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# NOTES ON A MODIFIED PLANAR MODEL: INTRINSIC PROPERTIES, SIMULATIONS

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**ABSTRACT:** A number of authors devote their research to the phase-space flow of a particle in a forced cubic potential with a specific Hamiltonian. In this paper, we focus on the Hamiltonians, which gives rise to the following modified dynamical system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -x + x^2 + \epsilon(-x + x^2)\displaystyle\sum_{j=1}^{N} g_j \cos(j\omega t) \end{cases}$$

where $0 \leq \epsilon < 1$, $g_i \geq 0$, and $N$ is integer. This will be included as an integral part of a planned much more general Web-based application for scientific computing. Some investigations in the light of Melnikov's approach is considered. The proposed new modification, contain many free parameters (the coefficients $g_i; i = 1, 2, \ldots, N$), which makes it attractive for use in mechanics, chemistry and engineering sciences.

**Key Words:** modified planar model, Melnikov function, chaos

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## 1. INTRODUCTION

A number of authors devote their research to the phase-space flow of a particle in a forced cubic potential with a Hamiltonian:

$$H_\epsilon(x, y, t) = \frac{1}{2}y^2 + \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)(1 + \epsilon\cos(\omega t))$$

where $\epsilon$ and $\omega$ are the two dimensionless parameters measuring the forcing strength and frequency, respectively.

This problem has very direct application in mechanics, chemistry and engineering sciences and can also be considered as a normal form of a more complex Hamiltonian system.

The publications on this topic are significant and varied (see for example [1]–[6]).

The Hamiltonian gives rise to the following dynamical system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -x + x^2 + \epsilon(-x + x^2)\cos(\omega t) \end{cases} \tag{1}$$

where $0 \leq \epsilon < 1$.

In [4] Rom–Kedar calculate the Melnikov function [7] and verify that it has simple zeros [8]–[11]. More precisely,

**Proposition** (Rom-Kedar [4]). The following representation for the Melnikov function holds:

$$M(t_0) = -\frac{3}{5}\pi\omega^2(-1 + \omega^4)\operatorname{csch}(\pi\omega)\sin(t_0\omega). \tag{2}$$

For example, for $\omega = 0.1$, $M(t_0)$ is depicted on Fig.1

In the serious studies cited above, the reader can find a considerable investigations on the topic mixing and transport in the considered driven flow.

In this paper, we suggest a modified planar model with more free parameters. Several simulations are composed.

We demonstrate also some specialized modules for investigating the dynamics of the proposed model.

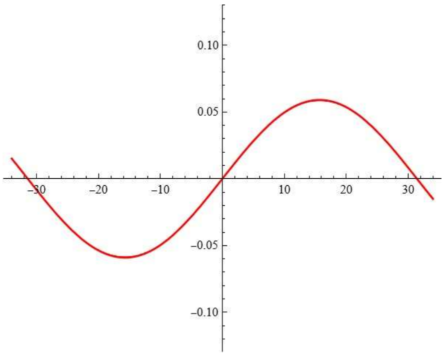


Figure 1: The function $M(t_0)$ from (2).

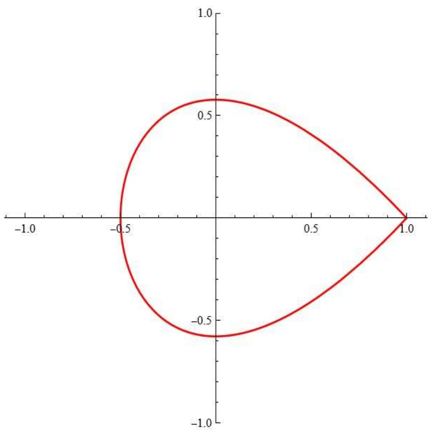


Figure 2: The homoclinic orbit (4).

The derived results can be used as an integral part of a much more general application for scientific computing – for some details see [12]–[22].

## 2. THE NEW MODIFIED MODEL

We consider the following modified planar model

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -x + x^2 + \epsilon(-x + x^2)\displaystyle\sum_{j=1}^{N} g_j \cos(j\omega t) \end{cases} \tag{3}$$

where $0 \leq \epsilon < 1$, $g_i \geq 0$, and $N$ is integer. The Hamiltonian of this system ($\epsilon = 0$) is

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3.$$

The form of the heteroclinic orbit for the unperturbed dynamical system (3) is (see Fig. 2):

$$\begin{aligned} x_0(t) &= 1 - \tfrac{3}{1+\cosh t} \\ y_0(t) &= \tfrac{3\sinh t}{(1+\cosh t)^2} \end{aligned} \tag{4}$$

### 2.1. CONSIDERATIONS IN THE LIGHT OF MELNIKOV APPROACH

The Melnikov function [7] is of the form

$$M(t_0) = \int_{-\infty}^{\infty} y_0(t)(-x_0(t) + x_0^2(t)) \sum_{j=1}^{N} g_j \cos(j\omega(t+t_0))dt. \tag{5}$$

We note that for $g_1 = 1$, $g_2 = g_3 = \ldots = d_N = 0$ we find the known representation (2). We can prove the following proposition when $N = 2$.

**Proposition 1.** *If $N = 2$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation*

$$\begin{aligned} M(t_0) \quad &= \quad -\tfrac{3}{5}\pi\omega^2 \operatorname{csch}(\pi\omega)\left(g_1(-1+\omega^4)\sin(t_0\omega)+\right. \\ &\left. +2g_2(-1+16\omega^4)\operatorname{sech}(\pi\omega)\sin(2t_0\omega)\right) = 0. \end{aligned} \tag{6}$$

For example, the equation $M(t_0) = 0$ (for $N = 2$; $\omega = 0.25$; $g_1 = 0.31$; $g_2 = 0.16$) is depicted on Fig. 3.

From Proposition 1 (see also Fig. 3) the reader may formulate the Melnikov's condition for chaotic behavior of the dynamical model.

We can prove the following proposition when $N = 3$.

**Proposition 2.** *If $N = 3$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation*

$$\begin{aligned} M(t_0) \quad = \quad &\tfrac{3ie^{-3it_0\omega}(-1+e^{2it_0\omega})\pi\omega^2}{20(1+2\cosh(2\pi\omega))}\left(\left(9g_3(-1+81\omega^4) + 9e^{4it_0\omega}g_3(-1+81\omega^4)\right.\right. \\ &+e^{2it_0\omega}\left(2g_1(-1+\omega^4) + 9g_3(-1+81\omega^4)\right)\big)\cosh(\pi\omega) \\ &+e^{it_0\omega}\left(2(1+e^{2it_0\omega})g_2(-1+16\omega^4) + 4(1+e^{2it_0\omega})g_2(-1+16\omega^4)\cosh(2\pi\omega)\right. \\ &\left.\left. +e^{it_0\omega}g_1(-1+\omega^4)\cosh(3\pi\omega)\right)\right)\operatorname{csch}(\pi\omega/2)\operatorname{sech}(\pi\omega/2)\operatorname{sech}(\pi\omega). \end{aligned} \tag{7}$$

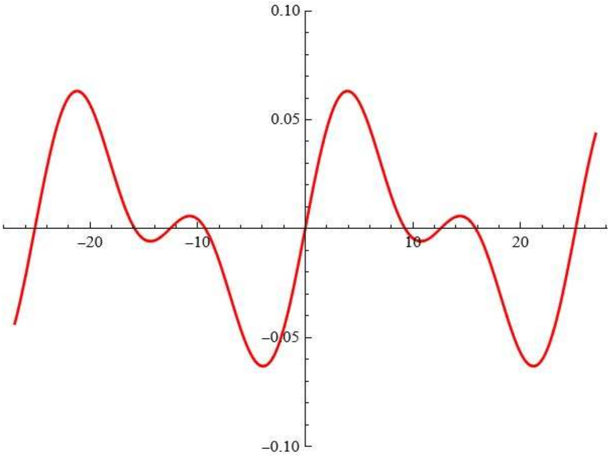


Figure 3: The function $M(t_0)$ (from Proposition 1).

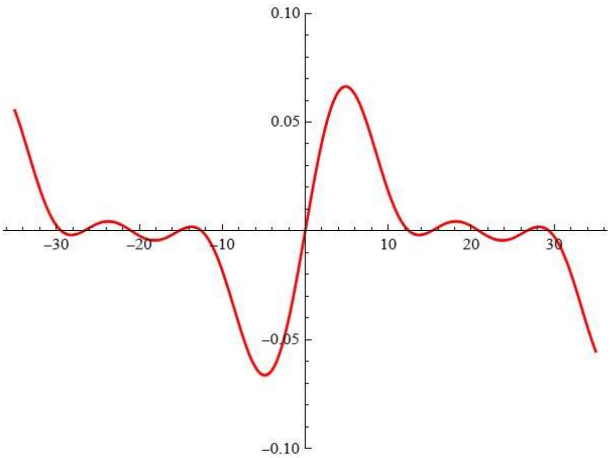


Figure 4: The function $M(t_0)$ (from Proposition 2).

For example, the equation $M(t_0) = 0$ (for $N = 3$; $\omega = 0.15$; $g_1 = 0.35$; $g_2 = 0.21$; $g_3 = 0.09$) is depicted on Fig. 4.

We can prove the following proposition when $N = 4$.

**Proposition 3.** *If $N = 4$, then the roots of Melnikov function $M(t_0)$ are given as* *solutions of the equation*

$$\begin{aligned}
M(t_0) \ = \ & -\tfrac{3}{20} i e^{-4it_0\omega}\pi\omega^2 \left(-e^{3it_0\omega}(-1+e^{2it_0\omega})g_1(-1+\omega^4)\coth(\pi\omega/2)\right. \\
& -4e^{2it_0\omega}(-1+e^{4it_0\omega})g_2(-1+16\omega^4)\coth(\pi\omega) - 9e^{it_0\omega}g_3\coth(3\pi\omega/2) \\
& +9e^{7it_0\omega}g_3\coth(3\pi\omega/2) + 729e^{it_0\omega}g_3\omega^4\coth(3\pi\omega/2) \\
& -729e^{7it_0\omega}g_3\omega^4\coth(3\pi\omega/2) - 16g_4\coth(2\pi\omega) + 16e^{8it_0\omega}g_4\coth(2\pi\omega) \\
& +4096g_4\omega^4\coth(2\pi\omega) - 4096e^{8it_0\omega}g_4\omega^4\coth(2\pi\omega) + e^{3it_0\omega}g_1\tanh(\pi\omega/2) \\
& -e^{5it_0\omega}g_1\tanh(\pi\omega/2) - e^{3it_0\omega}g_1\omega^4\tanh(\pi\omega/2) + e^{5it_0\omega}g_1\omega^4\tanh(\pi\omega/2) \\
& +4e^{it_0\omega}g_2\tanh(\pi\omega) - 4e^{6it_0\omega}g_2\tanh(\pi\omega) \\
& -64e^{2it_0\omega}g_2\omega^4\tanh(\pi\omega) + 64e^{6it_0\omega}g_2\omega^4\tanh(\pi\omega) \\
& +9e^{it_0\omega}g_3\tanh(3\pi\omega/2) - 9e^{7it_0\omega}g_3\tanh(3\pi\omega/2) \\
& -729e^{it_0\omega}g_3\omega^4\tanh(3\pi\omega/2) + 729e^{7it_0\omega}g_3\omega^4\tanh(3\pi\omega/2) \\
& +16g_1\tanh(2\pi\omega) - 16e^{8it_0\omega}g_4\tanh(2\pi\omega) \\
& \left.-4096g_4\omega^4\tanh(2\pi\omega) + 4096e^{8it_0\omega}g_4\omega^4\tanh(2\pi\omega)\right). \qquad (8)
\end{aligned}$$

For example, the equation $M(t_0) = 0$ (for $N = 4$; $\omega = 0.35$; $g_1 = 0.4$; $g_2 = 0.34$; $g_3 = 0.23$; $g_4 = 0.12$) is depicted on Fig. 5.

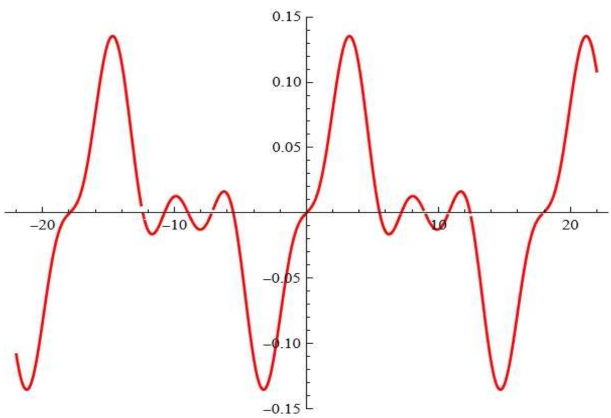


Figure 5: The function $M(t_0)$ (from Proposition 3).

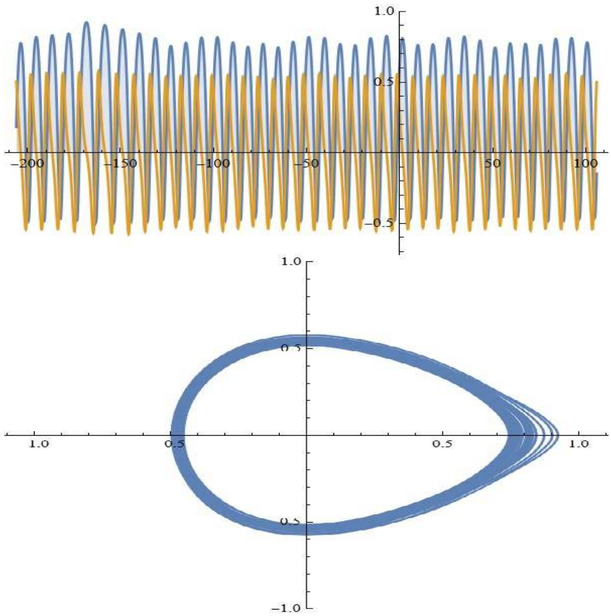


Figure 6: a) The solutions of the system (3); b) phase plot (Example 1).

## 2.2. SOME SIMULATIONS

**Example 1**. For given $N = 4$, $\omega = 0.9$, $g_1 = 2$, $g_2 = 0.01$, $g_3 = 0.02$, $g_4 = 3.9$, $\epsilon = 0.02$ the simulations on the system (3) for $x_0 = 0.3$; $y_0 = 0.089$ are depicted on Fig. 6.

**Example 2**. For given $N = 6$, $\omega = 0.9$, $g_1 = 2$, $g_2 = 0.01$, $g_3 = 0.02$, $g_4 = 0.05$, $g_5 = 0.08$, $g_6 = 3$, $\epsilon = 0.02$ the simulations on the system (3) for $x_0 = 0.5$; $y_0 = 0.2$ are depicted on Fig. 7.

# 3. CONCLUDING REMARKS

If $M(t_0) = 0$ and $\frac{M(t_0)}{dt_0} \neq 0$ for some $t_0$ and some sets of parameters, then chaos occurs.

From Propositions 1–3 the reader may formulate the Melnikov's condition for chaotic behavior of the dynamical model (3).

We note that with large values of the parameter $N$, the work of the dedicated modules implemented in the existing computer algebraic systems for scientific research for

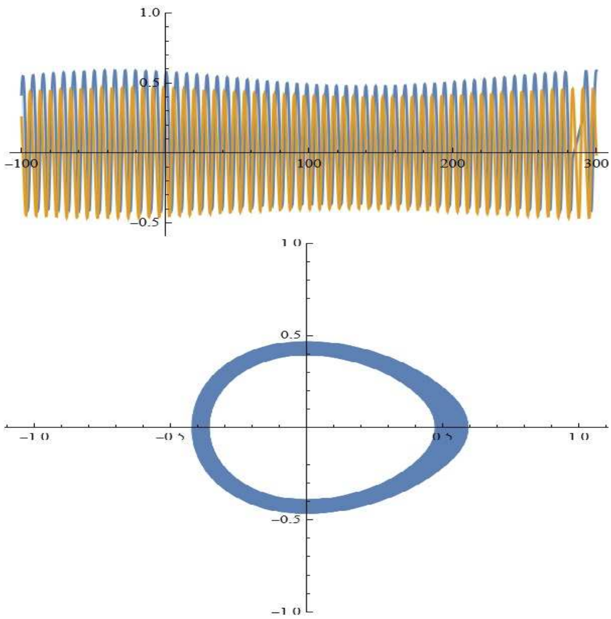


Figure 7: a) The solutions of the system (3); b) phase plot (Example 2).

solving the Melnikov's integrals is very difficult, and the user must also perform serious preliminary preparation and choose appropriate restrictions regarding the parameter $\omega$, for example $|Im(\omega)| \leq \frac{1}{2}$.

Nonstandard numerical methods connected to the investigation of the roots of non-linear equation $M(t_0) = 0$ can be found in [23].

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