International Journal of Differential Equations and Applications
Volume **19**, No. 1 (2020), pages: 133-141
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu
ijdea.eu

DISCRETE MODEL OF MULTI-AGENT SYSTEMS WITH NONLINEAR DYNAMICS BY CHUA'S CIRCUIT VIA LONG LASTING IMPULSIVE PROTOCOLS

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ABSTRACT: In this paper we define a discrete model of a multi-agent system which non-linearity of the velocity is given by Chua's circuit and in which the interaction between the agents is oy at some long time impulses. We study the second-order consensus problem of the defined model. Numerical simulations are presented to support the theoretical results.

AMS Subject Classification: 92B20, 39A30, 93C55
Key Words: multi agent system, discrete models, consensus, non-instantaneous impulses

Received: May 13, 2020 **Revised:** November 29, 2020
Published: December 7, 2020 **doi:** 10.12732/ijdea.v19i1.9
Academic Publications, Ltd. https://acadpubl.eu

1. INTRODUCTION

The first step in modeling of multi-agent systems is the statement up of the model for the considered system. We consider the multi-agent system and we model discretely the velocity and the position of each agent. The non-linearity of the velocity in the multi-agent system sis described by Chua's circuit. Also, we consider the case when there are interactions between the each agents only at some finite intervals of time. We study one of the most important problems so called consensus algorithm. It is connected with the behavior of the agents receiving information from the set of other agents in the group and then all agents adjust their own information states depending on the information received from other agents. The goal is to reach an agreement. This behavior is widespread in the nature. A consensus algorithm describes the information transfers between agents and it varies depending on the model. In the literature, many different consensus algorithms have been proposed (see for example, [2], [3], [4], [5], [6], [7]). In this paper we define and study second order consensus for the defined model and we obtain some sufficient conditions ensuring a second order consensus. By intensive application of computer simulation the influence of the impulses on the discrete second order consensus is illustrated.

2. DESCRIPTION OF THE DISCRETE MODEL WITH NON-INSTANTANEOUS IMPULSES

In this paper for any vector $x \in \mathbb{R}^N : x = (x_1, x_2, \dots, x_N)^T$ we define $||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$ and $|x_i|$ is the absolute value.

Let $\mathbb{Z}_+$ denote the set of all natural numbers. Let two increasing sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be given such that $n_k, m_k \in \mathbb{Z}_+$ and $m_k < n_{k+1} - n_k,\ k = 0, 1, 2, \dots,$ be given. We denote $\mathbb{Z}[a, b] = \{z \in \mathbb{Z}_+ : a \le z \le b\}$, $a, b \in \mathbb{Z}_+,\ a < b$ and $I_k = \mathbb{Z}[n_k + m_k, n_{k+1} - 1],\ k = 0, 1, 2, \dots,$ and $J_k = \mathbb{Z}[n_k, n_k + m_k - 1],\ k = 1, 2, \dots,$ where $m_0 = 1, n_0 = 0$.

The intervals $J_k,\ k = 1, 2, \dots,$ will be called intervals of non-instantaneous impulses. Note the length of interval of non-instantaneous impulses is $m_k,\ k = 1, 2, \dots.$

Denote the length of each interval without impulses I_k by $l_k = n_{k+1} - n_k - m_k,\ k = 0, 1, 2, \dots.$

In this paper, we consider a discrete-time second order multi-agent system consisting of 3 agents:

$$\begin{aligned}
&x_i(n) = x_i(n-1) + v_i(n-1)\\
&v_i(n) = f_i(v(n-1)) \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,\ i = 1, 2, 3,\\
&x_i(n) = \mu_k x_i(n_k - 1) + b_k \sum_{j=1}^{3} A_{ij}\big(x_j(n-1) - x_i(n-1)\big), \quad n \in J_k,\ k = 1, 2, \dots, \\
&v_i(n) = \mu_k v_i(n_k - 1) + c_k \sum_{j=1}^{3} B_{ij}\big(v_j(n-1) - v_i(n-1)\big), \quad n \in J_k,\ k = 1, 2, \dots,\\
&x_i(0) = x_i^0,\ v_i(0) = v_i^0 \quad i = 1, 2, 3.
\end{aligned} \tag{1}$$

where the discrete time instants $n \in \mathbb{Z}_+$, $x_i(n)$ and $v_i(n)$ represent the position and velocity of i-th agent at time n, respectively, $v = (v_1, v_2, v_3)$, and the nonlinear function $f(v) = (f_1(v), f_2(v), f_3(v))$ is a nonlinear function, $b_k, c_k, \mu_k,\ k = 1, 2, \ldots$ are the impulsive gains.

We will consider the dynamics of Chua's circuit ([1]) as the nonlinear function, i.e. the nonlinear functions

$$\begin{aligned} f_1(v) &= \xi\Big(- v_1 + v_2 - L(v_1)\Big) \\ f_2(v) &= v_1 - v_2 + v_3, \\ f_3(v) &= -\rho v_2, \end{aligned}$$

where ξ, ρ are real constants and $L(v_1) = bv_1 + 0.5(a - b)\Big(|v_1 + 1| - |v_1 - 1|\Big)$, a, b are constants.

Note that the impulsive protocols of multi- agent system (1), indicating that agents can only exchange information at impulsive instants, are defined by both equations on $J_k,\ k = 1, 2, \ldots$ in (1)

Definition 1. The multi-agent system (1) is said to achieve second-order consensus if for any initial state:

$$\lim_{n\to\infty} |x_j(n) - x_i(n)| = 0, \quad \lim_{n\to\infty} |v_j(n) - v_i(n)| = 0$$

where $i, j = 1, 2, 3$.

Theorem 2. *Let the* 3×3 *dimensional matrices* $\{A_{ij}\}$ *and* $\{B_{i,j}\}$ *and the constants* $b_k, c_k,\ k = 1, 2, \ldots$ *be such that*

$$(|\mu_k|\frac{1 - (4|b_k|A)^{m_k}}{1 - 4|b_1|A} + (4|b_k|A)^{m_k}))M^{l_k+l_{k+1}} < 1, \quad k = 1, 2, \ldots$$

and

$$(|\mu_l|\frac{1 - (4|c_k|B)^{m_k}}{1 - 4|c_k|B} + (4|c_k|B)^{m_k}))M^{l_k+l_{k+1}} < 1, \ k = 1, 2, \ldots$$

where $l_k = n_k - n_{k-1} - m_{k-1}$, $A = \max_{i,j} |A_{ij}|, \quad B = \max_{i,j} |B_{ij}|$ *and*

$$M = \max\{|\xi + b\xi + 1| + |\xi|\ |a - b|, |\xi + 1|, |\xi|(|1 + b| + |a - b|), |\xi + \rho|, 1\} \geq 2.$$

Then the multi-agent system (1) achieves the second-order consensus.

Proof. Using that $f_1(v) = -\xi(1 + b)v_1 + \xi v_2 - 0.5\xi(a - b)(|v + 1| - |v - 1|)$ and

$|v_1+1|-|v_1-1|\leq 2|v_1|$ we get for the nonlinear function $f(v)$

$$\begin{aligned}
|f_1(v)-f_2(v)| &= |-\xi(1+b)v_1+\xi v_2-0.5\xi(a-b)(|v+1|-|v-1|)-v_1+v_2-v_3| \\
&\leq |\xi+b\xi+1|v_1+|\xi+1|v_2+|\xi||a-b||v_1|+|v_3| \\
&= (|\xi+b\xi+1|+|\xi|\ |a-b|)v_1+|\xi+1|v_2+|v_3| \\
&\leq \max\{|\xi+b\xi+1|+|\xi|\ |a-b|,|\xi+1|,1\}||v||\leq M||v||, \\
|f_1(v)-f_3(v)| &= |-\xi(1+b)v_1+\xi v_2-0.5\xi(a-b)(|v+1|-|v-1|)+\rho v_2| \\
&\leq |\xi|(|1+b|+|a-b|)v_1+|\xi+\rho|v_2| \\
&\leq \max\{|\xi|(|1+b|+|a-b|),|\xi+\rho|\}||v||\leq M||v||, \\
|f_2(v)-f_3(v)| &= |v_1+(\rho-1)v_2+v_3|\leq \max\{1,|1-\rho|\}||v|| \\
&\leq M||v||.
\end{aligned} \tag{2}$$

Note $\frac{1-M^n}{1-M}\leq M^n=\frac{M^n}{M-1}-\frac{1}{M-1}\leq\frac{M^n}{M-1}\leq M^n$ for $M\geq 2$.
For $n\in\mathbb{Z}[1,n_1-1]$ using (2) we get

$$\begin{aligned}
&|v_j(n)-v_i(n)|\leq |f_j(n-1,v(n-1))-f_i(n-1,v(n-1))|\leq M||v(n-1)|| \\
&\leq M^n||v_0||.
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&|x_j(n)-x_i(n)|\leq |x_j(n-1)-x_i(n-1)|+|v_j(n-1)-v_i(n-1)| \\
&\leq |x_j(n-1)-x_i(n-1)|+M^{n-1}||v_0|| \\
&\leq |x_j(0)-x_i(0)|+(1+\dots+M^{n-2}+M^{n-1})||v_0|| \\
&\leq X_0+\frac{1-M^n}{1-M}||v_0||\leq M^n(X_0+||v_0||)
\end{aligned} \tag{4}$$

where $X_0=\max_{i,j}|x_j(0)-x_i(0)|$.

From the impulsive conditions of (1) and inequalities (3) and (4) we have

$$\begin{aligned}
&|x_i(n_1)-x_j(n_1)| \\
&= |\mu_1(x_i(n_1-1)-x_j(n_1-1))+b_1\sum_{l=1}^{3}A_{il}(x_l(n_1-1)-x_i(n_1-1)) \\
&\quad -b_1\sum_{l=1}^{3}A_{jl}(x_l(n_1-1)-x_j(n_1-1))| \\
&\leq |\mu_1|\ M^{n_1-1}(X_0+||v_0||)+2|b_1|A\ 2M^{n_1-1}(X_0+||v_0||) \\
&\leq (|\mu_1|+4|b_1|A)M^{l_1}\Big(X_0+||v_0||\Big)
\end{aligned} \tag{5}$$

and

$$
\begin{aligned}
|v_i(n_1) - v_j(n_1)| &= |\mu_1 v_i(n_1 - 1) + c_1 \sum_{l=1}^{3} B_{il}\big(v_l(n_1 - 1) - v_i(n_1 - 1)\big) \\
&- \mu_1 v_j(n_1 - 1) - c_1 \sum_{l=1}^{3} B_{jl}\big(v_l(n_1 - 1) - v(n_1 - 1)\big)| \\
&\le (|\mu_1| + 4|c_1|B) M^{l_1} ||v_0||
\end{aligned} \tag{6}
$$

Let $n \in \mathbb{Z}[n_1 + 1, n_1 + m_1 - 1]$. Then applying inequalities (5) and (6) we get

$$
\begin{aligned}
&|x_i(n) - x_j(n)| = |x_i(n + (n - n_1)) - x_j(n + (n - n_1))| \\
&\le (|\mu_1|(\sum_{j=0}^{n-n_1} (4|b_1|A)^j + (4|b_1|A)^{n-n_1+1}) M^{l_1} \Big(X_0 + ||v_0||\Big) \\
&= (|\mu_1| \frac{1 - (4|b_1|A)^{n-n_1+1}}{1 - 4|b_1|A} + (4|b_1|A)^{n-n_1+1})) M^{l_1} \Big(X_0 + ||v_0||\Big) \\
&\le (|\mu_1| \frac{1 - (4|b_1|A)^{m_1}}{1 - 4|b_1|A} + (4|b_1|A)^{m_1})) M^{l_1} \Big(X_0 + ||v_0||\Big)
\end{aligned} \tag{7}
$$

and

$$
|v_i(n) - v_j(n)| \le (|\mu_1| \frac{1 - (4|c_1|B)^{m_1}}{1 - 4|c_1|B} + (4|c_1|B)^{m_1})) M^{l_1} ||v_0|| \tag{8}
$$

Similarly, for $n \in \mathbb{Z}[n_k + m_k, n_{k+1} - 1]$ we get

$$
|x_i(n) - x_j(n)| \le M^{l_{k+1}} \Big(X_0 + ||v_0||\Big) \prod_{j=1}^{k} (|\mu_j| \frac{1 - (4|b_j|A)^{m_j}}{1 - 4|b_j|A} + (4|b_j|A)^{m_j})) M^{l_j} \tag{9}
$$

and

$$
|v_i(n) - v_j(n)| \le M^{l_{k+1}} ||v_0|| \prod_{j=1}^{k} (|\mu_j| \frac{1 - (4|c_j|B)^{m_j}}{1 - 4|c_j|B} + (4|c_j|B)^{m_j})) M^{l_j}. \tag{10}
$$

For $n \in \mathbb{Z}[n_{k+1}, n_{k+1} + m_{k+1} - 1]$ we obtain

$$
|x_i(n) - x_j(n)| \le \Big(X_0 + ||v_0||\Big) \prod_{j=1}^{k+1} (|\mu_j| \frac{1 - (4|b_j|A)^{m_j}}{1 - 4|b_j|A} + (4|b_j|A)^{m_j})) M^{l_j} \tag{11}
$$

and

$$
|v_i(n) - v_j(n)| \le ||v_0|| \prod_{j=1}^{k+1} (|\mu_j| \frac{1 - (4|c_j|B)^{m_j}}{1 - 4|c_j|B} + (4|c_j|B)^{m_j})) M^{l_j}. \tag{12}
$$

From conditions of Theorem 2 it follows that for all $k = 1, 2, \dots$ the inequalities $(|\mu_k|\frac{1-(4|b_k|A)^{m_k}}{1-4|b_1|A} + (4|b_k|A)^{m_k}))M^{l_k} < 1$, and $(|\mu_l|\frac{1-(4|c_k|B)^{m_k}}{1-4|c_k|B} + (4|c_k|B)^{m_k}))M^{l_k} < 1$ hold. Therefore, we get $\lim_{n\to\infty} |x_i(n) - x_j(n)| = 0$ and $\lim_{n\to\infty} |v_i(n) - v_j(n)| = 0$ which proves the claim.

□

Remark 1. From (3) and (4) it follows that in the case there are no impulses in the model (1) and therefore, there are no interactions between the agents, then it could be $\lim_{n\to\infty} |x_i(n) - x_j(n)| = \infty$ and $\lim_{n\to\infty} |v_i(n) - v_j(n)| = \infty$, i.e. there is no second order consensus.

3. APPLICATION

Case 1. Model of a multi-agent system with nonlinear part described by dynamics of Chua's circuit:

Initially we will consider a multi-agent system with no impulses and no interactions between agents:

$$\begin{aligned}
&x_i(n) = x_i(n-1) + v_i(n-1) \quad \text{for } n \geq 1,\ i = 1, 2, 3,\\
&v_1(n) = -1.25 v_1(n-1) + 2 v_2(n-1)\\
&\qquad - 1/24\Big(|v_1(n-1) + 1| - |v_1(n-1) - 1|\Big) \ \text{ for } n \geq 1,\\
&v_2(n) = v_1(n-1) - v_2(n-1) + v_3(n-1) \ \text{ for } n \geq 1,\\
&v_3(n) = -v_2(n-1) \ \text{ for } n \geq 1,\\
&x_i(0) = 1,\ v_i(0) = 1 \ \ i = 1, 2, 3.
\end{aligned} \tag{13}$$

where $a = -1/3, b = -3/8$.

Then $\xi = 2$, $\rho = 1$, $a = -1/3, b = -3/8$, $M = \max\{7/3, 3, 4/3, 3, 1\} = 3 > 2$ and the second-order consensus is not reached (see Figure 1 and 2).

Case 2. Model of a multi-agent system with nonlinear part described by Chua's circuit and non-instantaneous impulses.

Now we will consider a model of a multi-agent system with interactions between agents only at some non-instantaneous impulsive times. Let $m_k = 4$, $l_k = 10$, $k = 1, 2, \dots$. Then the intervals are $I_1 = \mathbb{Z}[1, 10], J_1 = \mathbb{Z}[11, 14], I_2 = \mathbb{Z}[15, 24], J_2 = \mathbb{Z}[25, 28], I_3 = \mathbb{Z}[29, 38], J_3 = \mathbb{Z}[39, 42], I_4 = \mathbb{Z}[43, 52], J_4 = \mathbb{Z}[53, 56], I_5 = \mathbb{Z}[57, 66], \dots$. Note that the intervals J_k are the impulsive intervals.

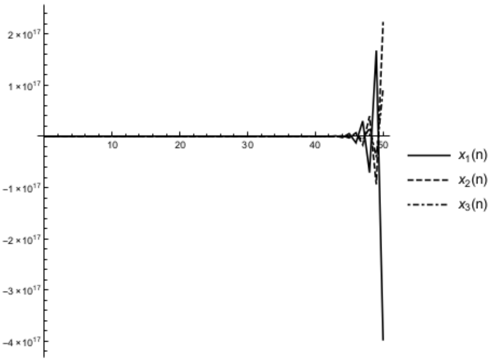

Figure 1. Graphs of the positions $x_i(n),\ i=1,2,3$ of (13) with initial values $(1,1,1)$.

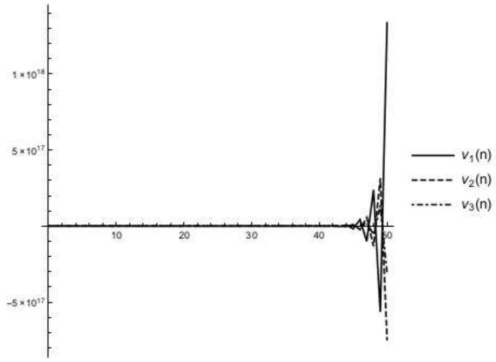

Figure 2. Graphs of the speeds $v_i(n),\ i=1,2,3$ of (13) with initial values $(1,1,1)$.

Consider the following discrete dynamical model describing the positions and the velocities of the three agents

$$
\begin{aligned}
&x_i(n) = x_i(n-1) + v_i(n-1), \quad \text{for } \ n \in I_k, \quad k=1,2,3\dots, \quad i=1,2,3,\\
&v_1(n) = -1.25v_1(n-1) + 2v_2(n-1)\\
&\qquad - 1/24\Big(|v_1(n-1)+1| - |v_1(n-1)-1|\Big)\\
&\qquad \text{for } \ n \in I_k, \quad k=1,2,3\dots,\\
&v_2(n) = v_1(n-1) - v_2(n-1) + v_3(n-1) \ \text{ for } \ n \in I_k, \quad k=1,2,3\dots,\\
&v_3(n) = -v_2(n-1) \ \text{ for } \ n \in I_k, \quad k=1,2,3\dots,\\
&x_1(n) = -0.2^{14}x_1(n_k-1) - (1/3)^{10}\Big(0.5\big(x_2(n-1)-x_1(n-1)\big)\\
&\qquad + 0.1x_3(n-1) - x_1(n-1))\Big),\\
&x_2(n) = -0.2^{14}x_2(n_k-1) - (1/3)^{10}(0.15)\big(x_1(n-1)-x_2(n-1)\big),\\
&x_3(n) = -0.2^{14}x_3(n_k-1) - (1/3)^{10}\big(x_1(n-1)-x_3(n-1)\big),\\
&v_1(n) = -0.2^{14}v_1(n_k-1),\\
&v_2(n) = -0.2^{14}v_2(n_k-1) - 0.5^{10}\Big(0.5\big(v_1(n-1)-v_2(n-1)\big)\\
&\qquad + 0.1\big(v_3(n-1)-v_2(n-1)\big),\\
&v_3(n) = -0.2^{14}v_3(n_k-1) - 0.5^{10}(-0.5)\big(v_2(n-1)-v_3(n-1)\big),\\
&\qquad\qquad \text{for } \ n \in J_k, \ k=1,2,\dots, \ i=1,2,3,\\
&x_i(0)=1, \ v_i(0)=1 \ \ i=1,2,3,
\end{aligned}
\tag{14}
$$

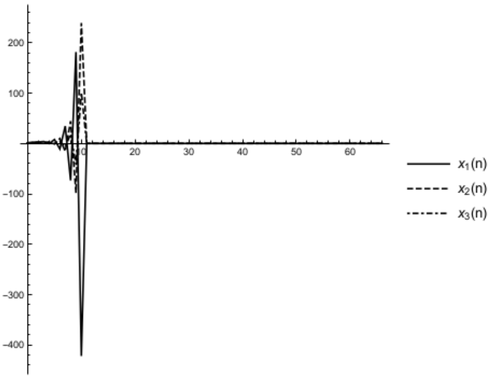

Figure 3. Graphs of the positions $x_i(n),\ i = 1,2,3$ of (14) with initial values $(1,1,1)$.

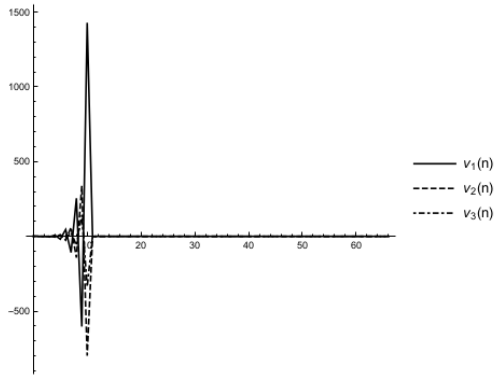

Figure 4. Graphs of the speeds $v_i(n),\ i = 1,2,3$ of (14) with initial values $(1,1,1)$.

i.e. the speed and the position of the first agent are determined by the interaction with the second and third agents and with the second agent, respectively. Similarly, the speed and the position of the second agent are determined by the interaction with the first agent and with the third agent, respectively.

Therefore, $\mu_k = -0.2^{14},\ \ b_k = -(1/3)^{10},\ c_k = -0.5^{10},\ k = 1,2,\dots,$ and

$$A_k = \begin{bmatrix} 0 & 0.5 & 0.1 \\ 0.15 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B_k = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0.1 \\ 0 & -0.5 & 0 \end{bmatrix},\ k = 1,2,\dots.$$

Then, $A = \max_{i,j} |A_{ij}| = 1$, $B = \max_{i,j} |B_{ij}| = 0.5$, $M = 3 = \frac{23}{6}$,

$$(|\mu_k|\frac{1-(4|b_k|A)^{m_k}}{1-4|b_k|A} + (4|b_k|A)^{m_k}))M^{l_k+l_{k+1}} \approx 0.57 < 1,$$

and

$$(|\mu_k|\frac{1-(4|c_k|B)^{m_k}}{1-4|c_k|B} + (4|b_k|B)^{m_k}))M^{l_k+l_{k+1}} \approx 0.62 < 1,$$

i.e. the conditions of Theorem 2 are satisfied. Therefore, the multi-agent system (14) achieves the second-order consensus (see, Figures 3 and 4).

This example illustrative the presence of impulses with the interactions between agent only on the impulsive times in spite the small weights of the interactions, can cause the achievement of second-order consensus.

ACKNOWLEDGMENTS

The research is supported by the Project FP19-FMI-002 ”Innovative ICT for Digital Research Area in Mathematics, Informatics and Pedagogy of Education” of the Scientific Fund of the University of Plovdiv Paisii Hilendarski, Bulgaria.

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