t)$ ($t \in [0, t_0]$) be a time-local curve of maximal slope constructed by Theorem 7. Then by the definition of curve of maximal slope, $|\nabla J|(u(t)) < +\infty$ holds for almost all $t \in (0, t_0)$. Therefore, by (iv) of Theorem 15, we have

$$|\nabla J|(u(t)) = \|\text{grad}^- J(u(t))\|_{L^2(\Omega)}$$

for almost all $t \in (0, t_0)$. Again by the general theory due to [6], this curve is a strong evolution curve. $\square$

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