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GEGENBAUER POLYNOMIALS AS CORRECTION IN THE LIENARD PLANAR SYSTEM: MELNIKOV'S APPROACH

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ABSTRACT: In this article a hypothetical oscillator model with Gegenbauer polynomials as corrections in the Lienard differential system is presented.

The model is considered in the light of Melnikov's approach.

Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, Melnikov's polynomial, Gegenbauer polynomials as "correcting factors" in the Lienard system, number and type of limit cycles

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1. INTRODUCTION

The Melnikov function [1] for the Lienard system [2]

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon\left(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1}\right) \\ \dfrac{dy}{dt} &= -x \end{cases}$$

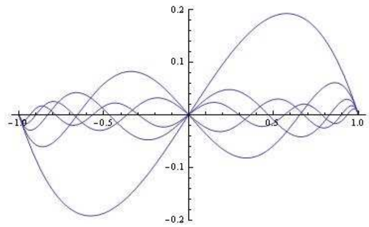

Figure 1: The polynomials $C_n^{(-\frac{1}{2})}(x)$ for $n = 3, 5, 7, 9, 11$.

is defined as

$$M(\alpha, \mu) = -2\pi\alpha^2 \left(\frac{a_1}{2} + \frac{3}{8}a_3\alpha^2 + \cdots + \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix} \frac{a_{2n+1}}{2^{2n+2}}\alpha^{2n} \right)$$

The *Melnikov polynomial* is defined as

$$P(r^2, n) = -\frac{1}{2\pi r^2} M(r, \mu).$$

The following result provides the necessary information about the number of limit cycles and their radii

Theorem [3]–[4]. The Lienard system for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii r_j, $j = 1, 2, \ldots, n$ as $\epsilon \to 0$ if and only if the nth degree polynomial in r^2,

$$P(r^2, n) = \frac{a_1}{2} + \frac{3}{8}a_3 r^2 + \cdots + \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix} \frac{a_{2n+1}}{2^{2n+2}} r^{2n}$$

has n positive roots $r^2 = r_j^2$, $j = 1, 2, \ldots, n$.

Denote by $C_n^{(-\frac{1}{2})}(x)$ the Gegenbauer polynomials [10]–[11]. In this paper we consider a new extended Lienard–type planar system with the polynomial $C_n^{(-\frac{1}{2})}(x)$ as correction. The type of limit cycles is also studied. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

2. MAIN RESULTS

2.1. GEGENBAUER POLYNOMIALS AS CORRECTIONS IN THE LIENARD DIFFERENTIAL SYSTEM

In this Section we consider formally the following model of the type:

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon C_n^{(-\frac{1}{2})}(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{1}$$

where $\epsilon > 0$ and $C_n^{(-\frac{1}{2})}(x)$ for $n = 3, 5, 7, 9, \dots,$ are the Gegenbauer polynomials [10]–[11].

For example we have (see Fig. 1)

$$\begin{aligned} C_3^{(-\frac{1}{2})}(x) &= \tfrac{x}{2} - \tfrac{x^3}{2} \\ C_5^{(-\frac{1}{2})}(x) &= -\tfrac{3x}{8} + \tfrac{5x^3}{4} - \tfrac{7x^5}{8} \\ C_7^{(-\frac{1}{2})}(x) &= \tfrac{5x}{16} - \tfrac{35x^3}{16} + \tfrac{63x^5}{16} - \tfrac{33x^7}{16} \\ C_9^{(-\frac{1}{2})}(x) &= -\tfrac{35x}{128} + \tfrac{105x^3}{32} - \tfrac{693x^5}{64} + \tfrac{429x^7}{32} - \tfrac{715x^9}{128} \\ C_{11}^{(-\frac{1}{2})}(x) &= \tfrac{63x}{256} - \tfrac{1155x^3}{256} + \tfrac{3003x^5}{128} - \tfrac{6435x^7}{128} + \tfrac{12155x^9}{256} - \tfrac{4199x^{11}}{256} \end{aligned}$$

Polynomials of this type can be used as correction factors in the Lienard differential system.

The solutions of the system

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(C_7^{(-\frac{1}{2})}(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{2}$$

for $\epsilon = 0.001; x_0 = 0, y_0 = 0.1$ are depicted on Fig. 2.

The solutions of the system

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(C_9^{(-\frac{1}{2})}(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{3}$$

for $\epsilon = 0.01; x_0 = 0, y_0 = 0.1$ are depicted on Fig. 3.

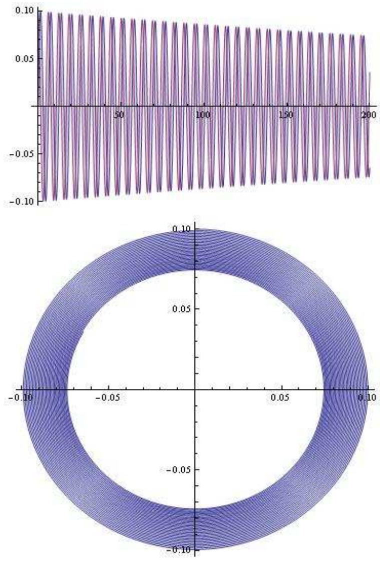

Figure 2: The solutions of the differential system (2).

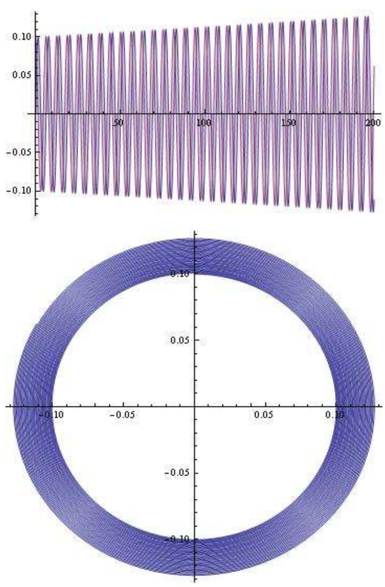

Figure 3: The solutions of the differential system (3).

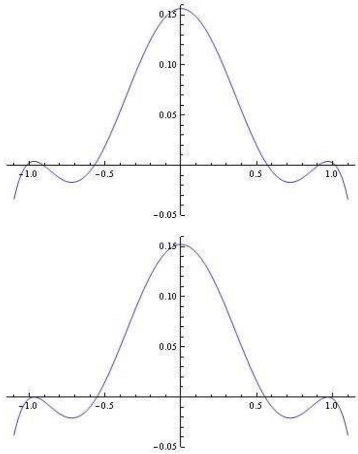

Figure 4: a) The Melnikov polynomial $P(r^2,3)$ for $n=7$ and $\mu=\frac{5}{16}$ (three limit cycles: 0.571222, 0.90548, 1.01768); b) The Melnikov polynomial $P(r^2,3)$ for $n=7$ and $\mu=0.304537955$ (simple limit cycle: 0.55442 and limit cycle 0.9681 with multiplicity – two).

2.2. THE NEW MODEL (1) IN THE LIGHT OF MELNIKOV'S CONSIDERATIONS

The case $n=7$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y-\epsilon(\mu x-\frac{35x^3}{16}+\frac{63x^5}{16}-\frac{33x^7}{16}) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{4}$$

where $\mu>0$, $\epsilon>0$.

The following is valid

Proposition 1. The Lienard–type system for $n=7$, and for all sufficiently small $\epsilon\neq 0$ for $\mu=0.304537955$ has simple limit cycle: 0.55442 and limit cycle 0.9681 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 4) we have:

$$P(r^2,3)=\frac{\mu}{2}-\frac{105}{128}r^2+\frac{315}{256}r^4-\frac{1155}{2048}r^6. \tag{5}$$

Evidently, for example $\mu=0.304537955$ we have simple limit cycle and limit cycle with multiplicity – two.

The case $n=11$.

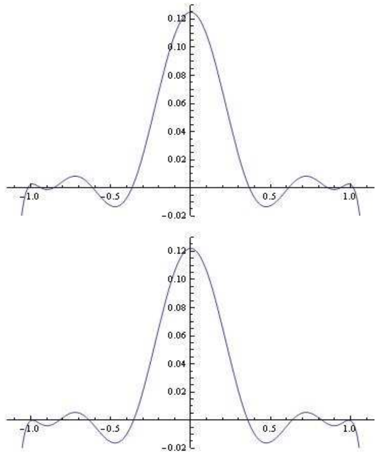

Figure 5: a) The Melnikov polynomial $P(r^2, 5)$ for $n = 11$ and $\mu = 0.25$ (simple limit cycles: 0.367826, 0.610487, 0.861024, 0.933275, 1.0186); b) The Melnikov polynomial $P(r^2, 5)$ for $n = 11$ and $\mu = 0.244308109$ (simple limit cycles: 0.357393, 0.63351, 0.82194 and limit cycle 0.9864 with multiplicity – two).

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(\mu x - \frac{1155x^3}{256} + \frac{3003x^5}{128} - \frac{6435x^7}{128} + \frac{12155x^9}{256} - \frac{4199x^{11}}{256}) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{6}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 2. The Lienard–type system for $n = 11$, and for all sufficiently small $\epsilon \neq 0$ for $\mu = 0.244308109$ has simple limit cycles: 0.357393, 0.63351, 0.82194 and limit cycle 0.9864 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 5) we have:

$$P(r^2, 5) = \frac{\mu}{2} - \frac{3465}{2048}r^2 + \frac{15015}{2048}r^4 - \frac{225225}{16384}r^6 + \frac{765765}{65536}r^8 - \frac{969969}{262144}r^{10}. \tag{7}$$

For $\mu = 0.244308109$ from (7) we have three simple limit cycles and limit cycle with multiplicity – two.

The catastrophe surfaces for $n = 7$ and $n = 9$

$$(x, y, p_1) = p_1 x - \frac{35x^3}{16} + \frac{63x^5}{16} - \frac{33x^7}{16} - y$$

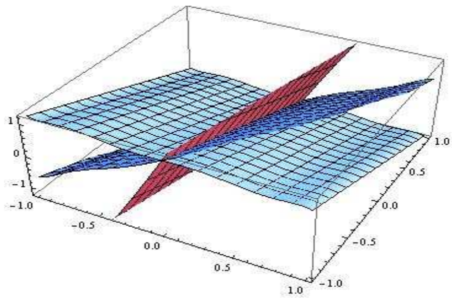

Figure 6: The catastrophe surface (x, y, p_1) for the following values of $p_1 = 0.1; 2; 6$.

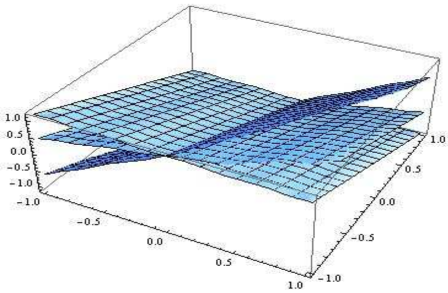

Figure 7: The catastrophe surface (x, y, p_2) for the following values of $p_2 = 0.1; 0.8; 1.9$.

$$(x, y, p_2) = -p_2 x + \frac{105x^3}{32} - \frac{693x^5}{64} + \frac{429x^7}{32} - \frac{715x^9}{128} - y$$

for the models are shown on Fig. 6–7.

The case $n = 15$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon\left(\mu x - \frac{15015}{2048}x^3 + \frac{15315}{2048}x^5 - \frac{692835}{2048}x^7 + \frac{1616615}{2048}x^9 - \right. \\ & & \left. - \frac{2028117}{2048}x^{11} + \frac{1300075}{2048}x^{13} - \frac{334305}{2048}x^{15}\right) \\ \dfrac{dy}{dt} & = & -x \end{cases} \quad (8)$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 3. The Lienard–type system for $n = 15$, and for all sufficiently small $\epsilon \neq 0$ for $\mu = 0.2088052$ has simple limit cycles: 0.26157, 0.470125, 0.644947, 0.804644, 0.905044 and limit cycle 0.994032 with multiplicity – two.

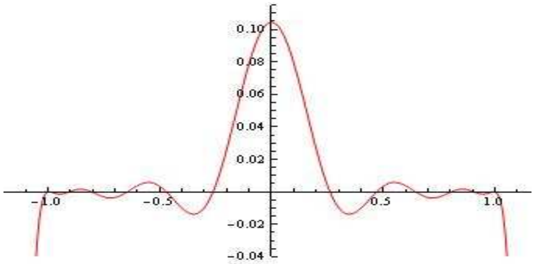

Figure 8: The Melnikov polynomial $P(r^2, 7)$ for $n = 15$ and $\mu = 0.2088052$ (simple limit cycles: 0.26157, 0.470125, 0.644947, 0.804644, 0.905044 and limit cycle 0.994032 with multiplicity – two).

Proof. For the Melnikov polynomial in r^2 (see Fig. 8) we have:

$$\begin{aligned} P(r^2,7) &= \tfrac{\mu}{2} - \tfrac{45045}{16384}r^2 + \tfrac{765765}{32768}r^4 - \tfrac{24249225}{262144}r^6 + \tfrac{101846745}{524288}r^8 - \\ &\quad - \tfrac{468495027}{2097152}r^{10} + \tfrac{557732175}{4194304}r^{12} - \tfrac{2151252675}{67108864}r^{14}. \end{aligned} \tag{9}$$

For $\mu = 0.2088052$ from (9) we have four simple limit cycles and limit cycle with multiplicity – two.

Consider a Lienard system of type

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = g(x) + \epsilon f(x) y \end{cases} \tag{10}$$

where $0 \le \epsilon \le 1$.

The solution of the system (10) for $x_0 = 0.9$, $y_0 = 0.2$, $\epsilon = 0.0001$, $g(x) = C_9^{(-\frac{1}{2})}(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ is visualized on Fig. 9.

For other results see [5]–[9], [12]–[15].

2.3. THE LEVEL CURVES

For more details of existing important results on the topic: Limit cycles bifurcations of some generalized polynomial Lienard system see [16]–[31].

Consider the Lienard polynomial systems of the type

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = Poly_i(x) + \epsilon h_i(x) y \end{cases} \tag{11}$$

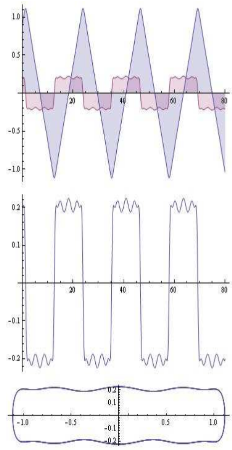

Figure 9: a) The solutions of the planar system (10) for $x_0 = 0.9$, $y_0 = 0.2$, $\epsilon = 0.0001$, $g(x) = C_9^{(-\frac{1}{2})}(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$; b) the y-component of the solution; c) the portrait of the planar system.

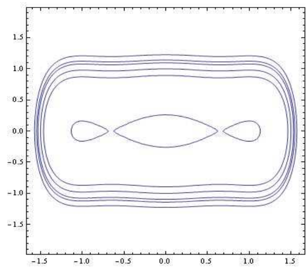

Figure 10: Level curves (the case A).

where $0 \leq \epsilon < 1$; $h_i(x)$ are specially chosen polynomials, and $Poly_i(x)$ are some of the polynomials discussed in this article.

Without going into details, we will note some more interesting level curves:

The case: A) $Poly_i(x)$ coincides with Gegenbauer polynomial

$$G_5^{-\frac{1}{2}}(x) = -\frac{7x^5}{8} + \frac{5x^3}{4} - \frac{3x}{8}.$$

The Hamiltonian of system (11) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} + \frac{7x^6}{48} - \frac{5x^4}{16} + \frac{3x^2}{16}.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted in Fig. 10.

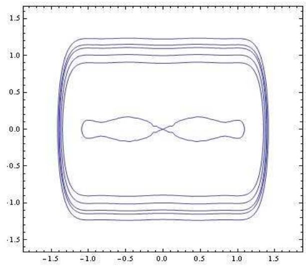

Figure 11: Level curves (the case B).

The case: B) $Poly_i(x)$ coincides with Gegenbauer polynomial

$$G_7^{-\frac{1}{2}}(x) = -\frac{33x^7}{16} + \frac{63x^5}{16} - \frac{35x^3}{16} + \frac{5x}{16}.$$

The Hamiltonian of system (11) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} + \frac{33x^8}{128} - \frac{63x^6}{96} + \frac{35x^4}{64} - \frac{5x^2}{32}.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted in Fig. 11.

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