International Journal of Differential Equations and Applications
Volume **24**, No. 1 (2025), pages: 1-11
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu



INVESTIGATIONS ON THE CHAOS IN A DYNAMICAL MODEL CORRESPONDING TO A SIMPLE SIR MODEL: ANDRONOV–MELNIKOV APPROACH, SOME SIMULATIONS

Nikolay Kyurkchiev[1,2], Anton Iliev[1,2], Vesselin Kyurkchiev[1]
Maria Vasileva[1], Angel Golev[1,2], Asen Rahnev[1,2]
[1] Faculty of Mathematics and Informatics,
University of Plovdiv Paisii Hilendarski,
24 Tzar Asen Str., 4000 Plovdiv, Bulgaria
[2] Centre of Excellence in Informatics and Information
and Communication Technologies, Sofia, Bulgaria

ABSTRACT: In this article, we consider a modified differential system with a specific potential, containing N number of free parameters, which could be of interest to specialists working in the field of epidemiology. The model correspond to a simple SIR model. With a specially developed software product, we generate the Melnikov equation $M(t) = 0$ and examine all its zeros. This opens up an opportunity for the researcher to correctly understand and formulate the classical Melnikov criterion for the possible occurrence of chaos in the dynamical system. Several simulations are composed.

AMS Subject Classification:
Key Words: dynamical systems with specific potential, homoclinic orbits, Melnikov function

Received: October 3, 2025 **Revised:** November 10, 2025
Published: November 16, 2025 **doi:** 10.12732/ijdea.v24i1.1
Math Team /MT/ – 1991, Ltd. https://www.acadpubl.eu/ap/

1. INTRODUCTION

A number of authors devote their research to the classical differential model:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= \nu_1 + x^2 + \epsilon F(x,y)y \end{cases} \tag{1}$$

where $0 \leq \epsilon \leq 1$ and $\nu_1 < 0$.

The publications on this topic are significant and varied.

The reader can find detailed information in the magnificent studies of Perko [1] and Guckenheimer and Holmes [2].

We direct the reader to [3, 4]; here, the reader can find additional related studies.

A classical SIR model can be described by the following system

$$\begin{cases} \dfrac{dS}{dt} &= \mu(1-S) - \beta(t)I^2S \\ \dfrac{dI}{dt} &= \beta(t)I^2S - (\gamma+\mu)I \\ \dfrac{dR}{dt} &= \gamma I - \mu R \end{cases} \tag{2}$$

which has been extensively studied in [5]–[7]. For some basic considerations on the epidemiological models see [8]–[11].

Melnikov analysis [12] of chaos in a simple SIR model with periodically or stochastically modulated nonlinear incidence rate is considered in [13].

After applying classical transformations and simplifications the normal form (2) reads as [13]

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -c^2 + x^2 + \epsilon\left(b_1 y + xy - 4b_3\sin(\Omega t) - 4b_4\xi(t)\right) + o(\epsilon^2) \end{cases} \tag{3}$$

where $4b_4$ is the intensity of the typical Winner process and $\xi(t)$ is the process itself, and occurs in epidemiology as, an extension of the SIR model (for some details see [13]).

We note that (3) with $b_3 = b_4 = 0$ is a standard normal form for the Takens-Bogdanov bifurcation, and an extensive analysis can be found in [14].

Some dynamical models with specific potential are considered in [15]–[20]. Investigations on some epidemiologycal models and n-stage growth models can be found in [21]–[31].

In this article, we suggest a new modified model. Investigations in the light of Melnikov's approach is considered. Several simulations are composed. We demonstrate also some specialized modules for investigating the dynamics of the modified hypothetical model. The derived results can be used as an integral part of a much more general application for scientific computing.

2. A NEW MODEL: MELNIKOV APPROACH, SOME SIMULATIONS

We consider the following new modified dynamical system of the form:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -1 + x^2 - \epsilon \left(Ay + xy + \displaystyle\sum_{j=1}^{N} g_j \sin(j\omega t) \right), \end{cases} \tag{4}$$

where $0 \leq \epsilon < 1$, A is the damping level, $g_i \geq 0$; $i = 1, \ldots, N$, and N is integer.

The Hamiltonian of this system ($\epsilon = 0$) is $H(x, y) = \dfrac{y^2}{2} + x - \dfrac{x^3}{3}$ and potential energy is $V(x) = x - \dfrac{x^3}{3}$.

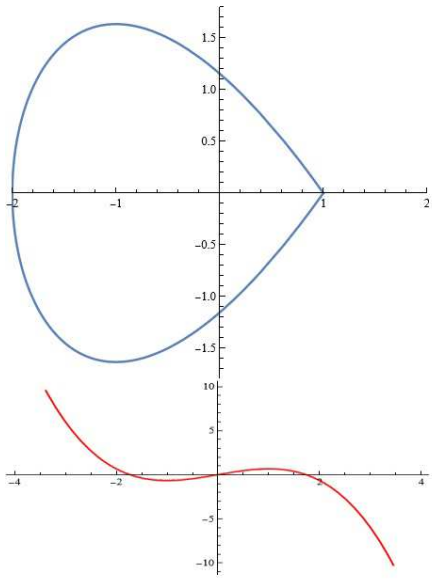

Figure 1. a) the orbits $(x_0(t), y_0(t))$; b) potential energy

The homoclinic orbit is given by (see Fig. 1):

$$\begin{cases} x_0(t) = 1 - 3sech^2\left(\frac{t}{\sqrt{2}}\right) \\ y_0(t) = 3\sqrt{2}sech^2\left(\frac{t}{\sqrt{2}}\right)tanh\left(\frac{t}{\sqrt{2}}\right). \end{cases} \tag{5}$$

Some details can be found in monograph [2].

2.1. CONSIDERATIONS IN THE LIGHT OF MELNIKOV'S APPROACH

The Melnikov function gives a measure of the leading order distance between the stable and unstable manifolds when $\epsilon \neq 0$ and can be used to tell where the stable and unstable manifolds intersect transversely.

By definition, the integral of Melnikov is given by

$$M(t_0) = \int_{-\infty}^{\infty} y_0(t)\left(Ay_0(t) + x_0(t)y_0(t) + \sum_{j=1}^{N} g_i \sin(j\omega(t+t_0))\right) dt \tag{6}$$

where the functions $x_0(t)$ and $y_0(t)$ are defined by equations (5).

From a numerical point of view, the task of finding a root of $M(t_0)$ is more interesting given that the parameters appearing in the proposed differential model are subject to a number of restrictions of a physical and practical nature.

We can prove the following proposition when $N = 2$.

Proposition 1. If $N = 2$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the nonlinear equation

$$\begin{aligned} M(t_0) = & 6g_1\pi\omega^2\cos(t_0\omega)csch\left(\frac{\pi\omega}{\sqrt{2}}\right) \\ & + \frac{24}{35}\left(\sqrt{2}(-5+7A) + 35g_2\pi\omega^2\cos(2t_0\omega)csch(\sqrt{2}\pi\omega)\right) = 0. \end{aligned} \tag{7}$$

For example, the Melnikov function for $N = 2$ in the case A.) $A = 0.72$, $\omega = 0.12$, $g_1 = 1.1$, $g_2 = 1.5$ is depicted on Fig. 2.

The case B.) $A = 0.677$, $\omega = 0.12$, $g_1 = 1.07$, $g_2 = 0.7$. The root with multiplicity two is $t_0 = 25.193$ (see Fig. 3).

If $M(t_0) = 0$ and $\frac{dM(t_0)}{dt_0} \neq 0$ for some t_0 and some sets of parameters, then *chaos* occurs.

We can prove the following proposition when and $N = 3$.

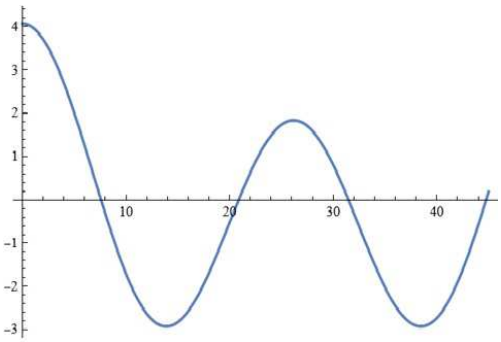

Figure 2. The equation $M(t_0) = 0$ (Proposition 1); the case A

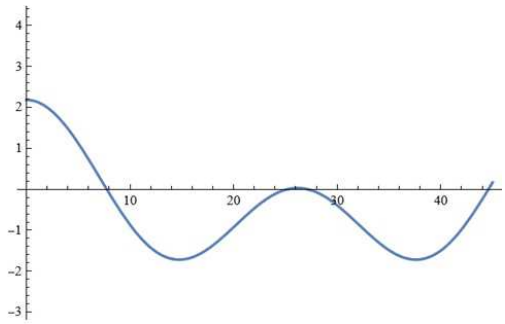

Figure 3. The equation $M(t_0) = 0$ (Proposition 1); the case B

Proposition 2. If $N = 3$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the nonlinear equation

$$\begin{aligned} M(t_0) = &- \frac{24\sqrt{2}}{7} + \frac{24\sqrt{2}A}{5} + 6g_1\pi\omega^2\cos(t_0\omega)csch\left(\frac{\pi\omega}{\sqrt{2}}\right) \\ &+ 24g_2\pi\omega^2\cos(2t_0\omega)csch(\sqrt{2}\pi\omega) \\ &+ 54g_3\pi\omega^2\cos(3t_0\omega)csch\left(\frac{3\pi\omega}{\sqrt{2}}\right) = 0. \end{aligned} \tag{8}$$

For fixed $N = 3$ the equation $M(t_0) = 0$ in the case $A = 0.8$, $\omega = 0.125$, $g_1 = 1.07$, $g_2 = 0.7$, $g_3 = 0.3$ is depicted on Fig. 4.

From Propositions 1–2, the reader may formulate the Melnikov's criterion for the appearance of the intersection between the perturbed and unperturbed separatrixes.

The reader can consider the corresponding approximation problem for arbitrarily chosen N.

We can prove the following proposition when and $N = 4$.

Proposition 3. If $N = 4$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the nonlinear equation

$$\begin{aligned} lclM(t_0) = &- \frac{24\sqrt{2}}{7} + \frac{24\sqrt{2}A}{5} + 6g_1\pi\omega^2\cos(t_0\omega)csch\left(\frac{\pi\omega}{\sqrt{2}}\right) \\ &+ 24g_2\pi\omega^2\cos(2t_0\omega)csch(\sqrt{2}\pi\omega) \\ &+ 54g_3\pi\omega^2\cos(3t_0\omega)csch\left(\frac{3\pi\omega}{\sqrt{2}}\right) \\ &+ 96g_4\pi\omega^2\cos(4t_0\omega)csch(2\sqrt{2}\pi\omega) = 0. \end{aligned} \tag{9}$$

For example, the equation $M(t_0) = 0$ in the case $N = 4$ for $A = 0.858$, $\omega = 0.15$, $g_1 = 1.07$, $g_2 = 0.75$, $g_3 = 0.5$, $g_4 = 0.3$ is depicted on Fig. 5.

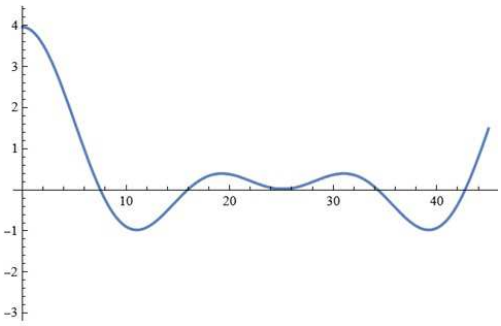

Figure 4. The equation $M(t_0) = 0$ (Proposition 2)

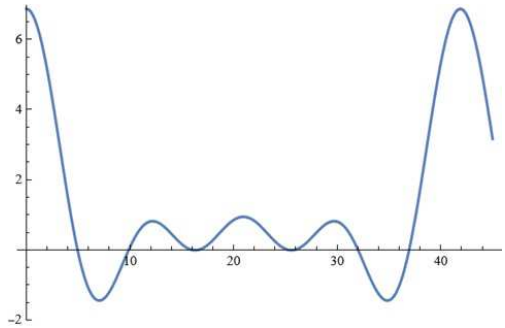

Figure 5. The equation $M(t_0) = 0$ (Proposition 3)

2.2. SOME SIMULATIONS AND APPLICATIONS

We will look at some interesting simulations on model (4):

Example 1. For given $N = 2$, $A = 0.7$, $\omega = 0.6$, $g_1 = 2.7$, $G - 2 = 0.5$, $\epsilon = 0.0095$ the simulations on the system (4) for $x_0 = 0.5$; $y_0 = 0.2$ are depicted on Fig. 6.

Example 2. For given $N = 3$, $A = 0.1$, $\omega = 0.1$, $g_1 = 0.07$, $g_2 = 0.4$, $g_3 = 0.09$, $\epsilon = 0.0095$ the simulations on the system (4) for $x_0 = 0.6$; $y_0 = 0.2$ are depicted on Fig. 7.

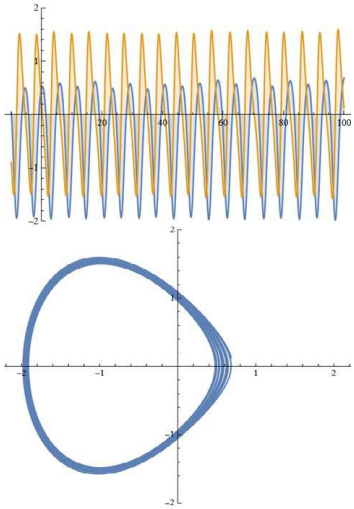

Figure 6. a) the solutions of the system (4) (Example 1)

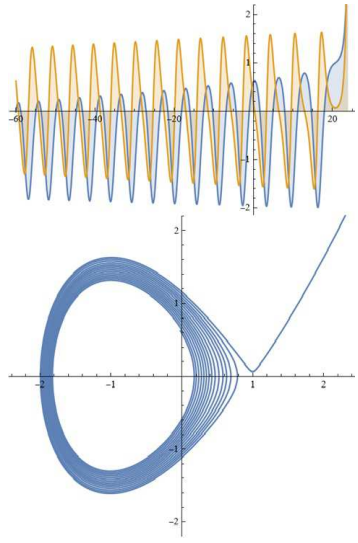

Figure 7. a) the solutions of the system (4) (Example 2)

Example 3. For given $N = 3$, $A = 0.1$, $\omega = 0.1$, $g_1 = 0.07$, $g_2 = 0.4$, $g_3 = 0.09$, $\epsilon = 0.0095$ the simulations on the system (4) for $x_0 = 0.6$; $y_0 = 0.2$ are depicted on Fig. 8.

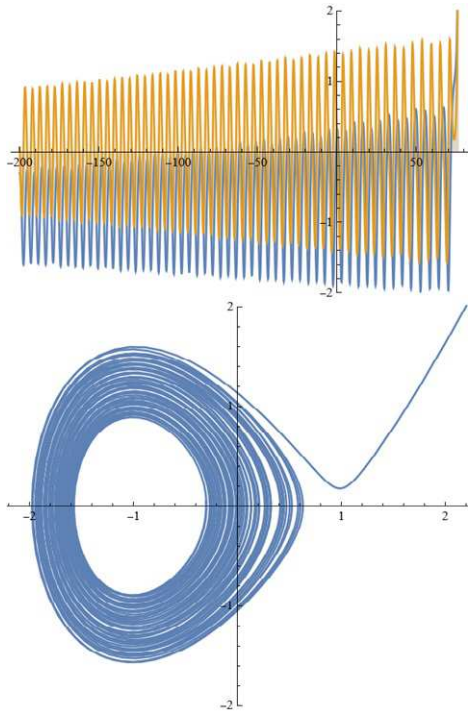

Figure 8. a) the solutions of the system (4) (Example 3)

We define the hypothetical normalized antenna factor as follows:

$$M^*(\theta) = \frac{1}{D}|M(K\cos\theta + k_1)|,$$

where θ is the azimuth angle; $K = kd$; $k = \frac{2\pi}{\lambda}$; λ is the wave length; d is the distance between emitters; k_1 is the phase difference (for some details see [20]).

Following the classical Dolph-Chebyshev, Gegenbauer-like, Jacoby-like, Soltis antenna array, we will call $M^*(\theta)$ the *Melnikov antenna factor*.

A typical Melnikov antenna factor (from Proposition 2) is depicted in Fig. 9.

A typical Melnikov antenna factor (from Proposition 3) is depicted in Fig. 10.

Of course, after serious consideration by specialists working in this scientific direction, the hypothetical Melnikov array proposed by us can be seen as a supplement to the Array Antenna Theory.

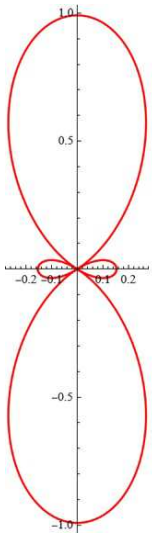

Figure 9. A typical Melnikov antenna array (from Proposition 2)



Figure 10. A typical Melnikov antenna array (from Proposition 3)

Remarks.

1. Some two-sided methods for simultaneous determination of all roots of trigonometric and exponential polynomials of the type $M(t_0) = 0$ are considered in [32].

2. The derived results can be used as an integral part of a much more general application for scientific computing – for some details see [33].

ACKNOWLEDGMENT

The work was supported by the Centre of Excellence in Informatics and ICT under the Grant No BG16RFPR002-1.014-0018-C01, financed by the Research, Innovation and Digitalization for Smart Transformation Programme 2021–2027 and co-financed by the European Union.

REFERENCES

[1] L. Perko, D*ifferential Equations and Dynamical Systems*, Springer: New York, NY, USA, 1991.

[2] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, New York, Springer, 1983.

[3] P. Holmes, J. Marsden, Horseshoes in perturbation of Hamiltonian systems with two degrees of freedom, *Comm. Math. Phys.*, 1982, 82, 523–544.

[4] P. Holmes, J. Marsden, A partial differential equation with infinitely many periodic orbits: Chaotic oscillations, *Arch. Ration. Mech. Anal.*, 1981, 76, 135–165.

[5] X. Liu, P. Stechlinski, Pulse and constant control schemes for epidemic models with seasonality, *Nonlinear Anal. Real World Appl.*, 2011, 12, pp. 931–946.

[6] X. Liu, P. Stechlinski, Infectious disease models with time-varying parameters and general nonlinear incidence rate, *Appl. Math. Model.*, 2012, 36, pp. 1974–1994.

[7] S. O'Regan, T. Kelly, A. Korobeinikov, M. O'Callaghan, A. Pokrovskii, D. Rachinskii, Chaos in a seasonally perturbed SIR model: Avian influenza in a seabird colony as a paradigm, *J. Math. Biol.*, 2013, 67, pp. 293–327.

[8] W. Kermack, A. McKendrick, Contributions to the mathematical theory of epidemics I, *Proc. R. Soc. A*, 1927, 115, 700–721.

[9] W. Kermack, A. McKendrick, Contribution to the mathematical theory of epidemics II, *Proc. R. Soc. A*, 1932, 138, 55–83.

[10] W. Kermack, A. McKendrick, Contributions to the mathematical theory of epidemics III, *Proc. R. Soc. A*, 1933, 141, 94–122.

[11] H. Hethcote, The mathematics of infectious diseases, *SIAM Rev.*, 2000, 42, 599–653.

[12] V. Melnikov, On the stability of the center for time periodic pertubbations, *Trans, Moskow Math, Soc.*, 1963, 12, 1–57.

[13] Yanxiang Shi, Melnikov analysis of chaos in a simple SIR model with periodically or stochastically modulated nonlinear incidence rate, *Journal of Biological Dynamics*, 2020, 14:1, 269–288.

[14] S. Chow, C. Li, D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge, Cambridge University Press, 1994.

[15] N. Kyurkchiev, T. Zaevski, A. Iliev, V. Kyurkchiev, A. Rahnev, Modeling of Some Classes of Extended Oscillators: Simulations, Algorithms, Generating Chaos, and Open Problems, *Algorithms*, 2024, 17, 121.

[16] N. Kyurkchiev, T. Zaevski, A. Iliev, V. Kyurkchiev, A. Rahnev, Generating Chaos

in Dynamical Systems: Applications, Symmetry Results, and Stimulating Examples, *Symmetry*, 2024, 16, 938.

[17] N. Kyurkchiev, T. Zaevski, A. Iliev, V. Kyurkchiev, A. Rahnev, Investigations of Modified Classical Dynamical Models: Melnikovs Approach, Simulations and Applications, and Probabilistic Control of Perturbations, *Mathematics*, 2025, 13, 231.

[18] N. Kyurkchiev, T. Zaevski, A. Iliev, V. Kyurkchiev, A. Rahnev, Dynamics of a class of chemical oscillators with asymmetry potential: Simulations and control over oscillations, *Mathematics*, 2025, 13, 1129.

[19] N. Kyurkchiev, A. Iliev, V. Kyurkchiev, A. Rahnev, One More Thing on the Subject: Generating Chaos via $x|x|^{a-1}$, Melnikovs Approach Using Simulations, *Mathematics*, 2025, 13, 232.

[20] N. Kyurkchiev, T. Zaevski, A. Iliev, V. Kyurkchiev, A. Rahnev, Studying homoclinic chaos in a class of piecewise smooth oscillators: Melnikovs approach, simulations and applications to generating antenna factors using approximation and optimization techniques, *Symmetry*, 17 (7), 2025, 1144.

[21] M. Borisov, S. Markov,The two-step exponential decay reaction network: Analysis of the solutions and relation to epidemiological SIR models with logistic and Gompertz type infection contact patterns, *J. Math. Chem.*, 2021, 59, 1283–1315.

[22] N. Kyurkchiev, S. Markov, On the numerical solution of the general kinetic K-angle reaction system, *J. Math. Chem.*, 2016, 54, 792–805.

[23] R. Anguelov, M. Borisov, A. Iliev, N. Kyurkchiev, S. Markov, On the chemical meaning of some growth models possessing Gompertziantype property, *Math. Methods Appl. Sci.*, 2018, 41, 8365–8376.

[24] S. Markov, A. Iliev, A. Rahnev, N. Kyurkchiev, A note on the n-stage growth model. Overview, *Biomath Commun.*, 2018, 5, 79–100.

[25] N. Kyurkchiev, A note on the new geometric representation for the parameters in the fibril elongation process, *C. R. Acad. Bulg. Sci.*, 2016, 69, 8, 963–972.

[26] N. Kyurkchiev, A. Iliev, A. Rahnev, On the Verhulst growth model with "polynomial variable transfer". Some simulations, *International Journal of Differential Equations and Applications*, 2020, Volume 19, No. 1, 15–32.

[27] N. Kyurkchiev, A. Iliev, A. Rahnev, On the half–logistic model with "polynomial variable transfer". Application to approximate the specific "data Corona virus",

International Journal of Differential Equations and Applications, 2020, Volume 19, No. 1, 45–61.

[28] N. Kyurkchiev, V. Kyurkchiev, A. Iliev, A. Rahnev, A new modifications of the SIR/SEIR models with "intervention polynomial factor". Methodological aspects, *International Journal of Differential Equations and Applications*, 2021, Volume 20, No. 1, 15–30.

[29] Kyurkchiev, N., Kyurkchiev, V., Iliev, A., Rahnev, A., A Look at the Modified SIRD Models with "Intervention Polynomial Factor". Methodological Aspects. II, *International Journal of Differential Equations and Applications*, 2021, 20, 1, 31–41.

[30] N. Kyurkchiev, G. Boyadjiev, Dynamics of modified LotkaVolterra model with "polynomial intervention factors". III, *International Journal of Differential Equations and Applications*, 2021, Volume 20, No. 1, 121–132.

[31] N. Kyurkchiev, V. Kyurkchiev, A. Iliev, A. Rahnev, *Another look at SIR/SIRD/-SEIR/GSEIR models: New trends, methodological aspects*, Plovdiv, Plovdiv University Press, 2021, ISBN: 978-619-202-685-1.

[32] I. Makrelov, N. Kyurkchiev, S. Tamburov, Two two-sided methods for simultaneous determination of all roots of trigonometric and exponential polynomials, *Travaux scientifiques Univ. Plovdiv, Fac. Math.*, 1985, 23, 2, 289–298, (in Bulgarian).

[33] A. Golev, T. Terzieva, A. Iliev, A. Rahnev, N. Kyurkchiev, Simulation on a generalized oscillator model: web-based application, *C. R. L'Acad. Bulg. Sci.*, 2024, 77, 2, 230–237.