International Journal of Differential Equations and Applications
Volume **20**, No. 2 (2021), pages: 169-178
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu
ijdea.eu

EXPONENTIAL STABILITY FOR DIFFERENCE EQUATIONS WITH MAXIMUM: THEORETICAL STUDY AND COMPUTER SIMULATIONS

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ABSTRACT: Difference equations with a special type of delay is studied. This type of delay is characterized by the maximum value of the unknown function on the past time interval with a constant length. It is studied the exponential stability. Some sufficient conditions are theoretically proved and computer simulated on several examples. The influence of the type of the coefficients is illustrated.

AMS Subject Classification: 34D20, 34D40, 34K20
Key Words: difference equations, maximum, exponential stability, Lyapunov functional

Received: August 23, 2021 **Revised:** November 4, 2021
Published: November 15, 2021 **doi:** 10.12732/ijdea.v20i2.3
Academic Publications, Ltd. https://acadpubl.eu

1. INTRODUCTION

In modeling one of the main problems is connected with time. In many case the real systems are studied as discrete time systems (see, for example, [11] for the first order systems in discrete-time approach, [2] for a discrete-time version of epidemic model is developed, [9] for the discrete-time predator prey model). Several numerical study of a real word models are given in [3, 4, 5, 6, 12, 13, 14]. Recently, the study of difference equations has caused a greater interest, for example, see [1, 7].

In the real world life there are many processes and phenomena that are characterized by significant dependence of the maximum value over a past time interval in their state.

The main purpose of this paper is the study of exponential stability of the zero solution of the initial value problem for difference equations with maximum. We construct a Lyapunov functional that yields exponential stability and provide computer simulations of several particular problems to illustrate the usefulness of the obtained criteria. There are not many papers dealing with the study of exponential stability of delay difference equations, especially with maximum delay.

2. STATEMENT OF THE PROBLEM

We will introduce basic notations used in this paper. Most of them are well known and used in the literature. Let $\mathbb{Z}$ be the set of integers; $\mathbb{Z}(a,b) = \{z \in \mathbb{Z} : a \le z \le b\}$, $a, b \in \mathbb{Z}$, $a < b$, $\mathbb{Z}(a,\infty) = \{z \in \mathbb{Z} : z \ge a\}$, $a \in \mathbb{Z}$.

Let $\phi : \mathbb{Z}(-h,0) \to \mathbb{R}$ with $||\phi||_0 = \max_{s \in \mathbb{Z}(-h,0)} |\phi(s)|$.

Consider the *initial value problem (IVP)* for the nonlinear *difference equation with maxima*

$$\begin{aligned} &x(n+1) = a(n)x(n) + b(n) \max_{\xi \in \mathbb{Z}(n-h,n)} x(\xi) \text{ for } n \in \mathbb{Z}(0,\infty), \\ &x(n) = \phi(n) \text{ for } n \in \mathbb{Z}(-h,0), \end{aligned} \tag{1}$$

where $h \in \mathbb{Z},\ h > 0$, and $\phi : \mathbb{Z}(-h,0) \to \mathbb{R}$.

Definition 1. The zero solution of (1) is said to be exponentially stable, if there exist positive constants $\lambda, \gamma : \lambda < 1$ and a function $C : \mathbb{R}_+ \to \mathbb{R}_+$ such that any solution $x(n;\phi)$ of (1) satisfies

$$|x(n;\phi)| \le C(||\phi||_0)\lambda^{\gamma n} \quad \text{for} \quad n \in \mathbb{Z}(0,\infty).$$

Denote $Q(n) = a(n) + b(n+h) - 1,\ n \in \mathbb{Z}(0,\infty)$.

We will introduce the assumptions:

A1. There exists a positive constant δ such that the inequality

$$-\frac{\delta}{h(1+\delta)} \le Q(n) \le -\Big(h\delta b^2(n+h) + Q^2(n)\Big), \quad n \in \mathbb{Z}(0,\infty)$$

holds.

A2. There exists a positive constant λ such that the inequality

$$a(n) + b(n+h) \le \lambda < 1, \quad n \in \mathbb{Z}(0,\infty)$$

holds.

Remark 2. If assumption (A1) is satisfied, then

$$\begin{aligned} Q(n)+1 &= a(n)+b(n+h) \\ &\geq 1-\frac{\delta}{h(1+\delta)} \\ &= \frac{h+(h-1)\delta-\delta}{h(1+\delta)} \\ &\geq 0, \quad n\in\mathbb{Z}(0,\infty) \end{aligned}$$

but $Q(n)<0$.

3. MAIN RESULT

For study of the exponential stability of the zero solution of (1), we will use the ideas of [10] and we define in an appropriate way connected with the presence of maximum the following Lyapunov functional

$$\begin{aligned} V(n) = {}& \Big[x(n)+\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]^2 \\ &+\delta\sum_{s\in\mathbb{Z}(-h,\,-1)}\ \sum_{j\in\mathbb{Z}(n+s,\,n-1)} b^2(j+h)\Big[\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\Big]^2, \end{aligned} \tag{2}$$

where $x(n)=x(n;\phi)$ is a solution of (1).

Lemma 3. *Let the assumption (A1) be satisfied. Then*

$$V(n+1)\leq \big(Q(n)+1\big)V(n), \quad n\in\mathbb{Z}(0,\infty). \tag{3}$$

Proof. For any $n\in\mathbb{Z}(0,\infty)$ from (1) we obtain

$$\begin{aligned} \Delta x(n) &= x(n+1)-x(n) \\ &= (a(n)+b(n+h)-1)x(n) \\ &\quad+\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi) \\ &\quad-\sum_{s\in\mathbb{Z}(n-h+1,\,n)} b(s+h)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi) \\ &= Q(n)x(n) \\ &\quad-\Delta\Big(\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big), \quad n\in\mathbb{Z}(0,\infty). \end{aligned} \tag{4}$$

Therefore,

$$\Delta x(n)+\Delta\Big(\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big)=Q(n)x(n). \tag{5}$$

From the definition (2) and $\Delta u^2(n)=u(n+1)\Delta u(n)+u(n)\Delta u(n)$ we obtain

$$\begin{aligned}\Delta V(n)=&\Big[x(n+1)+\sum_{s\in\mathbb{Z}(n+1,\,n+h)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]\times\\ &\times\Big[\Delta x(n)+\Delta\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]\\ &+\Big[x(n)+\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]\times\\ &\times\Big[\Delta x(n)+\Delta\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]\\ &+\delta\Delta\sum_{s\in\mathbb{Z}(-h,\,-1)}\sum_{j\in\mathbb{Z}(n+s,\,n-1)} b^2(j+h)\Big[\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\Big]^2.\end{aligned} \tag{6}$$

From (6) and (5) it follows

$$\begin{aligned}\Delta V(n)=&\Big[(Q(n)+1)x(n)\\ &+\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]Q(n)x(n)\\ &+\Big[x(n)+\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]Q(n)x(n)\\ &+\delta\sum_{s\in\mathbb{Z}(-h,\,-1)}\Big[b^2(n+h)[\max_{\xi\in\mathbb{Z}(n,\,n)} x(\xi)]^2\\ &-b^2(n+s+h)[\max_{\xi\in\mathbb{Z}(n+s,\,n)} x(\xi)]^2\Big]\\ =&\,Q(n)x^2(n)+2Q(n)x(n)\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\\ &+Q(n)\Big[\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]^2\\ &+Q(n)\delta\sum_{s\in\mathbb{Z}(-h,\,-1)}\sum_{j\in\mathbb{Z}(n+s,\,n-1)} b^2(j+h)[\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)]^2\\ &+(Q^2(n)+Q(n)+h\delta b^2(n+h))x^2(n)\\ &-Q(n)\Big[\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]^2\end{aligned}$$

$$
\begin{aligned}
&- \delta \sum_{s\in\mathbb{Z}(-h,\,-1)} b^2(n+h+s)\Big[\max_{\xi\in\mathbb{Z}(n+s,\,n)} x(\xi)\Big]^2 \\
&- Q(n)\delta \sum_{s\in\mathbb{Z}(-h,\,-1)} \sum_{j\in\mathbb{Z}(n+s,\,n-1)} b^2(j+h)\Big[\max_{\xi\in\mathbb{Z}(j,n)} x(\xi)\Big]^2 \\
\le{}& Q(n)V(n) + \big(Q^2(n)+Q(n)+h\delta b^2(n+h)\big)x^2(n) \\
&- Q(n)\Big[\sum_{s\in\mathbb{Z}(n-h,\,n-1)} b(s+h) \max_{\xi\in\mathbb{Z}(s,\,n)} x(\xi)\Big]^2 \\
&- \delta \sum_{j\in\mathbb{Z}(n-h,\,n-1)} b^2(j+h)\Big[\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\Big]^2 \\
&- Q(n)h\delta \sum_{j\in\mathbb{Z}(n-h,\,n-1)} b^2(j+h)\Big[\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\Big]^2.
\end{aligned}
\tag{7}
$$

By Holder's inequality and Remark 2 we get

$$
\begin{aligned}
&- Q(n)h \sum_{j\in\mathbb{Z}(n-h,\,n-1)} b^2(j+h)\Big[\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\Big]^2 \\
&\ge -Q(n)\Big[\sum_{j\in\mathbb{Z}(n-h,\,n-1)} b(j+h)\Big[\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\Big]\Big]^2.
\end{aligned}
\tag{8}
$$

Then, from equation (7) and inequality (8) we obtain

$$
\begin{aligned}
\Delta V(n) \le{}& Q(n)V(n) \\
&+ \big(Q^2(n)+Q(n)+h\delta b^2(n+h)\big)x^2(n) \\
&- \big(Q(n)h(1+\delta)+\delta\big) \sum_{j\in\mathbb{Z}(n-h,\,n-1)} b^2(j+h)\Big[\max_{\mathbb{Z}(j,\,n)} x(\xi)\Big]^2.
\end{aligned}
\tag{9}
$$

From assumption (A2) it follows $Q(n)h(1+\delta)+\delta \ge 0$ and $Q^2(n)+Q(n)+h\delta b^2(n+h) < 0$.

Therefore,

$$
\Delta V(n) \le Q(n)V(n). \tag{10}
$$

□

Corollary 4. *Let the conditions (A1) and (A2) be satisfied.*
Then the inequality

$$
V(n+1) \le V(0)\lambda^{n+1}, \quad n\in\mathbb{Z}(0,\infty) \tag{11}
$$

holds, where

$$
V(0) = \Big[x(0) + \sum_{s\in\mathbb{Z}(0,\,h-1)} b(s) \max_{\xi\in\mathbb{Z}(s-h,\,0)} x(\xi)\Big]^2
$$

$$+\,\delta \sum_{s\in\mathbb{Z}(-h,\,-1)}\ \sum_{j\in\mathbb{Z}(s,\,-1)} b^2(j+h)\Big[\max_{\xi\in\mathbb{Z}(j,\,0)} x(\xi)\Big]^2. \tag{12}$$

The proof follows from the inequality (3) and

$$V(n+1)\le \big(Q(n)+1\big)V(n)\le V(0)\prod_{j\in\mathbb{Z}(0,\,n)}\big[a(j)+b(j+h)\big]. \tag{13}$$

Theorem 5. *Let conditions (A1) and (A2) be satisfied. Then the zero solution of (1) is exponentially stable.*

Proof. From Holder's inequality we get

$$\begin{aligned} &h\sum_{j\in\mathbb{Z}(n+s,\,n-1)} b^2(j+h)[\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)]^2\\ &\ge\Big[\sum_{j\in\mathbb{Z}(n+s,\,n-1)} b(j+h)\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\Big]^2. \end{aligned}\tag{14}$$

Following the ideas of [10] and inequality (14) we obtain

$$\begin{aligned} V(n)&\ge\Big[x(n)+\sum_{s\in\mathbb{Z}(n,\,n+h-1)} b(s)\max_{\xi\in\mathbb{Z}(s-h,\,n)} x(\xi)\Big]^2\\ &\quad+\frac{\delta}{h}\Big(\sum_{j\in\mathbb{Z}(n-h,\,n-1)} b(j+h)\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\Big)^2\\ &=\frac{\delta}{\delta+h}x^2(n)+\bigg(\sqrt{\frac{h}{\delta+h}}x(n)\\ &\quad+\sqrt{\frac{h+\delta}{h}}\sum_{j\in\mathbb{Z}(n-h,\,n-1)} b(j+h)\max_{\xi\in\mathbb{Z}(j,\,n)} x(\xi)\bigg)^2\\ &\ge\frac{\delta}{\delta+h}x^2(n). \end{aligned}\tag{15}$$

From inequalities (11) and (15) we obtain

$$|x(n)|\le\sqrt{\frac{\delta+h}{\delta}V(0)\lambda^n}. \tag{16}$$

Since $\lambda<1$ from (16) it follows the exponential stability.

□

4. APPLICATIONS

Example 6. Consider the IVP (1) with $a = 1.2$, $b = -0.3$, and $h = 2$. Choose $\delta = 0.45$ and $\lambda = 0.95$. Then, the inequalities $-\frac{\delta}{h(1+\delta)} = -0.155172 < Q(n) = 1.2 - 0.3 - 1 = -0.1 \leq -(h\delta b^2(n+h) + Q^2(n)) = -0.091$ hold, i.e. assumption (A1) is satisfied.

Additionally, $a(n)+b(n+h) = 0.9 \leq \lambda = 0.95 < 1$, i.e. assumption (A2) is satisfied. According to Theorem 1, the zero solution of (1) for this special case is exponentially stable, i.e. the inequality

$$|x| \leq \sqrt{5.4444 * V(0)0.95^n}$$

holds with

$$\begin{aligned} V(0) = \Big[\phi(0) - 0.3\Big(&\max\{\phi(n),\ n = -2,-1,0\} \\ &+ \max\{\phi(n),\ n = -1,0\}\Big)\Big]^2 \\ &+ 0.45 * 0.09\Big(\phi^2(-2) + 2\phi^2(-1)\Big). \end{aligned} \tag{17}$$

For example, for $\phi(n) = n,\ n = -2,-1,0$, we get $V(0) = 0.45 * 0.09 * 6 = 0.243$. Therefore, $|x(n)| \leq \sqrt{1.323 * 0.95^n}$ (see Figure 1).

For $\phi(n) = 1,\ n = -2,-1,0$, we get $V(0) = \big[1-0.6\big]^2 + 0.1215 = 0.2815$. Therefore, $|x(n)| \leq \sqrt{1.5326 * 0.95^n}$ (see Figure 2).

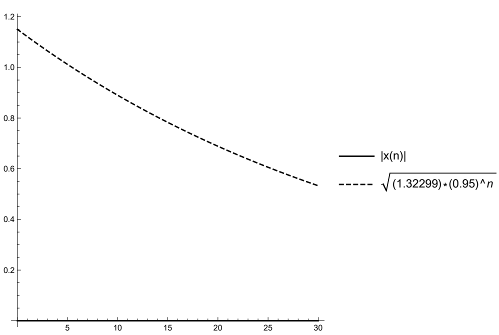

Figure 1. Graphs of $|x(n)|$ and the upper bound for $n \in [0,30]$.

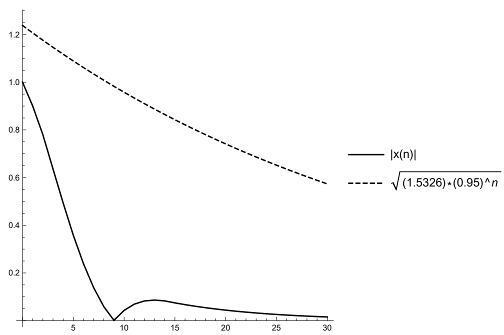

Figure 2. Graphs of $|x(n)|$ and the upper bound for $n \in [0,30]$.

For example, for $\phi(n) = n^2,\ n = -2,-1,0$, we get $V(0) = [-0.3 * 4 + 1]^2 + 0.45 * 0.09 * (4 = 2) = 2.979$. Therefore, $|x(n)| \leq \sqrt{16.2189 * 0.95^n}$ (see Figure 3).

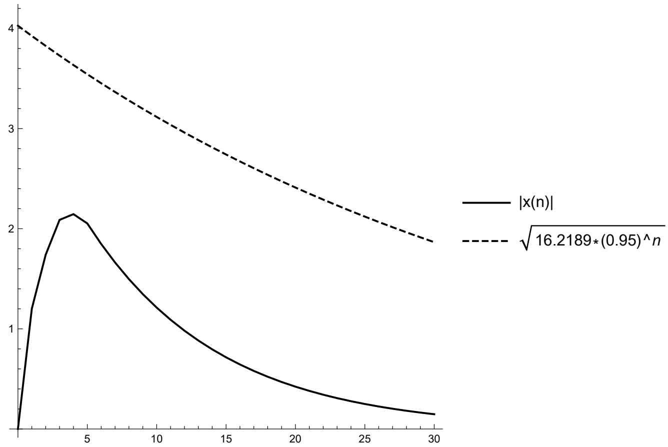

Figure 3. Graphs of $|x(n)|$ and the upper bound for $n \in [0, 30]$.

Example 7. Consider the IVP (1) with $a(n) = 1.6$, $b(n) = -0.3$ and $h = 2$. Then the assumption (A2) is not satisfied because $a(n) + b(n+h) = 1.3 \not< 1$

The zero solution is not exponentially stable. For the partial case of $\phi(n) = 1,\ n = -2, -1, 0$, the solution is graphed on Figure 4.

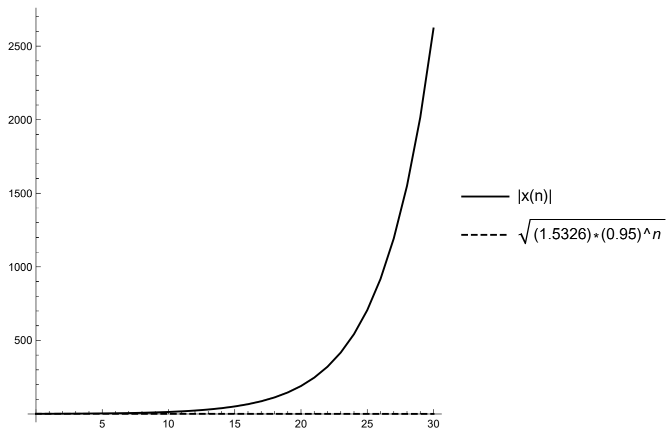

Figure 4. Graphs of $|x(n)|$ and the upper bound for $n \in [0, 30]$.

It proves the assumption (A2) is important for the exponential stability.

Example 8. Consider the IVP (1) with variable coefficients $a(n) = \frac{2}{n+2}$, $b(n) = -\frac{1}{n+1}$ and $h = 2$.

Then assumption (A1) is satisfied with $\lambda = \frac{2}{3}$ (see Figure 5).

Also, the inequality $Q(n) < hb^2(n+h) + \big(a(n) - \frac{1}{n+1+h}\big)^2$ holds (see Figure 6). But there is no $\delta > 0$ such that $-\frac{\delta}{h(1+\delta)} < Q(n)$, so the assumption (A2) is not satisfied.

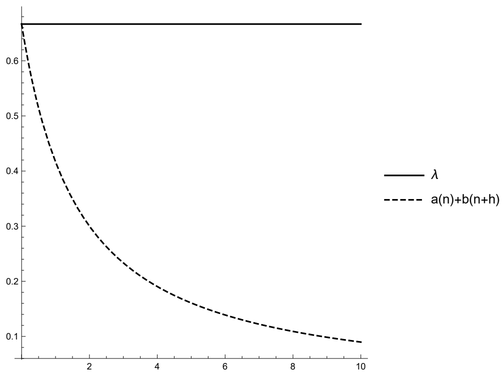

Figure 5. Graphs of λ and $a(n) + b(n + h)$ for $n \in [0, 10]$.

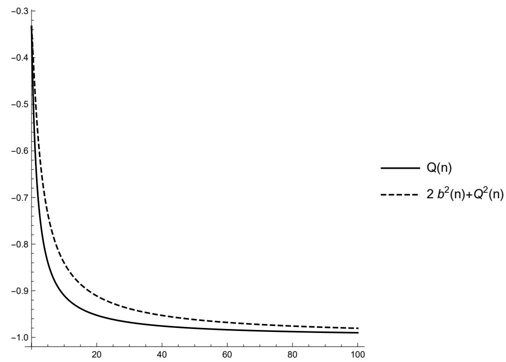

Figure 6. Graphs of $Q(n)$ and $2b^2(n) + Q^2(n)$ for $n \in [0, 100]$.

The graph of the solution for the partial case of $\phi(n) = 1$, $n = -2, -1, 0$, is given on Figure 7. Therefore, the assumption (A1) is important for the stability property.

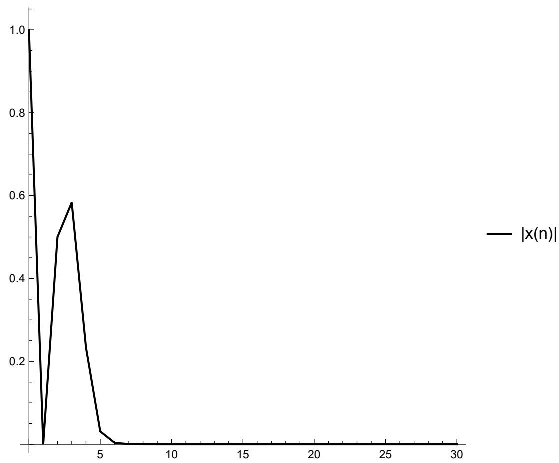

Figure 7. Graph of $|x(n)|$ for $n \in [0, 30]$.

ACKNOWLEDGEMENTS

K.S. was supported by the Science Fund of University of Plovdiv "Paisii Hilendarski" under Grant No. MU21FMI009.

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