International Journal of Differential Equations and Applications
Volume **22**, No. 1 (2023), pages: 137-154
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu

ijdea.eu

A LOOK AT THE DUFFING SYSTEM WITH PERIODIC PARAMETRIC EXCITATION: AN OVERVIEW (WEB–PLATFORM UPGRADE)

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ABSTRACT: In this article we demonstrate some specialized modules for investigating the dynamics of some generalized Duffing system with periodic parametric excitation, an integral part of a planned much more general Web–based application for scientific computing. We also study some new hypothetical oscillators. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: generalized Duffing system with periodic parametric excitation, Duffing oscillator, hypothetical oscillator, diagram factor

Received: October 4, 2023 **Revised:** November 10, 2023
Published: November 12, 2023 **doi:** 10.12732/ijdea.v22i1.10
Academic Publications, Ltd. https://acadpubl.eu

1. INTRODUCTION

In this paper we demonstrate some specialized modules for investigating the dynamics of some generalized Duffing system with periodic parametric excitation, an integral part of a planned much more general Web–based application for scientific computing (for some details see [10]–[17]).

We also study some new hypothetical oscillators.

More precisely, this WEB Platform envisages research on: Duffing system with periodic parametric excitation; modified Duffing systems; new hypothetical oscillators; a modification of the basic Duffing system with periodic parametric excitation.

Numerical examples, illustrating our results using *CAS MATHEMATICA* are also given.

In [7]–[8] some investigations and simulations on the planar Rayleigh–Lienard system are given.

We note that the use of normalized diagram factor $\frac{y(b\cos\theta+c)}{N}$ (where θ is the azimuthal angle and c is the phase difference) is very complicated.

Where possible, the corresponding diagram-functions (with application in the field of antenna analysis and synthesis) generated by the model oscillators investigated in this article have been generated and visualized.

A natural extension of the planned much more general Web–based application for scientific computing involves further consideration and simulations on generalized Duffing systems with periodic parametric excitation.

2. MAIN RESULTS. SIMULATIONS

2.1. A LOOK AT THE DUFFING SYSTEM WITH PERIODIC PARAMETRIC EXCITATION

The Rayleigh–Duffing oscillator models are widely used in physics, electronics, and many other disciplines.

Consider the following planar system (see for example [1])

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = f\cos(gt)x - x^3 - hy \end{cases} \tag{1}$$

where h is the damping coefficient, and f and g are the amplitude and frequency of excitation.

I. For given $f = 6.25$, $g = 0.1$; $h = 0.3$, the simulations on the system (1) for $x_0 = 0.2$; $y_0 = 0.1$ are depicted on Fig. 1.

I.1 For given $f = 5$, $g = 0.05$; $h = 0.01$, the simulations on the system (1) for $x_0 = 0.3$; $y_0 = 0.2$ are depicted on Fig. 2.

2.2. A LOOK AT THE MODIFIED DUFFING SYSTEM

Chaotic motions of a Rayleigh–Duffing oscillator with periodically external and parametric excitations are investigated rigorously.

For some results see [24]–[25].

Consider the planar system (see for example [2])

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= x - x^3 + \epsilon(a_1 \cos(at) - a_2 y) \end{cases} \tag{2}$$

where $0 \leq \epsilon \leq 1$.

II. For given $a_1 = 6$, $a_2 = 2$, $a = 0.005$, $\epsilon = 0.005$, the simulations on the system (2) for $x_0 = 0.2$; $y_0 = 0.1$ are depicted on Fig. 3.

II.1 For given $a_1 = 0.1$, $a_2 = 0.003$, $a = 0.005$, $\epsilon = 0.1$, the simulations on the system (2) for $x_0 = 0.3$; $y_0 = 0.2$ are depicted on Fig. 4.

For fixed $b = 0.92$, $c = 0.73$ the normalized diagram factor is depicted on Fig. 4 d).

Remark. In [6] the authors study the dynamics of a generalized oscillator model on the base of model (2) in the light of Melnikov's approach.

Consider the planar system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= a_1 x - a_2 x^3 + \epsilon((1 - y^2)y + (1 + x)\cos(gt)) \end{cases} \tag{3}$$

where $0 \leq \epsilon \leq 1$.

III. For given $a_1 = 0.2$, $a_2 = 0.9$, $g = 0.02$, $\epsilon = 0.01$, the simulations on the system (3) for $x_0 = 0.3$; $y_0 = 0.2$ are depicted on Fig. 5.

For fixed $b = 0.9$, $c = 0.45$ the normalized diagram factor is depicted on Fig. 5 d).

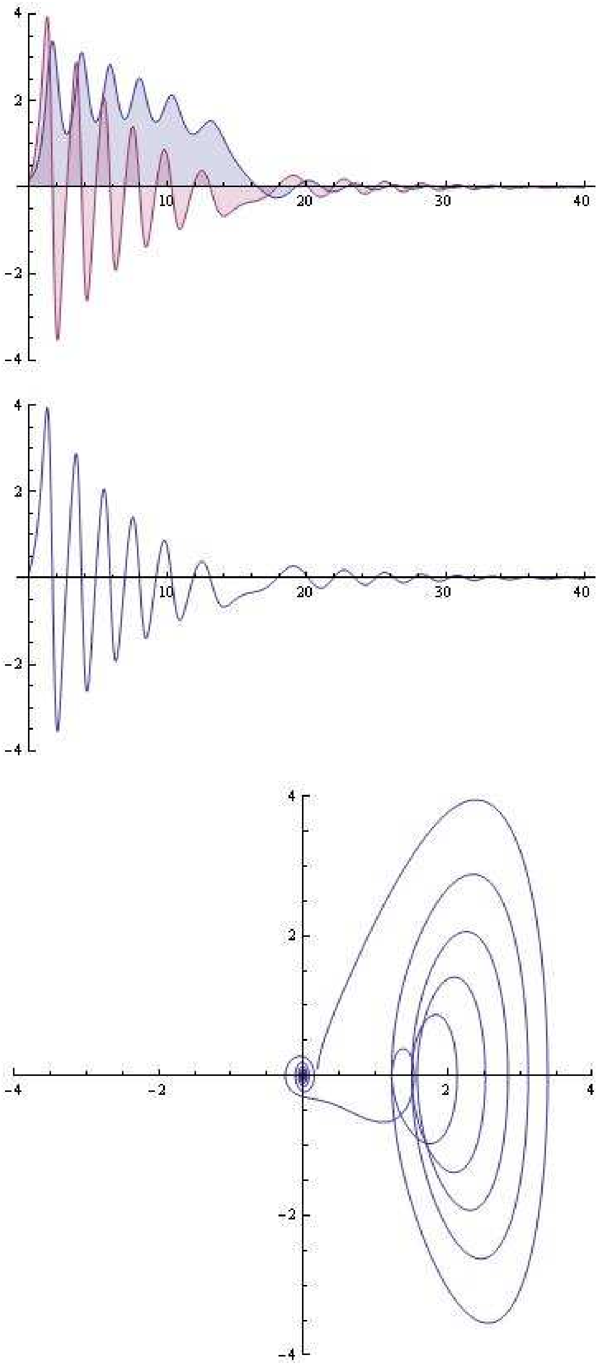

Figure 1: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait (example I).

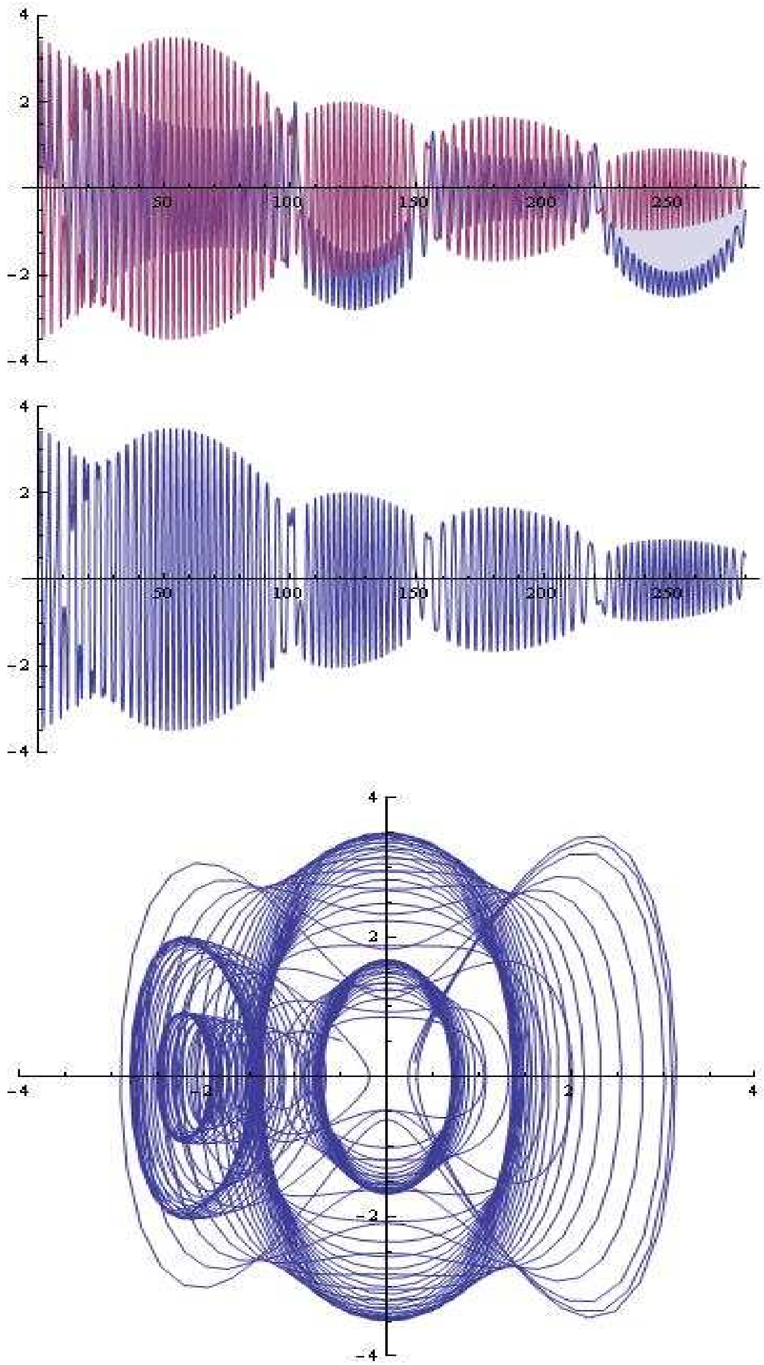

Figure 2: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait (example I.1).

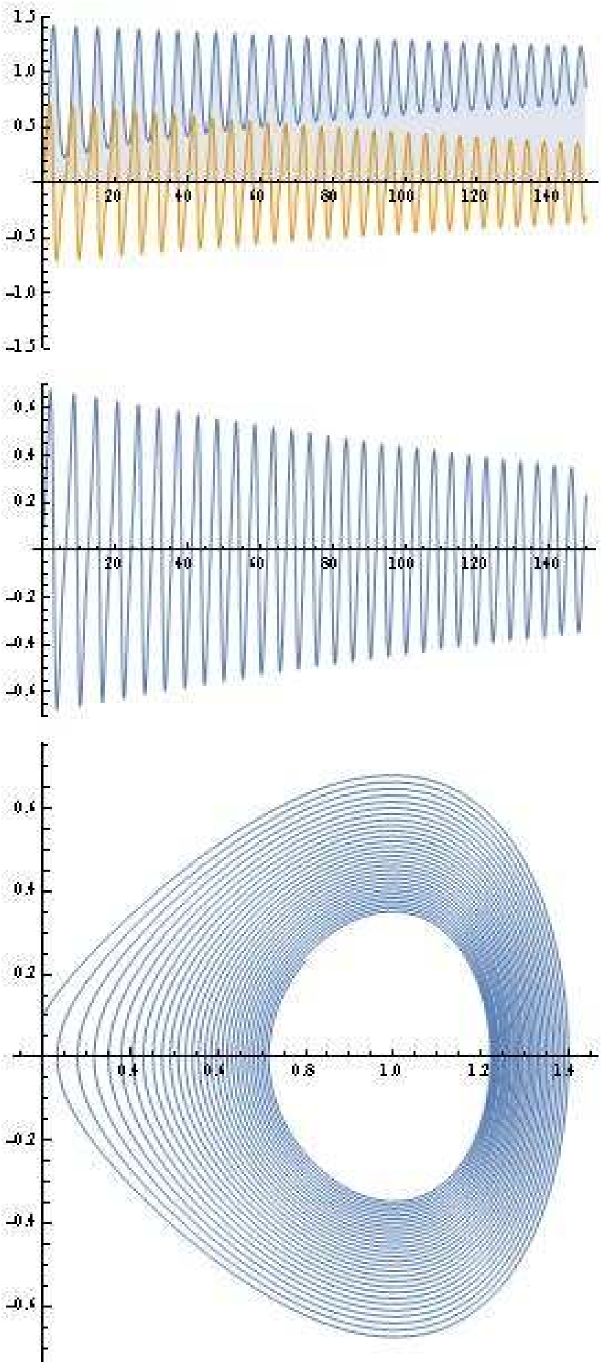

Figure 3: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait (example II).

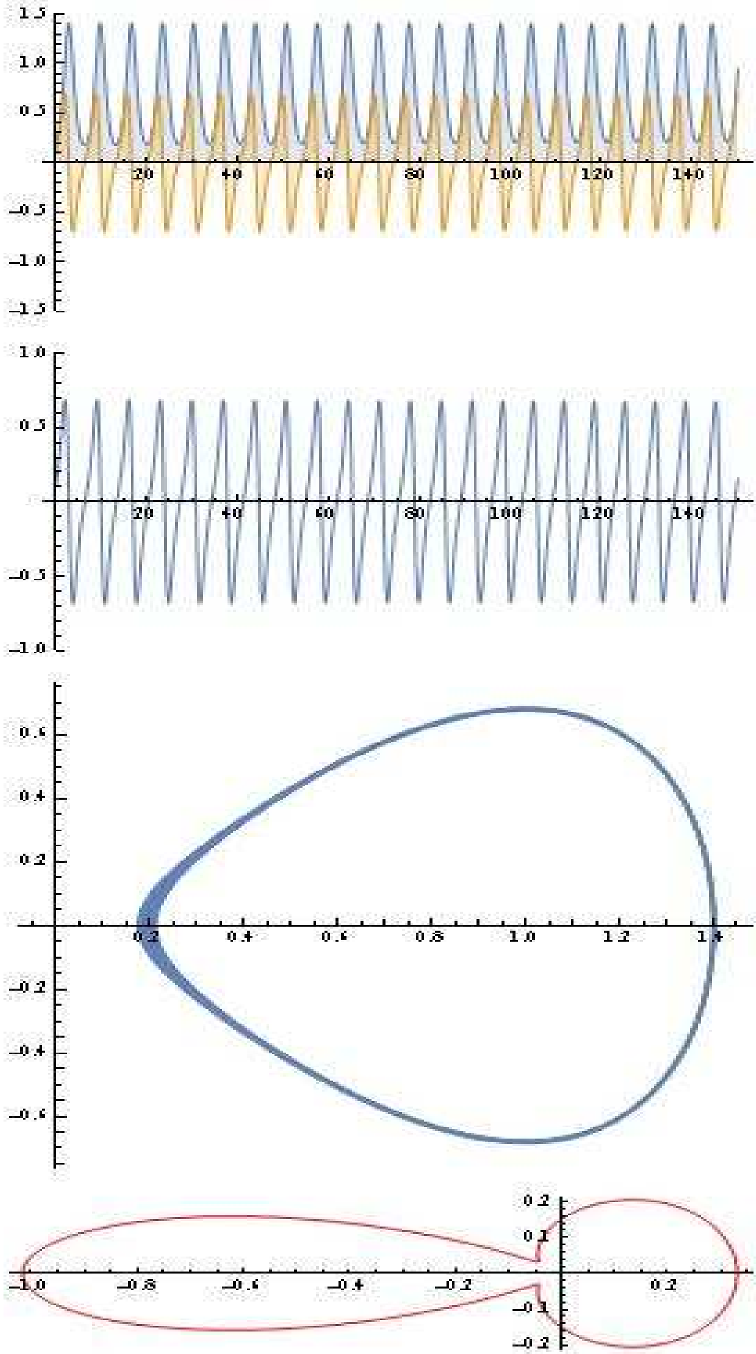

Figure 4: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) diagram factor (example II.1).

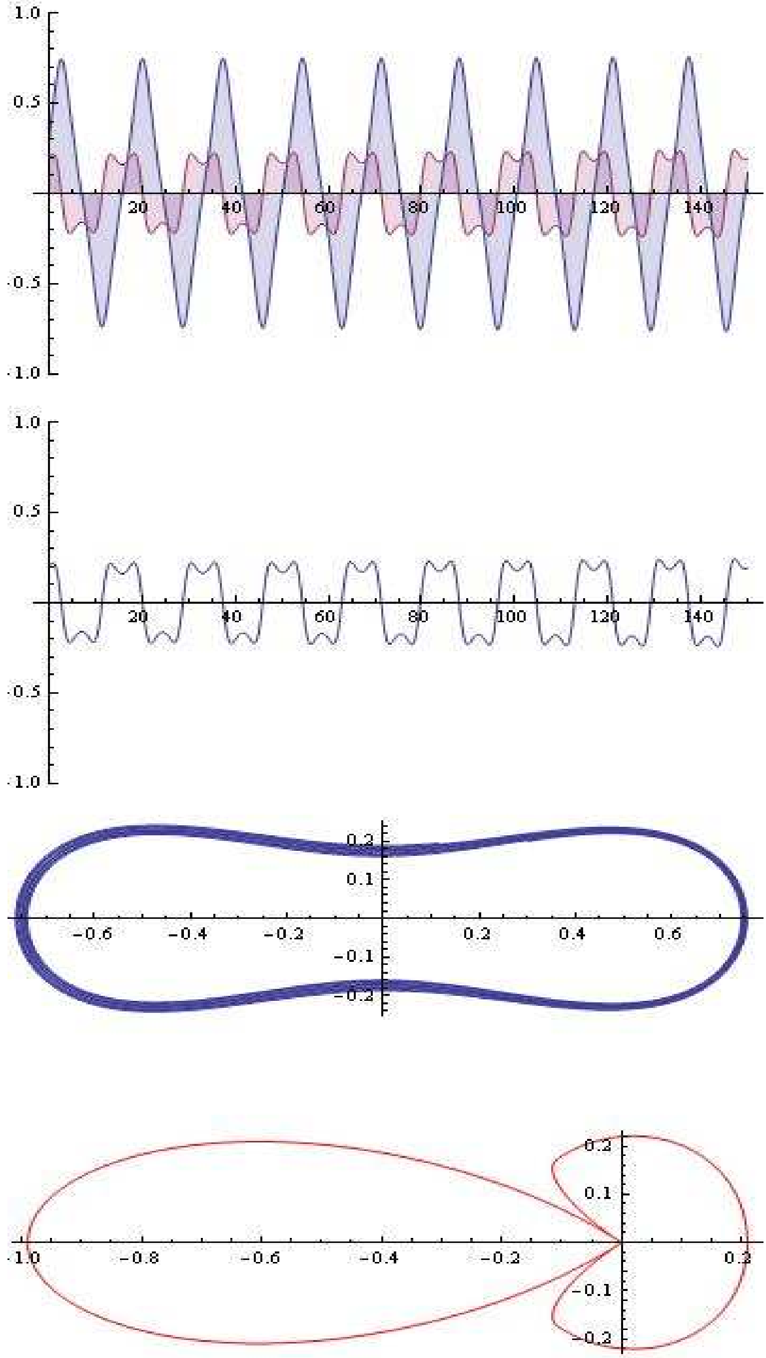

Figure 5: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) diagram factor (example III).

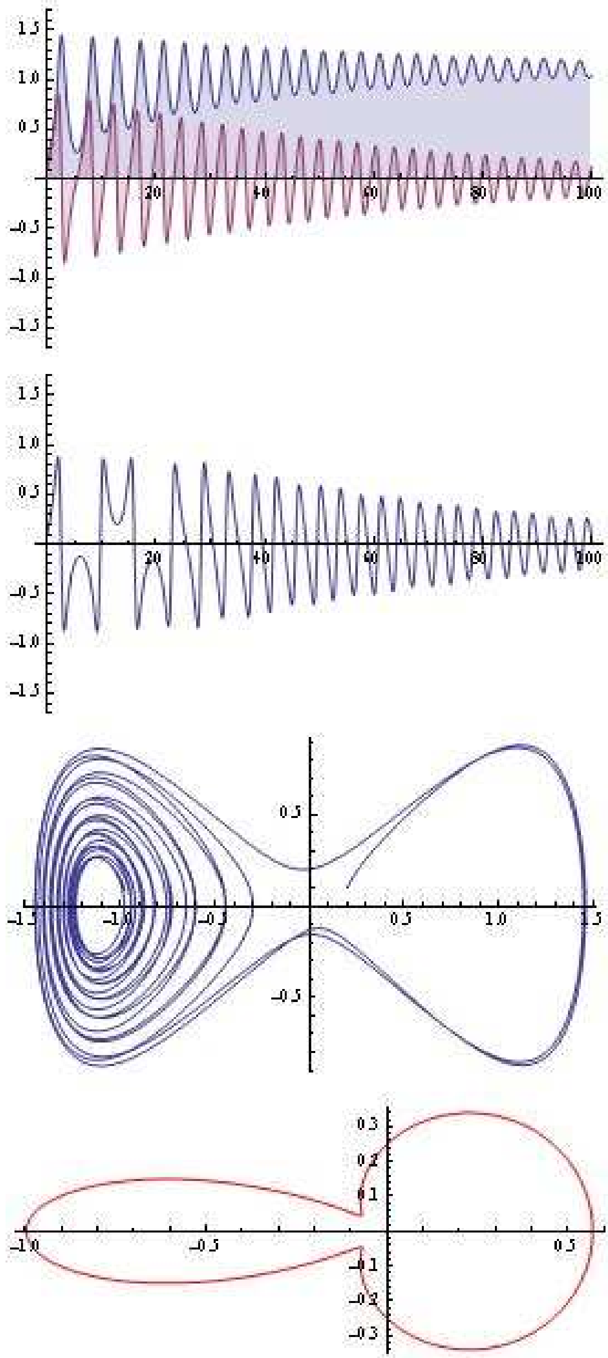

Figure 6: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) diagram factor (example IV).

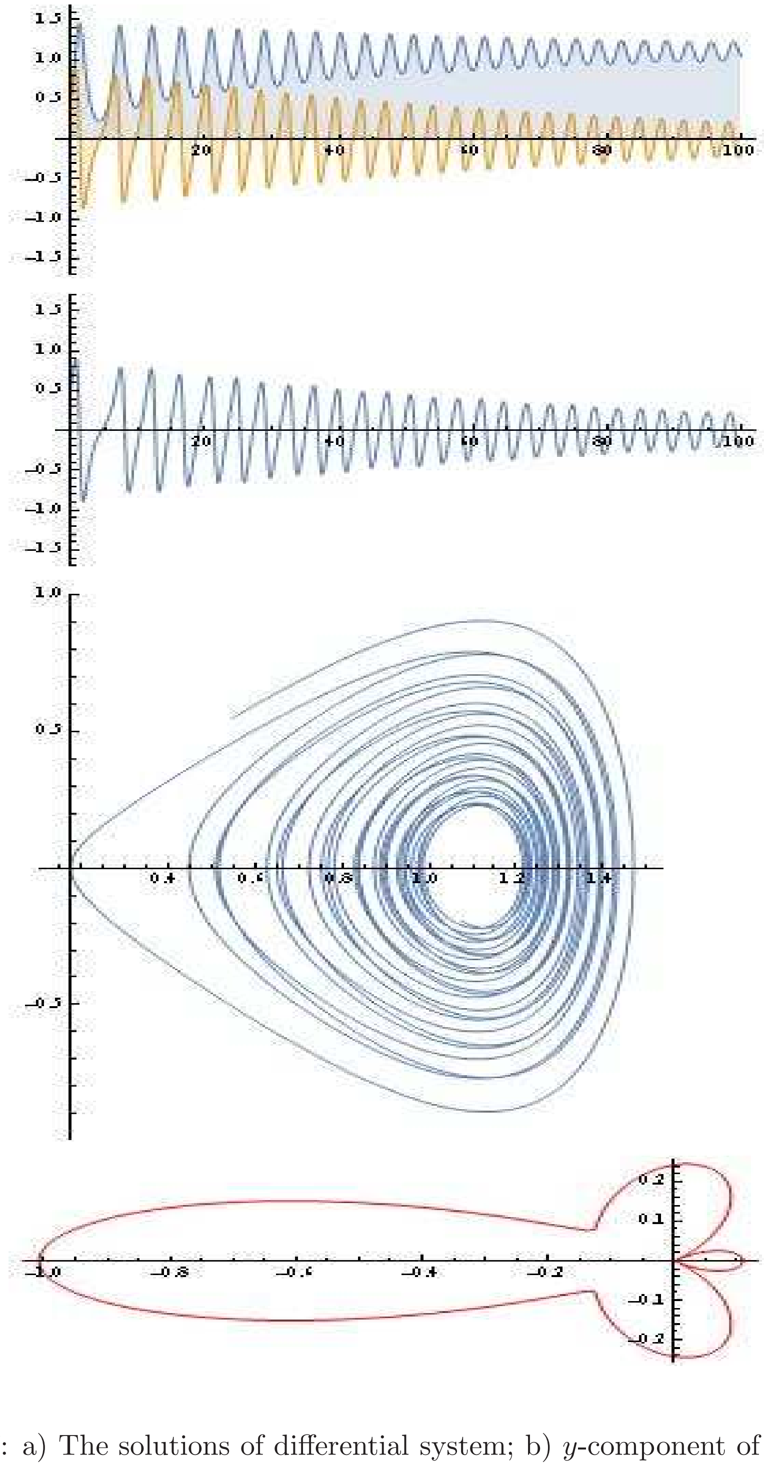

Figure 7: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) diagram factor (example IV.1).

2.3. A LOOK AT THE NEW HYPOTHETICAL OSCILLATORS

Consider the following planar system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \\ \dfrac{dy}{dt} &= x - \displaystyle\sum_{i=0}^{[\frac{n}{2}]-1} \frac{x^{n-2i}}{n-2i} + \epsilon(a_1\cos(at) - a_2 y) \end{cases} \tag{4}$$

where $0 \leq \epsilon \leq 1$.

Remark. Our considerations on the Melnikov homoclinic integral are summarized in Appendix 1.

IV. For given $n = 7$, $a_1 = 0.25$, $a_2 = 0.35$, $a = 0.55$, $b = 0.81$, $c = 0.66$, $\epsilon = 0.005$, the simulations on the system (4) for $x_0 = 0.2$; $y_0 = 0.1$ are depicted on Fig. 6.

IV.1 For given $n = 7$, $a_1 = 0.2$, $a_2 = 0.3$, $a = 0.5$, $b = 0.8$, $c = 0.665$, $\epsilon = 0.005$, the simulations on the system (4) for $x_0 = 0.55$; $y_0 = 0.55$ are depicted on Fig. 7.

2.4. A MODIFICATION OF THE MODEL (1)

Consider the following modification of model (1):

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \\ \dfrac{dy}{dt} &= f\cos(gt)x - \displaystyle\sum_{i=0}^{[\frac{n}{2}]-1} x^{n-2i} - hy \end{cases} \tag{5}$$

V. For given $n = 7$, $f = 4$, $g = 0.07$, $h = 0.03$, the simulations on the system (5) for $x_0 = 0.4$; $y_0 = 0.3$ are depicted on Fig. 8.

V.1 For given $n = 9$, $f = 5$, $g = 0.005$, $h = 0.01$, the simulations on the system (5) for $x_0 = 0.3$; $y_0 = 0.2$ are depicted on Fig. 9.

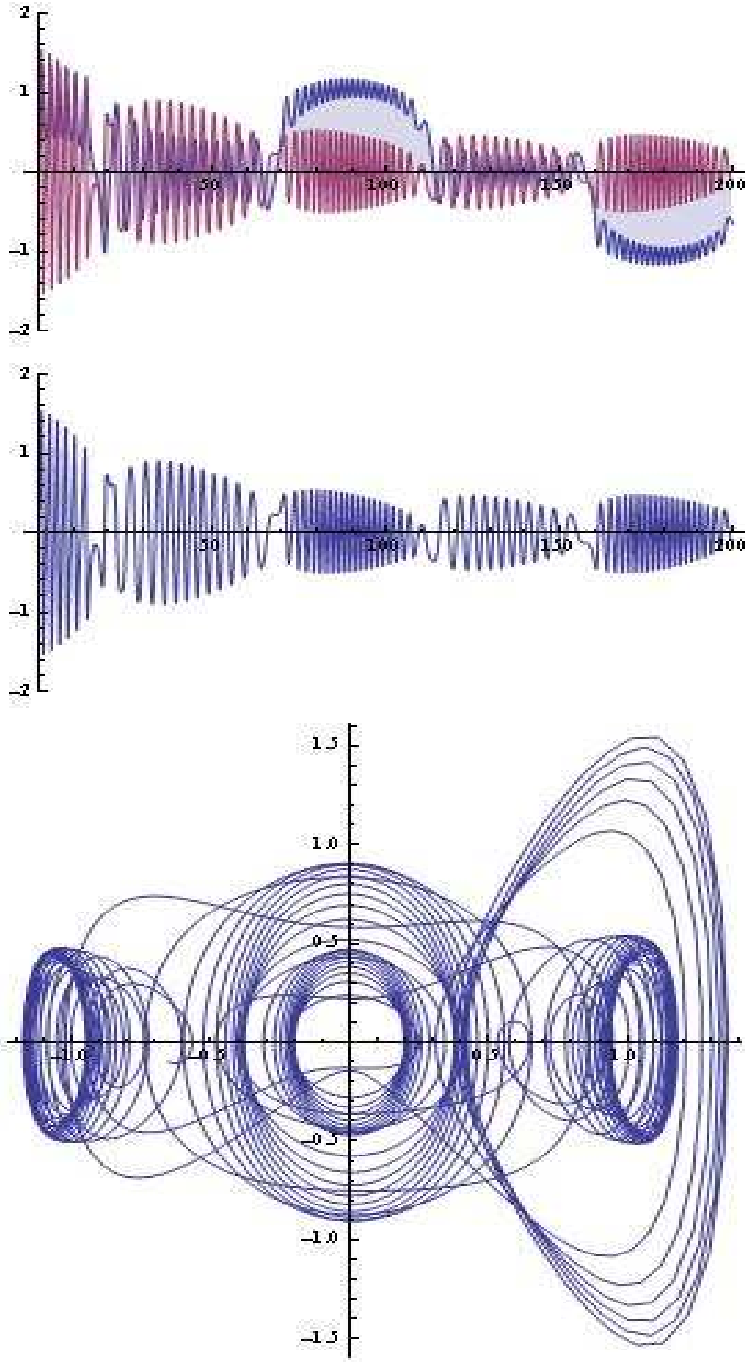

Figure 8: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait (example V).

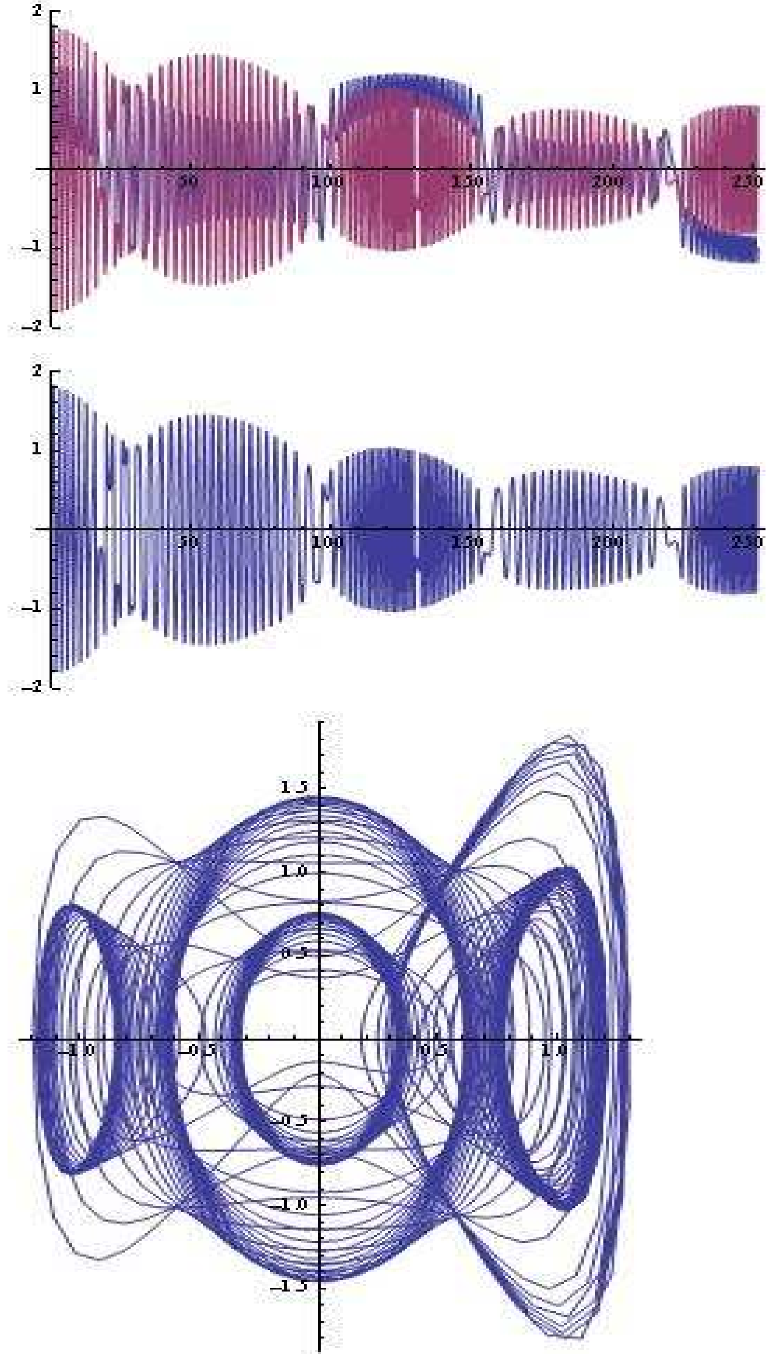

Figure 9: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait (example V.1).

3. CONCLUDING REMARKS

In this article we demonstrate some specialized modules for investigating the dynamics of some generalized Duffing system with periodic parametric excitation, an integral part of a planned much more general Web–based application for scientific computing. More precisely, this WEB Platform envisages research on: Duffing system with periodic parametric excitation; modified Duffing systems; new hypothetical oscillators; a modification of the basic Duffing system with periodic parametric excitation and others. Where possible, we employ various optimization techniques for highperformance calculations, including multi–processor and multi–threading calculations, and hardware intrinsics [15]–[17]. We presented only a small part of the platform's capabilities. We will be grateful to all colleagues who, with their critical remarks, will contribute to its significant improvement. We fully understand that the construction of such an ambitious Web–based platform for scientific computing can only be realized with the active participation of specialists from various branches of scientific knowledge.

Remark. The study of corresponding critical levels of $H(x,y) = \frac{1}{2}y^2 - P(x)$ is very complicated.

In this regard, we recommend the excellent study by Gavrilov and Iliev [28].

Acknowledgment

This study is financed by the project No FP23-FMI-002 "Intelligent software tools and applications in research in Mathematics, Informatics, and Pedagogy of Education" of the Scientific Fund of the Paisii Hilendarski University of Plovdiv, Bulgaria.

4. APPENDIX 1.

The case $n = 3$. The system of the type (4)

$$\begin{cases} \dfrac{dx}{dt} &= & y \\ \dfrac{dy}{dt} &= & x - \frac{1}{3}x^3 + \epsilon(a_1 \cos(at) - a_2 y) \end{cases}$$

has the following Hamiltonian ($\epsilon = 0$)

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{12}x^4.$$

The homoclinic orbit is given by (see Fig.10)

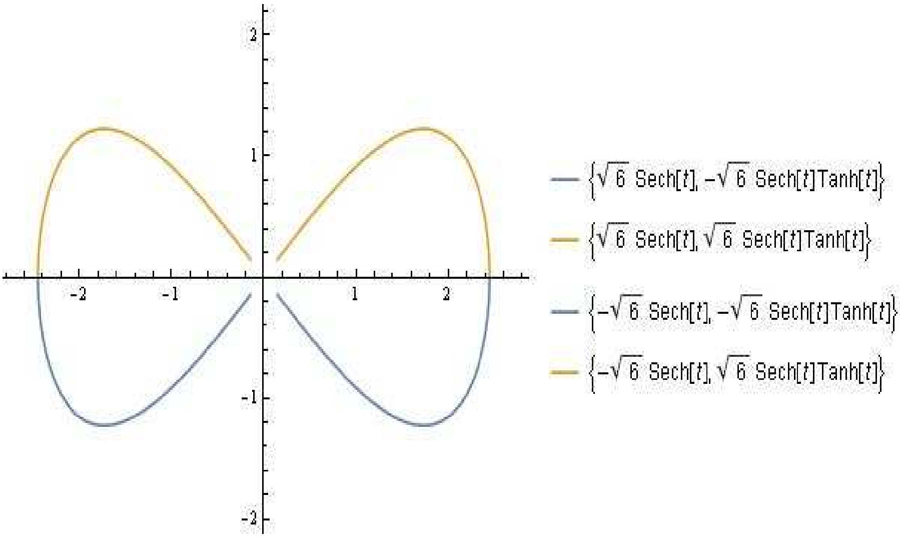

Figure 10: The homoclinic orbit.

$$x_0(t) = \pm\sqrt{6} sech\, t$$
$$y_0(t) = \mp\sqrt{6} sech\, t\, tanh\, t.$$

The Melnikov homoclinic integral is given by:

$$M(t_0) = \int_{-\infty}^{\infty} y_0(t)\,(a_1 \cos(a(t+t_0)) - a_2 y_0(t))\; dt.$$

From a numerical point of view, the task of finding a multiple root of $M(t_0)$ is more interesting given that the parameters appearing in the proposed differential model are subject to a number of restrictions of a physical nature.

The following is valid

Proposition 1. For $n = 3$ and $D = K$ where

$$D = \frac{4a_2}{\sqrt{6}\pi} cosh\left(\frac{a\pi}{2}\right);\;\; K = aa_1$$

the Melnikov function $M(t_0)$ has root with multiplicity two.

Proof. We have

$$\begin{aligned}
M(t_0) &= \int_{-\infty}^{\infty} y_0(t)\left(a_1\cos(a(t+t_0)) - a_2 y_0(t)\right)\, dt \\
&= \int_{-\infty}^{\infty} \left(-a_2 y_0(t)^2 + y_0(t) a_1 \cos(a(t+t_0))\right)\, dt \\
&= -4a_2 + \sqrt{6}aa_1\pi sech\left(\tfrac{a\pi}{2}\right)\sin(at_0) \\
&= \sqrt{6}aa_1\pi sech\left(\tfrac{a\pi}{2}\right)\left(\sin(at_0) - \tfrac{D}{K}\right)
\end{aligned}$$

This completes the proof of Proposition 1.

REFERENCES

[1] X. Li, Y. Shen, J. Sun, S. Yang, New periodic attractors is slow-fast Duffing system with periodic parametric excitation, *Scientific Reports*, 2019, 9:11185.

[2] P. J. Holmes, Chaos in Duffing's equation: Strange Attractors?, (in Encyclopedia of Vibration, (2001).

[3] L. Gavrilov, I. Iliev, The limit cycles in a generalized Rayleigh–Lienard oscillator, *Discrete and Continuous Dynamical Systems*, 43 (6) (2023), 2381–2400.

[4] L. Perko, *Differential Equations and Dynamical Systems*, Springer–Verlag, New York (1991)

[5] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, Springer-Verlag, New York, 1983.

[6] Kyurkchiev, N., T. Zaevski, On a hypothetical oscillator: investigations in the light of the Melnikov's approach, some simulations, *International Journal of Differential Equations and Applications*, (2023).

[7] Kyurkchiev, N., Iliev, A., Kyurkchiev, V., Vasileva, M., Rahnev, A., Some investigations and simulations on the planar Rayleigh–Lienard differential system (Web Platform upgrade), *International Journal of Differential Equations and Applications*, (2023).

[8] Kyurkchiev, N., Iliev, A., Kyurkchiev, V., Vasileva, M., Rahnev, A., Dynamics and simulations of some generalized Rayleigh systems (Web Platform upgrade), *International Journal of Differential Equations and Applications*, (2023).

[9] Golev, A., Arnaudova, V., Dynamics of a generalized hypothetical Rayleigh–Duffing–like oscillator: investigations in the light of Melnikov's approach, simulations (Web–Platform upgrade), *International Journal of Differential Equations and Applications*, (2023).

[10] Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., On a class of orthogonal polynomials as corrections in Lienard differential system. Applications, *Algorithms*, 2023.

[11] Kyurkchiev, V., Kyurkchiev, N., Iliev, A., Rahnev, A., On some extended oscillator models: a technique for simulating and studying their dynamics. Plovdiv University Press: Plovdiv, 2022; ISBN 978-619-7663-13-6.

[12] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On the Lienard System with Some "Corrections of Polynomial-type": Number of Limit Cycles, Simulations and Possible Applications. Part II.*, Plovdiv, Plovdiv University Press (2022), ISBN: 978-619-7663-41-9.

[13] M. Vasileva, V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Applications of Some Orthogonal Polynomials and Associated Polynomials of Higher Kind, Plovdiv, Plovdiv University Press (2023), ISBN: 978-619-7663-72-3.

[14] Kyurkchiev, N., Iliev, A., On the hypothetical oscillator model with second kind Chebyshev's polynomial–correction: number and type of limit cycles, simulations and possible applications, *Algorithms* 2022, 15, 12.

[15] Pavlov N., Efficient Matrix Multiplication Using Hardware Intrinsics And Parallelism With *C#*, *International Journal of Differential Equations and Applications*, 2021, 20, 217–223.

[16] Duffy, J., Concurrent Programming on Windows. Addison Wesley, (2009).

[17] Miller, W., Computational Complexity and Numerical Stability, *SIAM Journal on Computing*, 1975, 4, 97–107.

[18] Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., Dynamics of the Lienard Polynomial System Using Dickson Polynomials of the (M + 1)-th Kind. The Level Curves, *International Journal of Differential Equations and Applications* 2022, 21, 109–120.

[19] N. Kyurkchiev, A. Andreev, *Approximation and Antenna and Filters synthesis. Some Moduli in Programming Environment MATHEMATICA*, LAP LAMBERT Academic Publishing, Saarbrucken, 150 pp., (2014), ISBN: 978-3-659-53322-8.

[20] N. Kyurkchiev, Some Intrinsic Properties of Tadmor-Tanner Functions: Related Problems and Possible Applications, *Mathematics* (2020), 8, 1963.

[21] P. Apostolov and D. G. Valchev, Steering of Two-Element Array Antenna with Arbitrary Narrow Beam Array Factor, 2023 Photonics and Electromagnetics Research Symposium (PIERS), Prague, Czech Republic, 2023, pp. 89-95, doi: 10.1109 /PIERS59004.2023.10221407

[22] M. Baldissera, J. Libre, R. Olivera, Dynamics of a generalized Rayleigh system, *Differential Equations and Dynamical systems*, 2022.

[23] L. Gavrilov, A. Gargouri, B. Hamed, Special cubic perturbations of the Duffing oscillator $x'' = x - x^3$ near the eight–loop, arXiv:1907.00669v3 [math,DS] 24 Mar 2020.

[24] N. Sanchez, A. Nayfeh, Prediction of bifurcations in a parametrically excited Duffing oscillator, *Int. Journal of Nonlinear Mechanics*, 25 (2–3), 1990, 163–176.

[25] G. Michon, L. Manin, R. Parker, R. Dufour, Duffing oscillator with parametric excitation: analytical and experimental investigation on a Belt–Pulley systems (Preprint).

[26] X. Yue, Ge Lv, Y. Zhang, Rare and hidden attractors in a periodically forced Duffing system with absolute nonlinearity, *Chaos, Solitons and Fractals*, 150 (2021).

[27] K. Tang, K. Man, G. Zhong, G. Chen, Generating chaos via $x|x|$, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 48 (5), 2001, 636–641.

[28] L. Gavrilov, I.Iliev, Complete hyperelliptic integrals of the first kind and their non–oscillation, *Trans. of the Amer. Math. Soc.* (2003), 23 pp.