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NEW PROPERTIES OF THE ODD WEIBULL INVERSE TOPP-LEONE CUMULATIVE DISTRIBUTION FUNCTION

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ABSTRACT: In 2021 Almetwally introduced a new lifetime distribution named the odd Weibull inverted Topp-Leone (OWITL) distribution. In this note we study one of the important characteristics "saturation" of this new cumulative function to the horizontal asymptote with respect to Hausdorff metric as we prove some estimates. In additional we consider a new adaptive model with "polynomial variable transfer". The applicability of the model is proved in simulation study to "COVID-19 data". Some numerical examples and software modules within the programming environment *CAS MATHEMATICA* are presented.

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Key Words: odd Weibull inverted Topp-Leone, adaptive odd Weibull inverted Topp-Leone model with polynomial variable transfer, Hausdorff distance, Heaviside step function, Upper and lower bounds

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1. INTRODUCTION

A new lifetime distribution with a three-parameter named the odd Weibull inverted Topp-Leone (OWITL) distribution was introduced from Almetwally [1]. This new distribution is a combination of well-known inverted Topp-Leone distribution and the odd Weibull family. The main goal of this new distribution is to define an optimal statistical model that can modeling the COVID-19 data.

Definition 1. The odd Weibull inverted Topp-Leone (OWITL) distribution is associated with the cdf given as

$$F(t;\alpha,\lambda,\delta) = 1 - e^{-\lambda[(1+t)^{2\delta}(1+2t)^{-\delta}-1]^{\alpha}}, \quad t > 0,\ \alpha,\ \lambda,\ \delta > 0. \tag{1}$$

In the case $\lambda = 1$ OWITL distribution consist with two parameter modified Kies inverted Topp-Leone (MKITL) distribution. In [2] Almetwally, Alharbi, Alnagar, and Hafez introduced it as a combination of inverted Topp-Leone distribution and modified Kies family.

This paper deals with asymptotic behavior of some adaptive functions of the Hausdorff distance between Heaviside function and some novel distribution functions. The investigations on the "supersaturation" gives the opportunity to the researcher for choice of appropriate model when approximating cumulative specific data in areas of Biostatistics, Population dynamics, Growth theory, Debugging and Test theory, Computer viruses propagation, Financial and Insurance mathematics. For some modeling and approximation problems see related articles and monographs [3]–[17] and references therein.

Definition 2. The shifted Heaviside step function is defined by

$$h_{t_0}(t) = \begin{cases} 0 & \text{if } t < t_0, \\ [0,\,1] & \text{if } t = t_0, \\ 1 & \text{if } t > t_0. \end{cases}$$

Definition 3. [18, 19] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions $f,\ g$ on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \tag{2}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e.g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is

$$||A-B|| = \max(|t_A - t_B|, |x_A - x_B|).$$

In the next lemma we present one technical result.

Lemma 4. The following inequality holds

$$F_0(t;\alpha,\lambda,\delta) \le F(t;\alpha,\lambda,\delta) \le F_{00}(t;\alpha,\lambda,\delta),$$

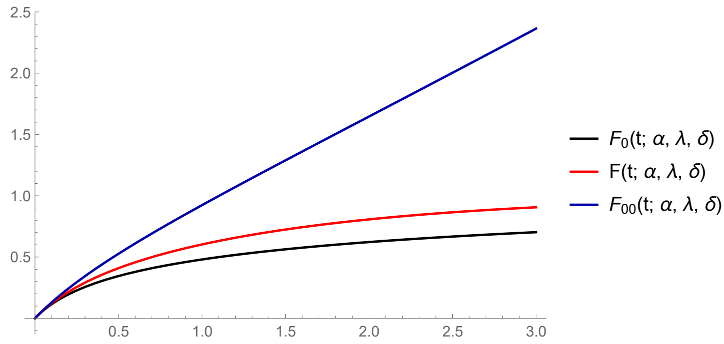

Figure 1: Confidential bounds for CDF function of OWITL distribution.

where

$$F_0(t;\alpha,\lambda,\delta) = \frac{\lambda\left((t+1)^{2\delta}(2t+1)^{-\delta}-1\right)^{\alpha}}{\lambda\left((t+1)^{2\delta}(2t+1)^{-\delta}-1\right)^{\alpha}+1} \quad \text{and}$$
$$F_{00}(t;\alpha,\lambda,\delta) = \lambda\left((t+1)^{2\delta}(2t+1)^{-\delta}-1\right)^{\alpha}.$$

Proof. The proof follows immediately from the following well know inequalities

$$\frac{x}{x+1} \leq 1-e^{-x} \leq x$$

that holds true for every $x > -1$.

□

Figure 1 is a graphical representation of Lemma 4. Note that functions F_0 and F_{00} can be used as “confidential bounds” for CDF function of OWITL distribution.

2. MAIN RESULT

In this Section we study the Hausdorff approximation of the Heaviside step function $h_{t_0}(t)$ where t_0 is the “median” by (1). The investigation of the characteristic “super-saturation” of the cdf (1) to the horizontal asymptote is important.

The quantile function is defined by [1]:

$$Q(u) = \left(\left(1-\left(1+\left(\frac{-\log(1-u)}{\lambda}\right)^{1/\alpha}\right)^{-1/\delta}\right)^{-0.5}-1\right)^{-1}. \qquad (3)$$

In Figure 2 we present quantile function for different values of parameters.

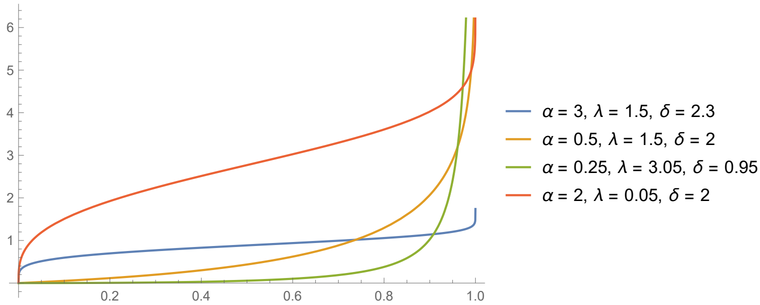

Figure 2: Quantile function of OWITL distribution.

The median is obtained by substituting $u = 0.5$ in (3). Let t_0 is the value for which $F(t_0, \alpha, \lambda, \delta) = \frac{1}{2}$, i.e. $t_0 = Q(0.5)$. Then the Hausdorff distance d between $F(t, \alpha, \lambda, \delta)$ defined by (1) and the Heaviside function $h_{t_0}(t)$ satisfies the following nonlinear equation

$$F(t_0 + d, \alpha, \lambda, \delta) = 1 - d \tag{4}$$

or

$$e^{-\lambda[(1+t)^{2\delta}(1+2t)^{-\delta}-1]^{\alpha}} = d.$$

In the next theorem we prove upper and lower estimates for the Hausdorff approximation d.

Theorem 5. *Let*

$$A = 1 + \frac{1}{x+1}\left(\alpha\delta\lambda\left(\frac{x}{x-1}\right)^{2\delta-1}\left(\frac{x+1}{x-1}\right)^{-\delta}\times \right. \\ \left. \times\left(\left(\frac{x}{x-1}\right)^{2\delta}\left(\frac{x+1}{x-1}\right)^{-\delta}-1\right)^{\alpha-1}\right) \tag{5}$$

$$\textit{with} \quad x = \left(1 - \left(\left(\frac{1}{\lambda}\right)^{1/\alpha}\sqrt[\alpha]{\log(2)} + 1\right)^{-1/\delta}\right)^{-0.5}$$

and $2.1A > e^{1.05}$. Then for the Hausdorff distance d between shifted Heaviside function $h_{t_0}(t)$ and the OWITL CDF function $F(t; \alpha, \lambda, \delta)$ defined by (1) the following inequalities hold true:

$$d_l = \frac{1}{2.1A} < d < \frac{\ln(2.1A)}{2.1A} = d_r.$$

Proof. Let us consider the function

$$H(d) = F(t_0 + d, \alpha, \lambda, \delta) - 1 + d \\ = d - e^{-\lambda[(1+t_0+d)^{2\delta}(1+2(t_0+d))^{-\delta}-1]^{\alpha}}.$$

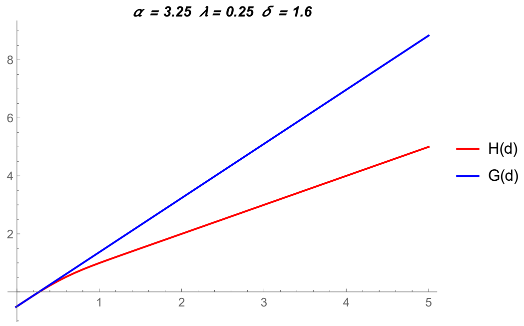

Figure 3: Functions $H(d)$ and $G(d)$.

It is easy to show that $H'(d) > 0$, so the function $H(d)$ is increasing. We examine the following approximation of $H(d)$ as we use the function

$$G(d) = -\frac{1}{2} + Ad.$$

Indeed from Taylor expansion, we get $G(d) - H(d) = \mathcal{O}(d^2)$. This means that $G(d)$ approximates $H(d)$ with $d \to 0$ as $\mathcal{O}(d^2)$ (see Figure 3). More over $G'(d) > 0$ and function $G(d)$ is also increasing. Let the following condition $2.1A > e^{1.05}$ holds. Then it is easy to show that

$$G(d_l) = -\frac{1}{2} + A\frac{1}{2.1A} < 0 \quad \text{and}$$
$$G(d_r) = -\frac{1}{2} + A\frac{\ln(2.1A)}{2.1A} > -\frac{1}{2} + \frac{1.05}{2.1} = 0.$$

This completes the proof.

□

In Table 1 we present some computational examples for different values of parameters α, λ and δ. We use Theorem 5 for computation of values of upper and lower estimates d_l and d_r. Several graphical representations are presented in Figure 4 and one can see that the "saturation" is faster.

3. SOME APPLICATIONS

Almetwally [1] analyzed data that represent a COVID-19 data belong to Canada of 56 days. In Figure 5 we present the results that we obtain with a software module (intellectual property) within the programming environment CAS Mathematica for the

Table 1: Bounds for Hausdorff distance d computed by Theorem 5.

α	λ	δ	d_l	d computed by (4)	d_r
3.64	0.45	6.56	0.0768882	0.0889537	0.197249
5.39	2.95	3.12	0.0847971	0.0970375	0.209236
1.24	0.95	9.12	0.129516	0.150581	0.264725
3	1.5	2.3	0.166549	0.183411	0.298533
0.37	4.05	1.12	0.136463	0.221791	0.271793
0.25	3.05	0.95	0.120697	0.235087	0.255211

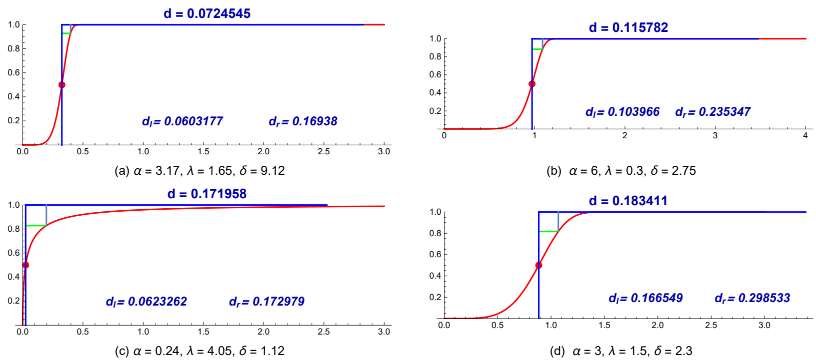

Figure 4: Approximation of CDF function of OWITL distribution.

analysis of the considered OWITL cumulative distribution function. We present an approximation the Heaviside step function and $F(t;\alpha,\lambda,\delta)$ with parameters $\alpha = 2.51$, $\lambda = 2.2$, and $\delta = 13.136$. Namely, we obtain the values of Hausdorff distance $d = 0.0676321$, its upper and lower estimates $d_l = 0.054953$ and $d_r = 0.159434$, respectively. Also we get the graphical visualization of the results. It is easy to see that the presented software module can be used from specialists when they make a choice for a model for approximation of cumulative data in a various modeling problems. More over upper and lower estimations from Theorem 5 can be used as "confidence bounds".

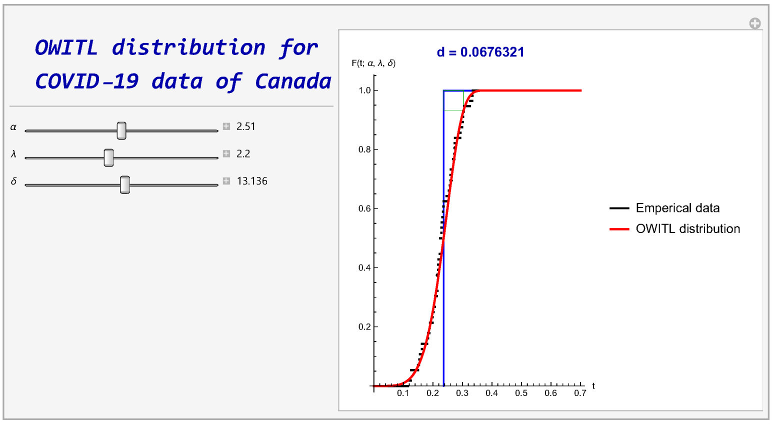

Figure 5: The model (1) for COVID-19 data (normalized) belong to Canada.

3.1. ADAPTIVE OWITL MODEL WITH POLYNOMIAL VARIABLE TRANSFER

Definition 6. Consider the following new "*adaptive OWITL model with polynomial variable transfer*":

$$F^*(t;\alpha,\lambda,\delta) = 1 - e^{-\lambda[(1+f(t))^{2\delta}(1+2f(t))^{-\delta}-1]^{\alpha}},$$
$$f(t) = \sum_{i=0}^{n} a_i t^i, \quad a_0 = 0. \tag{6}$$

The applicability of the model (6) is proved in simulation study to the data represent a COVID-19 data belong to The United Kingdom of 60 days, from 1 December 2020 to 29 January 2021 (see [1]). For the actual data in the specified period the our new model $F^*(t)$ for

$$\alpha = 1.233, \quad \lambda = 0.7076, \quad \delta = 13.136, \quad n = 4,$$
$$a_0 = 0,\ a_1 = 0.3223,\ a_2 = 4.7623,\ a_3 = -10.8312,\ a_4 = 7.9322$$

is depicted on Figure 6.

Remark 7. *The model is highly sensitive to the type and location of the zeros of the polynomial $f(t)$ (see [20]).*

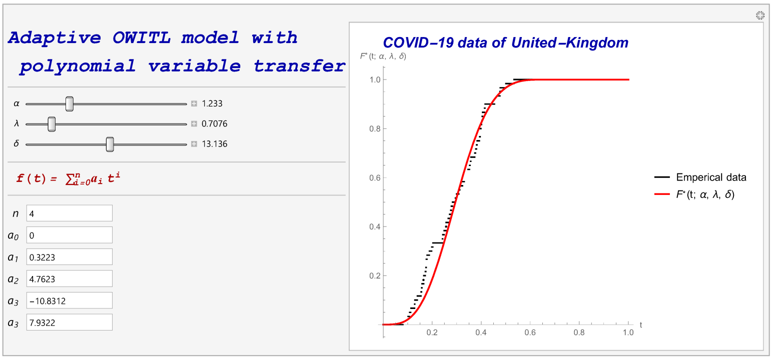

Figure 6: The model (6) for COVID-19 data (normalized) belong to United-Kingdom

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