CONJUGATE CONNECTIONS AND STATISTICAL STRUCTURES ON ALMOST NORDEN MANIFOLDS

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ABSTRACT: Relations between conjugate connections with respect to the pair of Norden metrics and to the almost complex structure on almost Norden manifolds are studied. Conjugate connections of the Levi-Civita connections induced by the Norden metrics are obtained. Statistical structures on almost Norden manifolds are considered.

1. INTRODUCTION

The concept of conjugate connections relative to a metric tensor field was originally introduced by A. P. Norden in the context of Weyl geometry [22]. Such linear connections were independently developed by H. Nagaoka and S. Amari [18] under the name dual connections and used by S. Lauritzen in the definition of statistical manifolds [15].

Statistical structures and relations between conjugate connections on Hermitian manifolds are considered in [9, 19]. Applications of conjugate connections and statistical structures to affine hypersurface theory have been studied in [8, 13, 14]. For more details on conjugate connections and their application to information theory, statistics

and other fields see also [2, 3, 7, 17, 21, 24].

Another kind of conjugate connections are the ones dual with respect to an invertible (1,1)-tensor field [1, 4]. Conjugate connections relative to an almost complex structure are studied by A. M. Blaga and M. Crasmareanu in [6]. Relations between conjugate connections with respect to a symplectic structure or to a complex structure on Kähler manifolds are investigated in [5].

The main purpose of the present work is to study relations between both aforementioned types of conjugate connections on almost complex manifolds with Norden metric (B-metric). These manifolds were introduced by A. P. Norden [23] and their geometry was studied for the first time by K. Gribachev, D. Mekerov and G. Djelepov [12] who termed them generalized B-manifolds.

Since on such manifolds, there exists a pair of Norden metrics, we can consider conjugate connections with respect to each of these metrical tensors and their relations to conjugate connections relative to the almost complex structure. Another aim of this work is to construct and study statistical structures on almost Norden manifolds.

The paper is organized as follows. In Section 2, we give some basic information about almost Norden manifolds and conjugate connections. In Section 3, we study the coincidence of conjugate connections with respect to the Norden metrics and the almost complex structure. The case of symmetric connections and completely symmetric connections is also investigated. In Section 4, we study curvature properties of the conjugate connections of the Levi-Civita connections induced by the pair of Norden metrics. In Section 5, we consider statistical structures on almost Norden manifolds by constructing families of linear connections with completely symmetric difference tensor and studying their curvature properties.

2. PRELIMINARIES

1.1. ALMOST NORDEN MANIFOLDS

The triple (M, J, g) is called an *almost Norden manifold* (almost complex manifold with Norden metric or B-metric) if M is a differentiable $2n$-dimensional manifold, J is an almost complex structure, and g is a pseudo Riemannian metric compatible with J such that

$$J^2X = -X, \qquad g(JX, JY) = -g(X, Y). \tag{1}$$

Here and further X, Y, Z, W will stand for arbitrary vector fields on M, i.e. elements in the Lie algebra $\mathfrak{X}(M)$, or vectors in the tangent space T_pM at an arbitrary point $p \in M$.

Equalities (1) imply $g(JX, Y) = g(X, JY)$ which means that the tensor $\widetilde{g}$ defined by $\widetilde{g}(X, Y) = g(X, JY)$ is symmetric and is known as the *associated (twin) metric* of g (g and $\widetilde{g}$ are called a pair of twin metrics). This tensor also satisfies the Norden metric property (1), and $(M, J, \widetilde{g})$ is also an almost Norden manifold. Both metrics, g and $\widetilde{g}$, are necessarily of neutral signature (n, n).

Let us denote by ∇^0 and $\widetilde{\nabla}^0$ the Levi-Civita connections of g and $\widetilde{g}$, respectively. The tensor field F defined by

$$F(X, Y, Z) = (\nabla^0_X \widetilde{g})(Y, Z) = g\left((\nabla^0_X J)Y, Z\right) \tag{2}$$

plays an important role in the geometry of almost Norden manifolds. It has the following properties $F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ)$.

Let $\{e_i\}$ $(i = 1, 2, ..., 2n)$ be an arbitrary basis of T_pM, and g^{ij} be the components of the inverse matrix of g with respect to this basis. The Lie 1-form associated with F and its corresponding vector Ω are given by

$$\theta(X) = g^{ij} F(e_i, e_j, X), \qquad \theta(X) = g(X, \Omega). \tag{3}$$

A classification of the almost Norden manifolds with respect to the properties of F is obtained by G. Ganchev and A. Borisov in [11]. This classification consists of eight classes: three basic classes $\mathcal{W}_i$ $(i = 1, 2, 3)$, their pairwise direct sums $\mathcal{W}_i \oplus \mathcal{W}_j$, the widest class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ and the class $\mathcal{W}_0$ of the Kähler Norden manifolds defined by $F = 0$ (i.e. $\nabla^0 J = 0$) which is contained in the intersection of each two classes. The basic classes are distinguished by the following characteristic conditions, respectively

$$\mathcal{W}_1 : F(X, Y, Z) = \frac{1}{2n}\Big\{g(X, Y)\theta(Z) + g(X, JY)\theta(JZ)$$
$$+ g(X, Z)\theta(Y) + g(X, JZ)\theta(JY)\Big\}; \tag{4}$$
$$\mathcal{W}_2 : F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \quad \theta = 0;$$
$$\mathcal{W}_3 : F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0.$$

The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ of the Norden manifolds (complex manifolds with Norden metric) is the widest integrable class (i.e. with a vanishing Nijenhuis tensor) and is characterized also by the condition $F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0$.

Let R^0 be the curvature tensor of ∇^0, i.e. $R^0(X, Y)Z = \nabla^0_X \nabla^0_Y Z - \nabla^0_Y \nabla^0_X Z - \nabla^0_{[X,Y]}Z$. Its corresponding (0,4)-tensor with respect to g is defined by $R^0(X, Y, Z, W) =$

$g(R^0(X,Y)Z,W)$ and has the following properties

$$R^0(X,Y,Z,W) = -R^0(Y,X,Z,W) = -R^0(X,Y,W,Z),$$
$$R^0(X,Y,Z,W) + R^0(Y,Z,X,W) + R^0(Z,X,Y,W) = 0. \tag{5}$$

Any tensor of type (0,4) which satisfies all three conditions in (5) is called a *curvature-like tensor*. Then, the Ricci tensor $\rho(L)$ and the scalar curvature $\tau(L)$ of L are obtained by

$$\rho(L)(X,Y) = g^{ij}L(e_i,X,Y,e_j), \qquad \tau(L) = g^{ij}\rho(L)(e_i,e_j). \tag{6}$$

A curvature tensor L is called a *Kähler tensor* if $L(X,Y)JZ = JL(X,Y)Z$. Then, for the corresponding (0,4)-type tensor with respect to g, i.e. $L(X,Y,Z,W) = g(L(X,Y)Z,W)$ we have $L(X,Y,JZ,JW) = -L(X,Y,Z,W)$.

Let S be a tensor of type (0,2), and denote by $\widetilde{S}(X,Y) = S(X,JY)$. Consider the following (0,4)-tensors:

$$\psi_1(S) = g \owedge S, \qquad \psi_2(S) = \widetilde{g} \owedge \widetilde{S},$$
$$\pi_1 = \tfrac{1}{2}\psi_1(g), \quad \pi_2 = \tfrac{1}{2}\psi_2(g), \quad \pi_3 = -\psi_1(\widetilde{g}) = \psi_2(\widetilde{g}), \tag{7}$$

where $\owedge$ is the Kulkarni-Nomizu product of two (0,2)-tensors, e.g. $(g\owedge S)(X,Y,Z,W) = g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(X,W)S(Y,Z) - g(Y,W)S(X,Z)$. The tensor $\psi_1(S)$ is curvature-like iff S is symmetric, and $\psi_2(S)$ is curvature-like iff S is symmetric and hybrid with respect to J, i.e. $S(X,Y) = S(Y,X) = -S(JX,JY)$.

On a pseudo-Riemannian manifold M ($\dim M = 2n \geq 4$) the Weyl tensor of a curvature-like tensor L is given by

$$W(L) = L - \frac{1}{2(n-1)}\left\{\psi_1(\rho(L)) - \frac{\tau(L)}{2n-1}\pi_1\right\}.$$

The square norm of $\nabla^0 J$ is defined by

$$||\nabla^0 J||^2 = g^{ij}g^{kl}g\big((\nabla^0_{e_i}J)e_k, (\nabla^0_{e_j}J)e_l\big).$$

An almost Norden manifold is called *isotropic Kählerian* if

$$||\nabla^0 J||^2 = 0.$$

1.2. CONJUGATE CONNECTIONS WITH RESPECT TO A METRIC TENSOR AND STATISTICAL MANIFOLDS

Let (M, g) be a pseudo Riemannian manifold, and ∇ be an arbitrary linear connection on M. Then the linear connection ∇^* defined by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z), \tag{8}$$

is called the *conjugate (dual) connection of* ∇ *with respect to g*. From (8) it is easy to see that $(\nabla^*)^* = \nabla$. Hence, ∇ and ∇^* are said to be mutually conjugate. Also, from (8) it follows that a connection ∇ is self-conjugate, i.e. $\nabla = \nabla^*$ if and only if it is a *metric (g-compatible) connection*, i.e. $\nabla g = 0$. The average connection $\overline{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$ of two mutually conjugate connections is a metric connection.

Let ∇ be a torsion free (symmetric) connection. Then, it is known that its conjugate connection ∇^* is also torsion free if and only if the tensor ∇g is completely symmetric, i.e.

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z). \tag{9}$$

Then the same is valid for $\nabla^* g$, i.e. (∇, g) and (∇^*, g) are both Codazzi pairs. Also, in this case the average connection of ∇ and ∇^* is the Levi-Civita connection of g. The triple (M, g, ∇) is called a *statistical manifold* if ∇ is torsion free and ∇g is completely symmetric. Equivalently, a statistical manifold is a pseudo Riemannian manifold (M, g) equipped with a pair of symmetric conjugate connections. Then, (g, ∇, ∇^*) is called a *statistical structure* on M. An almost Norden manifold (M, J, g) equipped with a statistical structure (g, ∇, ∇^*) will be called *a statistical almost Norden manifold*.

1.3. CONJUGATE CONNECTIONS WITH RESPECT TO AN ALMOST COMPLEX STRUCTURE

Let (M, g) be a pseudo Riemannian manifold, and J be an almost complex structure on M. If ∇ is an arbitrary linear connection then the connection ∇^* defined by

$$\nabla_X^* Y = -J\nabla_X JY = \nabla_X Y - J(\nabla_X J)Y \tag{10}$$

is called the *complex conjugate connection* of ∇ [1, 6]. From (10) it follows that $(\nabla^*)^* = \nabla$, i.e. ∇ and ∇^* are mutually conjugate relative to J. A connection ∇ is self-conjugate with respect to J if and only if it is an *almost complex connection* (*J-compatible connection*), i.e. $\nabla J = 0$. The average connection $\overline{\nabla} = \frac{1}{2}(\nabla + \nabla^*) = \nabla - \frac{1}{2}J\nabla J$ of two complex conjugate connections is J-compatible [1].

By the same manner as in [6], we prove that if g is a Norden metric then

$$(\nabla_X^* g)(JY, JZ) = -(\nabla_X g)(Y, Z).$$

Thus, $\nabla^* g = 0$ iff $\nabla g = 0$.

3. RELATIONS BETWEEN CONJUGATE CONNECTIONS ON ALMOST NORDEN MANIFOLDS

Let (M, J, g) be an almost Norden manifold. In this section, we study relations between the aforementioned types of conjugate connections on M. First, we study the coincidence of conjugate connections with respect the pair of Norden metrics. Let us remark that if ∇ and ∇^* are conjugate with respect to a Norden metric tensor g, then by (8) it follows that $g((\nabla_X J)Y, Z) = g((\nabla_X^* J)Z, Y)$. Hence, in this case $\nabla J = 0$ iff $\nabla^* J = 0$.

Proposition 1. *Let ∇ and ∇^* be linear connections on an almost Norden manifold (M, J, g). Then, each two of the following conditions imply the third one:*

(i) ∇ and ∇^* are conjugate relative to g;

(ii) ∇ and ∇^* are conjugate relative to $\widetilde{g}$;

(iii) $\nabla J = 0$ ($\nabla^* J = 0$).

Proof. Let us prove that conditions (i) and (ii) imply (iii). First, we take into account that ∇ and ∇^* are conjugate with respect to g and substitute $Y \to JY$ in (8). Hence, by covariant differentiation and the definition of $\widetilde{g}$, we obtain $X\widetilde{g}(Y, Z) = \widetilde{g}(\nabla_X Y, Z) + \widetilde{g}(Y, \nabla_X^* Z) + g((\nabla_X J)Y, Z)$. Then, keeping in mind that ∇ and ∇^* are also conjugate with respect to $\widetilde{g}$, the last equality implies $\nabla J = 0$. The truthfulness of the other two statements is proved analogously.

□

Proposition 1 yields the following

Corollary 2. *Let $(M, J, g, \nabla, \nabla^*)$ be a statistical almost Norden manifold. Then, $(M, J, \widetilde{g}, \nabla, \nabla^*)$ is also a statistical almost Norden manifold if and only if $\nabla J = 0$ ($\nabla^* J = 0$).*

Let us remark that if $(M, J, g, \nabla, \nabla^*)$ and $(M, J, \widetilde{g}, \nabla, \nabla^*)$ are simultaneously statistical manifolds, the Levi-Civita connections ∇^0 and $\widetilde{\nabla}^0$ of g and $\widetilde{g}$, respectively, coincide

with the average connection of ∇ and ∇^* and hence $\nabla^0 J = \widetilde{\nabla}^0 J = 0$. The last implies that (M, J, g) and $(M, J, \widetilde{g})$ are both Kähler Norden manifolds.

Next, we study the coincidence of conjugate connections relative to a Norden metric and to an almost complex structure. In this regard, we prove the following

Proposition 3. *Let (M, J, g) be an almost Norden manifold, and ∇ be a linear connection on M. Then: (i) the conjugate connections of ∇ relative to g and to J coincide if and only if $\nabla \widetilde{g} = 0$; (ii) the conjugate connections of ∇ relative to $\widetilde{g}$ and to J coincide if and only if $\nabla g = 0$.*

Proof. Let us prove (i) (the other statement is proved analogously). The conjugate connections of ∇ relative to g and to J coincide if and only if the connection ∇^* defined by (10) satisfies condition (8). Keeping in mind the properties of g and $\widetilde{g}$, the last condition is equivalent to $Xg(Y, Z) = g(\nabla_X Y, Z) - g(JY, \nabla_X JZ)$. Then, by substituting $Z \to JZ$ in the last equality, we obtain $X\widetilde{g}(Y, Z) = \widetilde{g}(\nabla_X Y, Z) + \widetilde{g}(Y, \nabla_X Z)$, i.e. $\nabla \widetilde{g} = 0$ which completes the proof. $\qquad\square$

It is well-known that the unique linear connection which is symmetric and metric with respect to a given metric tensor is the Levi-Civita connection induced by this metric tensor. In light of the last fact, Proposition 3 yields

Corollary 4. *Let (M, J, g) be an almost Norden manifold, and ∇ be a symmetric connection on M. Then: (i) the conjugate connections of ∇ relative to g and J coincide if and only if ∇ is the Levi-Civita connection $\widetilde{\nabla}^0$ of $\widetilde{g}$; (ii) the conjugate connections of ∇ relative to $\widetilde{g}$ and J coincide if and only if ∇ is the Levi-Civita connection ∇^0 of g.*

Thus, the conjugate connection of ∇^0 (resp. of $\widetilde{\nabla}^0$) relative to $\widetilde{g}$ (resp., relative to g) is its complex conjugate connection defined by (10).

The case of a completely symmetric connection ∇ (see Section 5) is considered in the following

Corollary 5. *Let (M, J, g) be an almost Norden manifold, and let ∇ and ∇^* be linear connections on M. Then:*

(i) *If $(M, J, g, \nabla, \nabla^*)$ is a statistical manifold, and ∇^* is the conjugate connection of ∇ relative to J then (M, J, g) is a Kähler manifold;*

(ii) *If $(M, J, \widetilde{g}, \nabla, \nabla^*)$ is a statistical manifold, and ∇^* is the conjugate connection of ∇ relative to J then (M, J, g) is a Kähler manifold.*

Proof. (i) Since $(M, J, g, \nabla, \nabla^*)$ is a statistical manifold, the average connection of ∇ and ∇^* is the Levi-Civita connection ∇^0 of g. But because it is also the average connection of two complex conjugate connections, ∇^0 should be an almost complex connection, i.e. $\nabla^0 J = 0$. Hence, (M, J, g) is a Kähler manifold.

(ii) By a similar manner, we deduce that $(M, J, \widetilde{g})$ is a Kähler Norden manifold, i.e. $\widetilde{\nabla}^0 J = \widetilde{\nabla}^0 \widetilde{g} = 0$ which implies $\widetilde{\nabla}^0 g = 0$. Because $\widetilde{\nabla}^0$ is symmetric, the last equality yields $\widetilde{\nabla}^0 = \nabla^0$ and hence (M, J, g) is also Kählerian. $\qquad\qquad\square$

Based on the results in this section, we conclude that a pair of linear connections ∇ and ∇^* is conjugate with respect to all three basic structural tensor g, $\widetilde{g}$ and J simultaneously iff $\nabla g = \nabla \widetilde{g} = \nabla J = 0$ (which implies $\nabla^* = \nabla$). Linear connections preserving the structural tensors of the manifold by covariant differentiation are called *natural(adapted)*. Hence, ∇ is such a connection.

4. CONJUGATE CONNECTIONS OF THE LEVI-CIVITA CONNECTIONS INDUCED BY THE PAIR OF NORDEN METRICS

As seen in the previous section (Corollary 4), the Levi-Civita connections induced by the Norden metrics are the unique symmetric linear connections on an almost Norden manifold for which the conjugate connections relative to the associated metric tensor and the almost complex structure coincide. In this section, we study curvature properties of these connections.

Let us consider the conjugate connection ∇^* of the Levi-Civita connection ∇^0 with respect to $\widetilde{g}$ and J, i.e. $\nabla_X^* Y = \nabla_X^0 Y - J(\nabla_X^0 J)Y$. We remark that ∇^* is a metric connection, i.e. $\nabla^* g = 0$. If we denote by R^0 and R^* the corresponding curvature tensors of ∇^0 and ∇^*, resp., then according to [6], we have $JR^*(X,Y)Z = R^0(X,Y)JZ$. Hence, the average curvature tensor P of R^0 and R^* defined by

$$P(X,Y)Z = \frac{1}{2}\left\{R^0(X,Y)Z + R^*(X,Y)Z\right\},$$

satisfies the property $P(X,Y)JZ = JP(X,Y)Z$, yielding that P is a Kähler curvature tensor of type (1,3). For (0,4)-type tensors we have

$$g(P(X,Y)Z,W) = \frac{1}{2}\{R^0(X,Y,Z,W) - R^0(X,Y,JZ,JW)\}. \qquad (11)$$

Next, let us focus on the average connection of ∇^0 and ∇^* which we denote by D, i.e. $D_X Y = \nabla_X^0 Y - \frac{1}{2}J(\nabla_X^0 J)Y$. Since ∇^0 and ∇^* are conjugate relative to $\widetilde{g}$ and

J simultaneously, their average connection is metric with respect to $\widetilde{g}$ and almost complex, i.e. satisfies the conditions $D\widetilde{g} = DJ = 0$ and hence $Dg = 0$, i.e. D is a natural connection. Moreover, it is the well-known Lichnerowicz first canonical connection [16]. In [26], we have obtained the form of the curvature tensor K of D on an almost Norden manifold as follows

$$g\big(K(X,Y)Z,W\big) = \frac{1}{2}\Big\{R^0(X,Y,Z,W) - R^0(X,Y,JZ,JW)\Big\}$$
$$+ \frac{1}{4}\Big\{g\big((\nabla_X^0 J)Z,(\nabla_Y^0 J)W\big)$$
$$- g\big((\nabla_X^0 J)W,(\nabla_Y^0 J)Z\big)\Big\}.$$

Then, the last equality and (11) yield:

Proposition 6. *On an almost Norden manifold, the average curvature tensor P of the conjugate connections ∇^0 and ∇^* (with resp. to $\widetilde{g}$ and J) and the curvature tensor K of their average connection D are related as follows*

$$g(K(X,Y)Z,W) = g(P(X,Y)Z,W)$$
$$+ \frac{1}{4}\Big\{g\big((\nabla_X^0 J)Z,(\nabla_Y^0 J)W\big) \tag{12}$$
$$- g\big((\nabla_X^0 J)W,(\nabla_Y^0 J)Z\big)\Big\}.$$

In [26], we have shown that

$$||\nabla^0 J||^2 = 2g^{il}g^{jk}g\big((\nabla_{e_i}^0 J)e_k,(\nabla_{e_j}^0 J)e_l\big)$$

on an almost Norden manifold in the class $\mathcal{W}_1 \oplus \mathcal{W}_2$, i.e. a Norden manifold. Then, if by $\tau(K)$ and $\tau(P)$ we denote the scalar curvatures of K and P, respectively, from (3) and (12), on a Norden manifold we have

$$\tau(K) = \tau(P) + \tfrac{1}{8}\big(||\nabla^0 J||^2 - 2\,\theta(\Omega)\big). \tag{13}$$

In [25], we have proved that on a manifold in the class $\mathcal{W}_1$ the relation $\theta(\Omega) = \frac{n}{2}||\nabla^0 J||^2$ is valid. Then, by (4) and (13) we get

Corollary 7. *On a Norden manifold (M,J,g) belonging to the class $\mathcal{W}_1$ (dim $M = 2n \geq 4$) or to $\mathcal{W}_2$ is isotropic Kählerian iff $\tau(K) = \tau(P)$.*

Analogous results are valid for the Levi-Civita connection $\widetilde{\nabla}^0$ of $\widetilde{g}$ and its conjugate connection $\widetilde{\nabla}^*$ relative to g and J simultaneously.

Next, using the characteristic condition (4) of the class $\mathcal{W}_1$, the form (7) of the tensors ψ_1 and ψ_2, and by straightforward calculations, we obtain

Proposition 8. *Let* (M, J, g) *be a* $\mathcal{W}_1$-*manifold. Then, the curvature tensors* R^* *and* $\widetilde{R}^*$ *of* ∇^* *and* $\widetilde{\nabla}^*$, *respectively, have the form:*

$$R^* = R^0 - \frac{1}{2n}[\psi_1 + \psi_2](S) - \frac{\theta(\Omega)}{4n^2}[\pi_1 + \pi_2],$$

$$\widetilde{R}^* = \widetilde{R}^0 - \frac{1}{2n}[\psi_1 + \psi_2](\widehat{S}) - \frac{\theta(J\Omega)}{4n^2}[\pi_1 + \pi_2],$$

where $\widetilde{R}^0$ *is the curvature tensor of* $\widetilde{\nabla}^0$, $S(X, Y) = (\nabla^0_X \theta)JY + \frac{1}{2n}\theta(X)\theta(Y)$ *and* $\widehat{S}(X, Y) = -S(X, JY)$.

We remark that R^* and $\widetilde{R}^*$ are not (0,4)-type curvature-like tensors.

5. STATISTICAL STRUCTURES ON ALMOST NORDEN MANIFOLDS

In this section, we consider statistical structures on almost Norden manifolds by constructing and studying families of completely symmetric linear connections. Let ∇ be a symmetric linear connection, and $Q(X, Y)$ be its difference tensor with respect to the Levi-Civita connection ∇^0 of g, i.e.

$$\nabla_X Y = \nabla^0_X Y + Q(X, Y), \tag{14}$$

where $Q(X, Y) = Q(Y, X)$. Denote by $Q(X, Y, Z) = g(Q(X, Y), Z)$.

Then, by covariant differentiation we obtain

$$(\nabla_X g)(Y, Z) = -Q(X, Y, Z) - Q(X, Z, Y).$$

If (g, ∇, ∇^*) is a statistical structure, the last equality and (9) imply that the tensor $Q(X, Y, Z)$ is completely symmetric, i.e.

$$Q(X, Y, Z) = Q(Y, X, Z) = Q(X, Z, Y) \quad \text{and} \quad \nabla g = -2Q.$$

In this case, the connection ∇ is called *completely symmetric*.

By (8) and (14) we have

$$\nabla^*_X Y = \nabla^0_X Y - Q(X, Y). \tag{15}$$

Let us remark that in the theory of statistical manifolds, the (0,3)-type tensor $C(X, Y, Z) = g(\nabla^*_X Y - \nabla_X Y, Z) = (\nabla_X g)(Y, Z)$, which differs from Q only by a factor, is called the *cubic form* (*skewness tensor*) of the manifold.

It is known that equality (14) and $\nabla^0 g = 0$ imply the following relation between the curvature tensors R and R^0 of ∇ and ∇^0, respectively

$$
\begin{aligned}
g(R(X,Y)Z,W) = & R^0(X,Y,Z,W) + (\nabla^0_X Q)(Y,Z,W) \\
& - (\nabla^0_Y Q)(X,Z,W) + Q(X,Q(Y,Z),W) \\
& - Q(Y,Q(X,Z),W).
\end{aligned}
\tag{16}
$$

Analogously, (15) yields

$$
\begin{aligned}
g(R^*(X,Y)Z,W) = & R^0(X,Y,Z,W) - (\nabla^0_X Q)(Y,Z,W) \\
& + (\nabla^0_Y Q)(X,Z,W) \\
& + Q(X,Q(Y,Z),W) \\
& - Q(Y,Q(X,Z),W),
\end{aligned}
\tag{17}
$$

where R^* is the curvature tensor of ∇^*.

Because the tensor Q is completely symmetric, we have

$$
Q(X,Q(Y,Z),W) = g(Q(X,W),Q(Y,Z)).
\tag{18}
$$

Let us denote

$$
\begin{aligned}
L(X,Y,Z,W) = & g(Q(X,W),Q(Y,Z)) \\
& - g(Q(X,Z),Q(Y,W)).
\end{aligned}
\tag{19}
$$

The tensor L satisfies properties (5), i.e. L is a curvature-like tensor.

By adding (16) and (17) we obtain

$$
\begin{aligned}
& R^0(X,Y,Z,W) + L(X,Y,Z,W) \\
& = \frac{1}{2}\left\{ g(R(X,Y)Z,W) + g(R^*(X,Y)Z,W) \right\}.
\end{aligned}
\tag{20}
$$

Let P be the average curvature tensor of R and R^* of type (1,3), i.e.

$$
P(X,Y)Z = \tfrac{1}{2}\left\{ R(X,Y)Z + R^*(X,Y)Z \right\}.
\tag{21}
$$

Then, because of (20), the corresponding (0,4)-type tensor of P, i.e. $P(X,Y,Z,W) = g(P(X,Y)Z,W)$ is curvature-like. The tensor P is known as the statistical curvature tensor of ∇ and ∇^* [10]. Thus, we verified the following

Proposition 9. *On a statistical manifold, the statistical curvature tensor P and the curvature tensor R^0 of the Levi-Civita connection of g are related by*

$$
P = R^0 + L.
\tag{22}
$$

If we consider P as the curvature tensor jointly generated by ∇ and ∇^* then in the next statement we give a necessary and sufficient condition for the Weyl tensor to be invariant under the transformation of the Levi-Civita connection ∇^0 into the pair of symmetric conjugate connections (∇, ∇^*).

Corollary 10. *On a statistical manifold, the Weyl tensors of P and R^0 coincide iff $W(L) = 0$ where L is given by (19).*

Let (M, J, g) be an almost Norden manifold, and (g, ∇, ∇^*) be a statistical structure on M. If we ask for this structure to be compatible with J, i.e. $\nabla J = 0$ (which implies $\nabla^* J = 0$) we immediately obtain $\nabla^0 J = 0$. Hence, almost complex completely symmetric connections exist only on Kähler manifolds. Thus, in order to study wider classes of statistical almost Norden manifolds we will not aim for J-compatibility.

4.1. COMPLETELY SYMMETRIC CONNECTIONS CONSTRUCTED BY THE METRICS AND THE LIE 1-FORMS

According to (4), an almost Norden manifold which is not in the class $\mathcal{W}_2 \oplus \mathcal{W}_3$ has non-vanishing Lie 1-forms θ and $\widetilde{\theta} = \theta \circ J$. Thus, on such manifolds, the pairs of Lie 1-forms and Norden metrics can be used to construct difference tensors of completely symmetric linear connections and thus statistical structures. One such family of connections is introduced in the next

Proposition 11. *On an almost Norden manifold $(M, J, g) \notin \mathcal{W}_2 \oplus \mathcal{W}_3$, there exists a four-parametric family of completely symmetric connections ∇ defined by (14) with difference tensor Q given by*

$$
\begin{aligned}
Q(X, Y) = &\lambda_1[\theta(X)Y + \theta(Y)X + g(X, Y)\Omega] \\
&+ \lambda_2[\theta(JX)Y + \theta(JY)X + g(X, Y)J\Omega] \\
&+ \lambda_3[\theta(X)JY + \theta(Y)JX + g(X, JY)\Omega] \\
&+ \lambda_4[\theta(JX)JY + \theta(JY)JX + g(X, JY)J\Omega],
\end{aligned}
\tag{23}
$$

$\lambda_i \in \mathbb{R}$ *($i = 1, 2, 3, 4$).*

By (7), (19), (23) and straightforward calculations we obtain

Proposition 12. *Let $(M, J, g, \nabla, \nabla^*)$ be the statistical almost Norden manifold with ∇ defined by (14) and (23). Then, the statistical curvature tensor P of the*

manifold has the form (22) where

$$L = \psi_1(S_1) + \psi_2(S_2) + [(\lambda_1^2 - \lambda_2^2)\theta(\Omega) + 2\lambda_1\lambda_2\theta(J\Omega)]\pi_1$$
$$+ [(\lambda_3^2 - \lambda_4^2)\theta(\Omega) + 2\lambda_3\lambda_4\theta(J\Omega)]\pi_2 \tag{24}$$
$$- [(\lambda_1\lambda_3 - \lambda_2\lambda_4)\theta(\Omega) + (\lambda_1\lambda_4 + \lambda_2\lambda_3)\theta(J\Omega)]\pi_3,$$

and

$$S_1(X,Y) = (\lambda_1^2 + \lambda_3^2 - 2\lambda_2\lambda_3)\theta(X)\theta(Y)$$
$$+ (\lambda_2^2 + \lambda_4^2 + 2\lambda_1\lambda_4)\theta(JX)\theta(JY)$$
$$+ (\lambda_1(\lambda_2 + \lambda_3)$$
$$+ \lambda_4(\lambda_3 - \lambda_2))[\theta(X)\theta(Y) + \theta(JX)\theta(JY)],$$
$$S_2(X,Y) = (\lambda_3^2 - \lambda_4^2)[\theta(X)\theta(Y) - \theta(JX)\theta(JY)]$$
$$- 2\lambda_3\lambda_4[\theta(X)\theta(Y) + \theta(JX)\theta(JY)].$$

Since for the Weyl tensor of $\psi_1(S)$ it is valid $W(\psi_1(S)) = 0$, by Corollary 10, equalities (7), (2) and (24) we get the following

Proposition 13. *Let ∇ be the family of linear connections defined by (14) and (23) with the condition $\lambda_3 = \lambda_4 = 0$. Then, the Weyl tensors of P and R^0 coincide.*

4.2. COMPLETELY SYMMETRIC CONNECTIONS CONSTRUCTED BY THE LIE 1-FORMS

A family of completely symmetric linear connections with difference tensor depending only on the Lie 1-forms θ and $\widetilde{\theta} = \theta \circ J$ is presented in the following

Proposition 14. *On an almost Norden manifold $(M, J, g) \notin \mathcal{W}_2 \oplus \mathcal{W}_3$, there exists a four-parametric family of completely symmetric connections ∇ defined by (14) with difference tensor Q given by*

$$Q(X,Y) = \lambda_1\theta(X)\theta(Y)\Omega + \lambda_2\theta(JX)\theta(JY)J\Omega$$
$$+ \lambda_3[\theta(X)\theta(Y)J\Omega + \theta(X)\theta(JY)\Omega$$
$$+ \theta(JX)\theta(Y)\Omega] \tag{25}$$
$$+ \lambda_4[\theta(JX)\theta(Y)J\Omega + \theta(JX)\theta(JY)\Omega$$
$$+ \theta(X)\theta(JY)J\Omega],$$

$\lambda_i \in \mathbb{R}$ *($i = 1, 2, 3, 4$).*

By (19), (25) and straightforward calculations we obtain

Proposition 15. *Let $(M, J, g, \nabla, \nabla^*)$ be the statistical almost Norden manifold with ∇ defined by (14) and (25). Then, the statistical curvature tensor P of the manifold has the form (22) where*

$$
\begin{aligned}
L(X, Y, Z, W) = \alpha \Big[\theta(X)\theta(JY) - \theta(JX)\theta(Y) \Big] \times \\
\times \Big[\theta(Z)\theta(JW) - \theta(JZ)\theta(W) \Big],
\end{aligned}
$$

where $\alpha = [\lambda_3^2 - \lambda_4^2 - \lambda_1\lambda_4 + \lambda_2\lambda_3]\theta(\Omega) - (\lambda_1\lambda_2 + \lambda_3\lambda_4)\theta(J\Omega)$.

A direct consequence of the last statement and (22) is that on manifolds with isotropic Lie vector field Ω with respect to both g and $\widetilde{g}$, i.e. satisfying $\theta(\Omega) = \theta(J\Omega) = 0$, we obtain $L = 0$, and thus the statistical curvature tensor P of the statistical structure defined by (14) and (25) coincides with the curvature tensor R^0 of the Levi-Civita connection ∇^0.

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