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THE EFFECTS ON THE DYNAMICS OF LIENARD EQUATION WITH MORSE–TYPE CORRECTIONS: LEVEL CURVES

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ABSTRACT: In this paper we consider a class of polynomial Lienard system

$$\begin{cases} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= Poly_i(x) + \epsilon f_i(x) y \end{cases}$$

where $0 \leq \epsilon < 1$; $f_i(x)$ and $Poly_i(x)$ are specially chosen polynomials. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard system, Morse–type polynomial, extended generalized model, level curves.

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1. INTRODUCTION

The classical polynomial Lienard equations, i.e. planar vector fields associated to second order differential equation $x'' + f(x)x' + x = 0$ where f is a polynomial of degree $2n$ is

$$\begin{cases} \frac{dx}{dt} &= y - F(x) \\ \frac{dy}{dt} &= -x \end{cases}$$

where $F(x) = \int_0^x f(\tau)d\tau$ is a polynomial of degree $2n + 1$.

Conjecture of Neto, de Melo and Pugh [2]: The number of limit cycles in a Lienard system, with polynomial degree $2n + 1$, is at most n. The conjecture, known to be correct for $n = 1$, was still open for $n \geq 2$. Dumortier, Panazzolo and Roussarie [1] show that this conjecture has counterexamples for $n \geq 3$. In general, the topic: More limit cycles than expected in Lienard equation traced by Dumortier, Panazzolo and Roussarie is extremely interesting and relevant. For other results see [3]–[15].

In this paper we consider a new extended Lienard–type system with the Morse-type polynomial as "correction". Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

2. MAIN RESULTS. SIMULATIONS

We consider a class of polynomial Lienard system

$$\begin{cases} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= Poly_i(x) + \epsilon f_i(x)y \end{cases} \tag{1}$$

where $0 \leq \epsilon < 1$ and $Poly_i(x)$ coincides with Morse-type polynomials as "correction".

Without going into details, we will consider the case: $Poly_i(x)$ coincides with polynomial:

$$P_5(x) = x^5 + \frac{15}{4}x^4 + \frac{9.5}{3}x^3 + \frac{0.75}{2}x^2.$$

We note that the roots of $P_5'(x)$ are

$$-2.14795 < -0.76019 < -0.091864 < 0.$$

Eq. (1) is a slow–fast system. $P_5(x)$ is the sufficiently differentiable function with

$$\begin{cases} P_5'(x) > 0; \ x > 0 \\ P_5'(x) < 0; \ -0.0918641 < x < 0 \\ P_5'(x) > 0; \ -0.76019 < x < -0.0918641 \\ P_5'(x) < 0; \ -2.14795 < x < -0.76019 \\ P_5'(x) > 0; \ x < -2.14785 \end{cases}$$

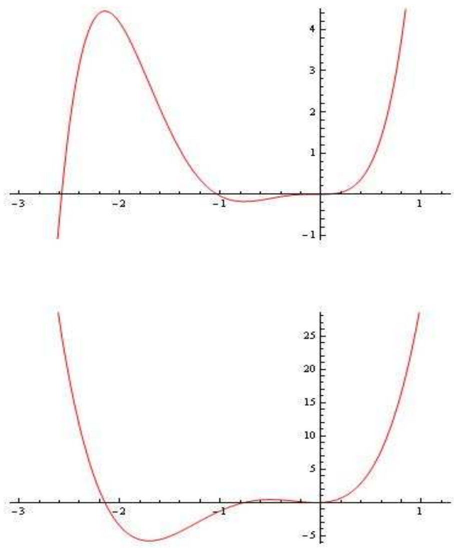

Figure 1: The polynomials: a) $P_5(x)$; b) $P_5'(x)$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - \frac{1}{6}x^6 - \frac{3}{4}x^5 - \frac{9.5}{12}x^4 - \frac{0.75}{6}x^3.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 2. For real Morse polynomials of degree 5 and 6 see [17].

Example 1. Consider a Lienard system of type:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -P_5(x) + \epsilon f(x) y \end{cases} \tag{2}$$

where $0 \leq \epsilon \leq 1$ and

$$f(x) = f_n(x) = \sum_{i=1}^{[\frac{n}{2}]} (-1)^{i+1} x^{n-2i} - \frac{x^n}{n}$$

for $n = 3, 7, 11, 15, 19, \ldots$, (see [20])

Let

$$f(x) = x - x^3 + x^5 - \frac{1}{7}x^7.$$

The simulations on the system (2) for $x_0 = 0.5$, $y_0 = 0.5$ and $\epsilon = 0.0001$ are depicted on Fig. 3.

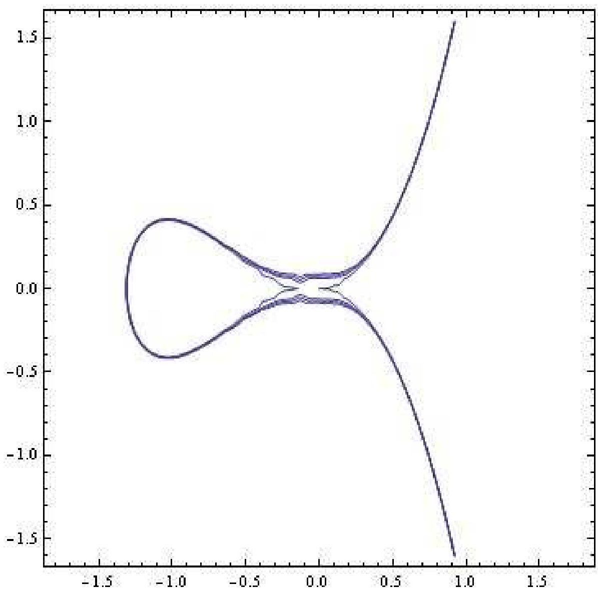

Figure 2: Level curves.

Consider the following model in the light of Zeeman's approach:

$$\begin{cases} \dfrac{dx}{dt} &= c(F(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{3}$$

with $c > 0$ and

$$F(x) = x^5 + \frac{15}{4}x^4 + \frac{9.5}{3}x^3 + px^2$$

The catastrophe surfaces $(x, y, p) = F(x) - y$ for the model is depicted on Fig. 4.

Let $Poly_i(x)$ in (1) coincides with polynomial:

$$P_8(x) = x^8 - 1.80902x^6 + 0.97264x^4 - 0.15169x^2.$$

We note that the roots of $P_8'(x)$ are

$$-0.905659 < -0.65475 < -0.328404 < 0 < 0.328404 < 0.65475 < 0.905659.$$

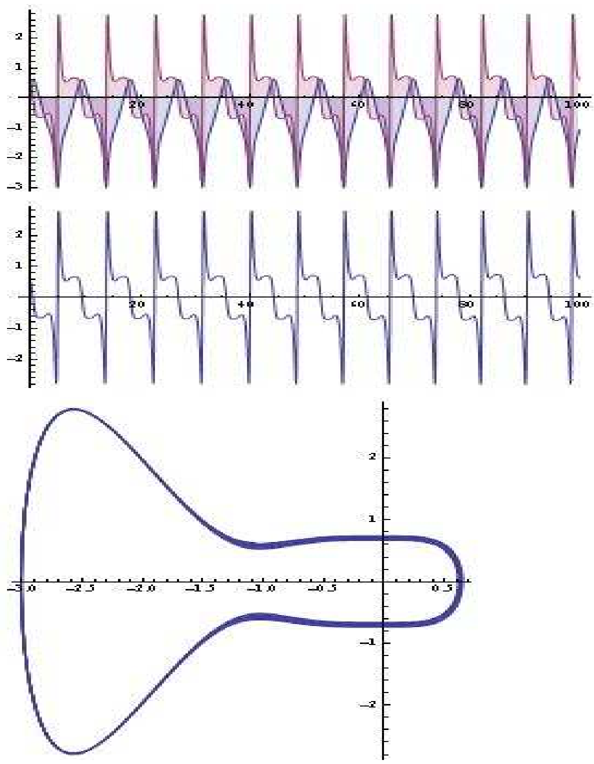

Figure 3: a) The solutions of the Lienard system (2); b) y-component of the solution; c) The portrait.

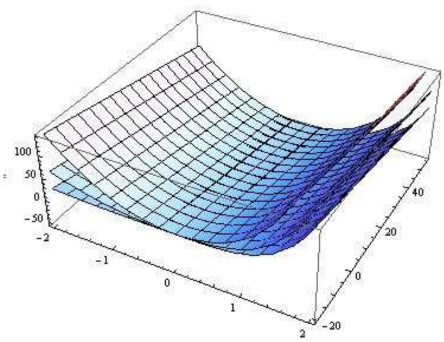

Figure 4: The catastrophe surfaces in the light of Zeeman considerations.

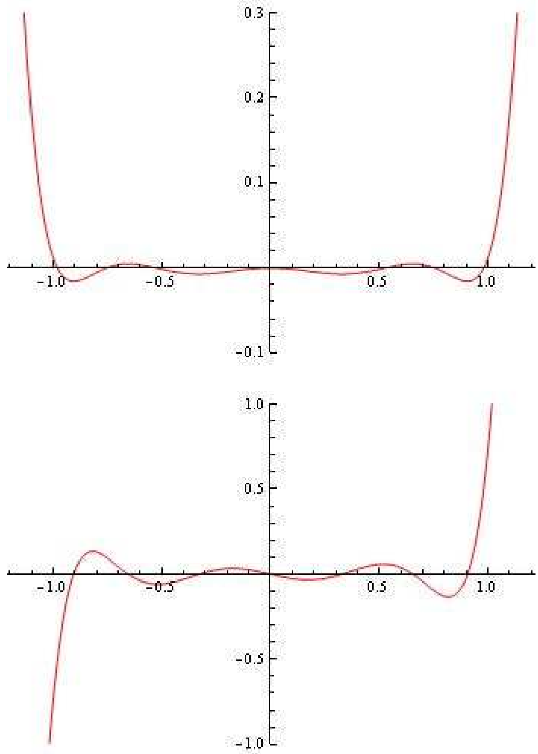

Figure 5: The polynomials: a) $P_8(x)$; b) $P_8'(x)$.

$P_8(x)$ is the sufficiently differentiable function with

$$\begin{cases} P_8'(x) > 0; \ x > 0.905659 \\ P_8'(x) < 0; \ 0.65475 < x < 0.905659 \\ P_8'(x) > 0; \ 0.328404 < x < 0.65475 \\ P_8'(x) < 0; \ 0 < x < 0.328404 \\ P_8'(x) > 0; \ -0.328404 < x < 0 \\ P_8'(x) < 0; \ -0.65475 < x < -0.328404 \\ P_8'(x) > 0; \ -0.905659 < x < -0.65475 \\ P_8'(x) < 0; \ x < -0.905659 \end{cases}$$

(see Fig. 5).

We note that the polynomial $P_8(x)$ can be used successfully in modeling and approximating of the "U-shaped transfer function" (see Fig. 6). If t_i, $i = 1, 2, \ldots$ are the critical points of P_8, we denote by $T_i = (t_i, P_8(t_i))$ the corresponding points on the slow curve of the layer equation. These points are the ending points of critical arcs (which are normally hyperbolic: attracting or repelling - see for more details [1]). With this

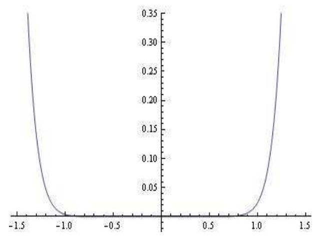

Figure 6: A typical "U-shaped transfer function".

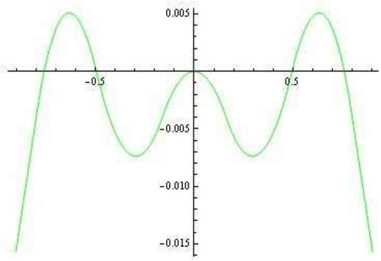

Figure 7: The "snakes" in Arnolds sense.

arrangement of critical zeros all "snakes" (in Arnolds sense [16]) are presented at Fig. 7.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} - \frac{1}{9}x^9 + \frac{1.80902}{7}x^7 - \frac{0.97264}{5}x^5 + \frac{0.15169}{3}x^3.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ are depicted at Fig. 8.

Example 2. Consider the hypothetical system of type:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -P_8(x) + \epsilon(x - x^3 + x^5 - \frac{1}{7}x^7)y \end{cases} \tag{4}$$

The simulations on the system (4) for $x_0 = 0.9$, $y_0 = 0.1$ and $\epsilon = 0.0001$ are depicted on Fig. 9. We will note that the $y(t)$-components of the differential systems discussed above can be used successfully in modeling and approximating functions and point sets (at appropriate intervals) in the field of signal theory and electrical circuits.

In the light of Zeeman's approach:

$$\begin{cases} \dfrac{dx}{dt} &= c(F(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases}$$

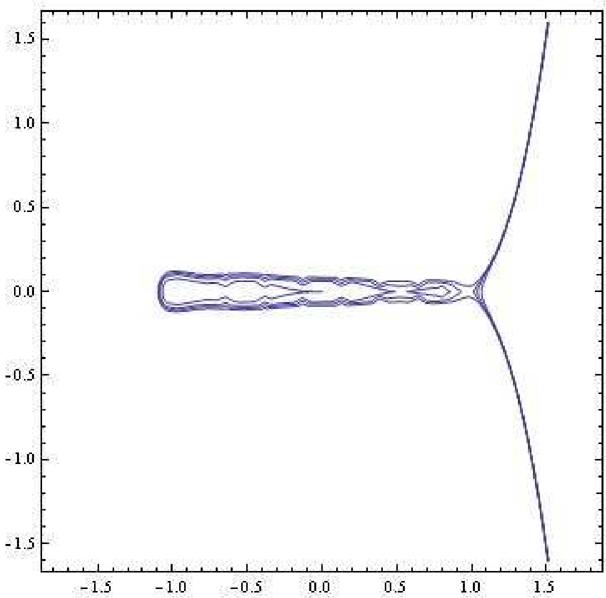

Figure 8: Level curves.

with $c > 0$ and

$$F(x) = x^8 - 1.80902x^6 + 0.97264x^4 - \alpha x^2.$$

the catastrophe surfaces $(x, y, \alpha) = F(x) - y$ for the model is depicted on Fig. 10. Let $Poly_i(x)$ in (1) coincides with polynomial:

$$\begin{aligned} P_{11}(x) \quad = \quad & 68.16624x^{11} - 62.48571x^{10} - 215.22858x^9 + 174.43928x^8 + \\ & +257.46429x^7 - 173.13750x^6 - 142.56964x^5 + 71.74821x^4 + \\ & +34.00774x^3 - 10.56429x^2 \end{aligned}$$

We note that the roots of $P'_{11}(x)$ are $-0.97598 < -0.86956 < -0.64488 < -0.48974 < 0 < 0.166666 < 0.656412 < 0.81155 < 1.03622 < 1.142653$.

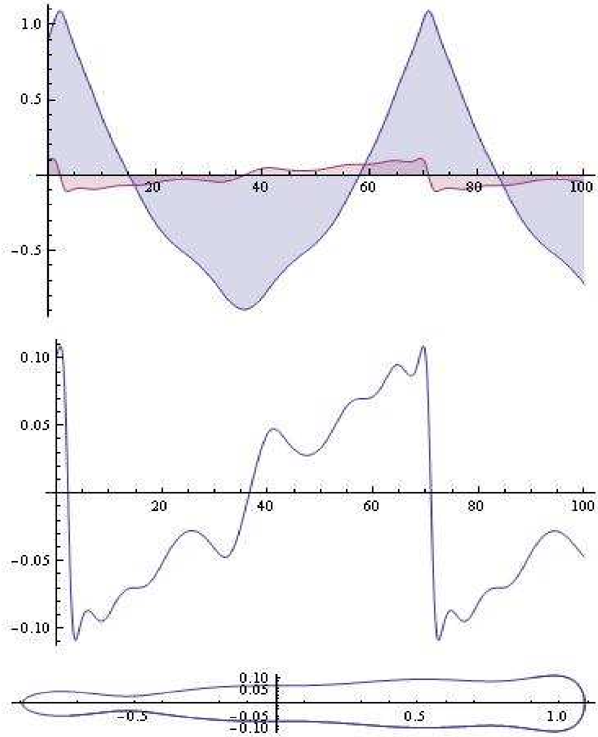

Figure 9: a) The solutions of the (4); b) y-component of the solution; c) The portrait.

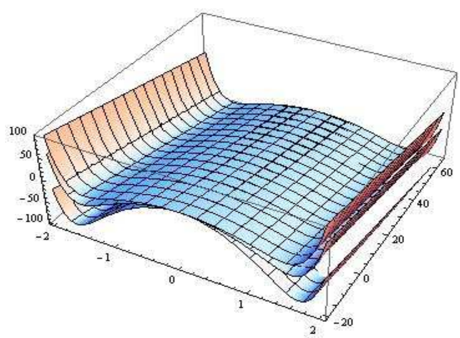

Figure 10: The catastrophe surfaces in the light of Zeeman considerations ($\alpha := 20,\ 30,\ 50$).

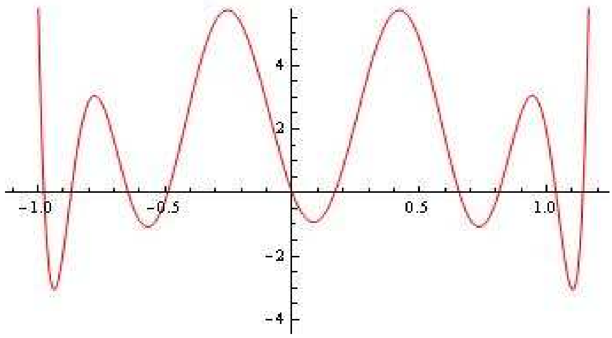

Figure 11: $P'_{11}(x)$.

$P_{11}(x)$ is the sufficiently differentiable function with

$$\begin{cases} P'_{11}(x) > 0;\ x < -0.97598 \\ P'_{11}(x) < 0;\ -0.97598 < x < -0.86956 \\ P'_{11}(x) > 0;\ -0.86956 < x < -0.64488 \\ P'_{11}(x) < 0;\ -0.64488 < x < -0.48974 \\ P'_{11}(x) > 0;\ -0.48974 < x < 0 \\ P_{11}(x) < 0;\ 0 < x < 0.166666 \\ P'_{11}(x) > 0;\ 0.166666 < x < 0.656412 \\ P'_{11}(x) < 0;\ 0.656412 < x < 0.81155 \\ P'_{11}(x) > 0;\ 0.81155 < x < 1.03622 \\ P'_{11}(x) < 0;\ 1.03622 < x < 1.142653 \\ P'_{11}(x) > 0;\ x > 1.142653 \end{cases}$$

(see Fig. 11). If t_i, $i = 1, 2, \ldots$ are the critical points of P_{11}, we denote by $T_i = (t_i, P_{11}(t_i))$ the corresponding points on the slow curve of the layer equation. With this arrangement of critical zeros all "snakes" (in Arnolds sense [16]) are presented at Fig. 12. Let $H(x, y)$ is the Hamiltonian of system (1) ($\epsilon = 0$). The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 13.

We will also offer specialized software tools (implemented in *CAS Mathematica*) for simulating the dynamics of the new families.

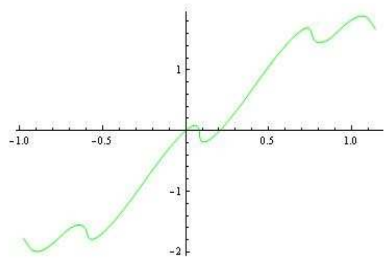

Figure 12: The "snakes" in Arnolds sense.

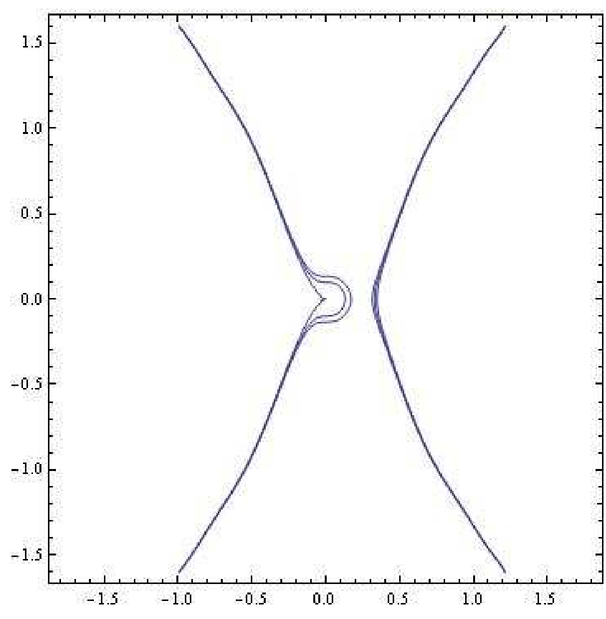

Figure 13: Level curves.

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REFERENCES

[1] F. Dumortier, D. Panazzolo, R. Roussarie, More limit cycles than expected in Lienard systems, Proc. Amer. Math. Soc. 135 (2007) 1895-1904

[2] Lins, A., de Melo, W., Pugh, C., On Lienard's equation, Lectures Notes in Math., vol. 597, Springer Verlag (1977).

[3] J. Llibre, Cl. Valls, Global centers of the generalized polynomial Lienard differential systems, *Journal of Differential Equations*, 330 (2022), 66–80.

[4] W. Xu, Limit cycle bifurcations of some polynomial Lienard system with symmetry, Nonlinear Anal. and Differential Equations, 8 (1), 2020, 77–87.

[5] W. Xu, C. Li, Limit cycles of some polynomial Lienard system, Journal of Math. Anal. and Appl., 389 (2012), 367–378.

[6] V. Gaiko, C. Vuik, H. Reijm, Bifurcation analysis of multi–parameter Lienard polynomial system, IFAC, Elsevier Ltd, 2018.

[7] J. Cai, M. Wei, H. Zhu, Nine limit cycles in a 5–degree polynomials Lienard system, Complexity, vol. 2020.

[8] X. Sun, H. Xi, Bifurcation of limit cycles in small perturbation of a class of Lienard systems, International Journal of Bifurcation and Chaos, 24 (1), 2014, 23 pp.)

[9] A. Zaghian, R. Kazemi, H. Zangenech, Bifurcation of limit cycles in a class of Lienard system with a cusp and nilpotent saddle, U.P.B. Sci. Bull. Series A, 78 (3), 216, 95–106.

[10] J. Hou, M. Han, Melnikov functions for planar near–Hamiltonian systems and Hopf bifurcations, J, Shanghai Normal University (Natural Sciences), 35, 2006, 1–10.

[11] M. Han, J, Yang, A. Tarta, Y. Gao, Limit cycles near homoclinic and heteroclinic loops, J. Dyn. Differential Equations, 20, 2008, 923–947.

[12] Y. An, M. Han, On the number of limit cycles near a homoclinic loop with a nilpotent singular point, J. Differential Equations, 258, 2015, 3194–3247.

[13] H. Chen, Z. Lie, R. Zhang, A sufficient and necessary condition of generalized polynomial Lienard systems with global centers, Preprint (August 2022).

[14] H. He, J. Llibre, D. Xiao, Hamiltonian polynomial differential systems with global centers in the plane, *Sci. China Math.*, 48 (2021) 2018.

[15] Y. Zhao, Z. Liang, G. Lu, On the global center of polynomial differential systems of degree 2k + 1, *Differential Equations and Control Theory*, 1996, 10 pp.

[16] V. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equation*, Springer (1988).

[17] Y. Kochetkov, Real Morse polynomials of order 5 and 6, arXiv:1911.02305V1 [math.CO] 6 Nov 2019.

[18] Lienard A., Etude des oscillations entretenues, *Revue generale de e'electricite*, **23** (1828), 901–912 and 946–954.

[19] N. Kyurkchiev, Some Intrinsic Properties of Tadmor-Tanner Functions: Related Problems and Possible Applications, *Mathematics* (2020), **8**, 1963.

[20] V. Kyurkchiev, N. Kyurkchiev, On an extended relaxation oscillator model: number of limit cycles, simulations. I, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[21] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, A technique for simulating the dynamics of some extended relaxation oscillator models. II, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[22] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Another extended polynomial Lienard systems: simulations and applications. III, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No. 1 (2022), 55–65.

[23] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Investigations on some polynomial Lienard-type systems: number of limit cycles, simulations, *International Journal of Differential Equations and Applications*, **21**, No. 1 (2022), 117–126.

[24] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, On some extended oscillator models: a technique for simulating and studying their dynamics, Plovdiv, Plovdiv University Press (2022), ISBN: 978-619-7663-13-6.

[25] N. Kyurkchiev, A. Iliev, On the hypothetical oscillator model with second kind Chebyshev's polynomial–correction: number and type of limit cycles, simulations and possible applications, *Algorithms*, 2022 (accepted).

[26] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Lienard system with first kind Chebyshev's polynomial–correction in the light of Melnikov's approach. Ssimulations and possible applications, Proc. of the Int. Conf. "Informatics, Mathematics, Education and their Application" (IMEA'2022), Pamporovo 2022 (accepted).

[27] A. Malinova, T. Terzieva, O. Rahneva, E. Angelova, Legendre polynomials as "correction factors" in the Lienard differential system. Simulations, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[28] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Simulations on the Lienard polynomial system with Dickson-type polynomial corrections. The level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 31–44.

[29] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Gegenbauer polynomials as correction in the Lienard planar system: Melnikov's approach, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 45–57.

[30] E. Angelova, V. Arnaudova, T. Terzieva, A. Malinova, Investigations on a differential system with correction of Zernike–type radial polynomials. Simulations, Proc. of the Int. Conf. "Informatics, Mathematics, Education and their Application" (IMEA'2022), Pamporovo 2022 (accepted).

[31] A. Golev, V. Arnaudova, Lienard system with "correcting factors" of the type of interpolating polynomials of some basic functions, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No 1 (2022), 67–80.

[32] Mladenov, V., A unified and open LTSPICE memristor model library, *Electronics*, **10** (2021), 28 pp.