

HOMOCLINIC BIFURCATION IN AN DUAL SIQR MODEL FOR CHILDHOOD DISEASES: SOME SIMULATIONS AND APPLICATIONS, CHALLENGES FOR LEARNERS

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ABSTRACT: Many authors consider a system of ODEs which describes the transmission dynamics of modified SIQR (susceptible, infected, isolated (quarantined), and recovered individuals) models for childhood diseases. In this article, we study homoclinic bifurcation in a modified differential system (the so-called by us – dual SIQR model for childhood diseases) containing N number of free parameters, which could be of interest to specialists working in the field of epidemiology. With a specially developed software product, we generate the Melnikov equation $M(t) = 0$ and examine all its zeros. This opens up an opportunity for the researcher to correctly understand and formulate the classical Andronov–Melnikov criterion for the possible occurrence of chaos in the dynamical system. We demonstrate also some specialized modules for investigating the dynamics of the new model. A specialized module has been developed and implemented in CAS Mathematica for generating a Melnikov antenna factor.

Key Words: dynamical systems with specific potential, homoclinic bifurcation, Melnikov function, dual SIQR model for childhood diseases, Melnikov antenna factor

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1. INTRODUCTION

Many authors consider modified SIQR models for childhood diseases. The reader can find detailed information in the studies [1, 2], as well as additional bibliography.

In [2] the authors derive an analytic understanding of the complex dynamics that may arise from the SIQR (susceptible, infected, isolated (quarantined), and recovered individuals) model.

Tens of thousands of publications are known in the literature on the topic of COVID-19.

Analysis of the COVID-19 epidemic in different countries using the SIQR model has been the subject of research by a number of authors (see, for example, [3, 4], where the reader can find relevant additional literature).

Using tools in dynamical systems theory such as center manifold reduction, normal form, and unfolding methods it is shown that homoclinic bifurcation exists in a biologically reasonable region of the parameter and state space.

We will not dwell on these classic techniques here.

For the global theory of Hopf bifurcation for differential equations, see, e.g., [5]–[15].

Melnikov analysis of chaos in a general epidemiological model can be found in [16]–[18].

More precisely, in [2] the authors consider the following differential model corresponding to the SIQR model

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = -y + x^2 - \epsilon(\alpha y + xy) \end{cases} \quad (1)$$

where $0 \leq \epsilon < 1$, $\alpha > 0$ and studied the homoclinic bifurcation.

In [19] the following modified model is considered

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = -y + x^2 - \epsilon \left(\alpha y + xy + y \sum_{j=1}^N a_j \cos(j\omega t) \right) \end{cases} \quad (2)$$

where $0 \leq \epsilon < 1$, $\alpha > 0$, $\omega > 0$, $a_i \geq 0$, $i = 1, 2, \dots, N$, and N is integer.

Research in this interesting direction can be successfully expanded.

In this article, we consider a modified differential system containing N number of free parameters, which could be of interest to specialists working in the field of epidemiology.

We pay special attention to the Melnikov function [20], corresponding to the proposed new model.

It is known that if $M(t_0) = 0$ and $\frac{dM(t_0)}{dt_0} \neq 0$ for some t_0 and some sets of parameters, then chaos occurs in the considered differential model.

With a specially developed software product, we generate the Melnikov equation $M(t) = 0$ and examine all its zeros.

This opens up an opportunity for the researcher to correctly understand and formulate the classical Melnikov criterion for the possible occurrence of chaos in the dynamical system.

We demonstrate also some specialized modules for investigating the dynamics of the new model.

In this article, we offer an efficient strategy for training our master's and doctoral students on this interesting topic and at the same time we hope to provoke a discussion on teaching methodology.

2. A NEW MODEL: MELNIKOV APPROACH, SOME SIMULATIONS

We consider the following new modified dynamical system of the form:

$$\begin{cases} \frac{dx}{dt} = y - x \\ \frac{dy}{dt} = y - x^2 - \epsilon \left(\alpha y + xy + y \sum_{j=1}^N a_j \cos(j\omega t) \right) \end{cases} \quad (3)$$

where $0 \leq \epsilon < 1$, $\alpha > 0$, $\omega > 0$, $a_i \geq 0$, $i = 1, 2, \dots, N$, and N is integer.

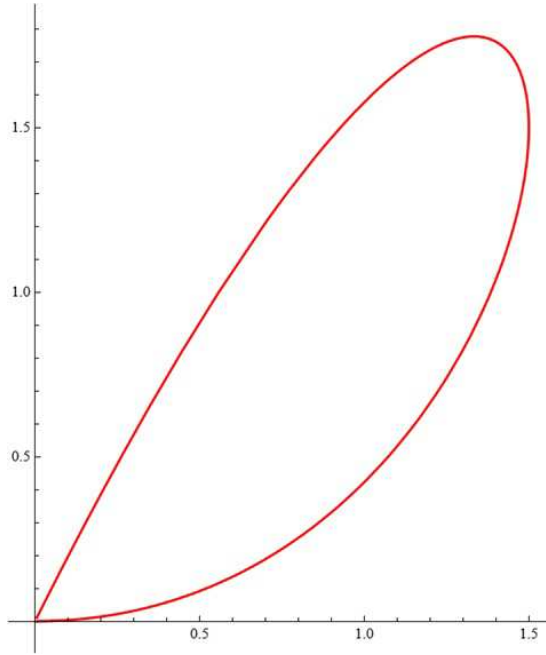


Figure 1. The homoclinic orbit

2.1. MELNIKOV APPROACH

The Hamiltonian ($\epsilon = 0$) of the system (3) is

$$H(x, y) = \frac{y^2}{2} - xy + \frac{x^3}{3}.$$

We obtain the homoclinic orbit through the following theorem:

Theorem 1. The homoclinic orbit of the unperturbed version of system (3) is given by

$$\begin{cases} x_h = \frac{3}{2} \operatorname{sech}^2\left(\frac{t}{2}\right) \\ y_h = \frac{3}{2} \operatorname{sech}^2\left(\frac{t}{2}\right) \left(1 - \tanh\left(\frac{t}{2}\right)\right) \end{cases} \quad (4)$$

Proof. To find equilibrium points, we use the Hamiltonian equation of motion:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} = y - x \\ \dot{y} &= -\frac{\partial H}{\partial x} = y - x^2. \end{aligned}$$

Setting $\dot{x} = 0$ gives $y = x$.

Substituting this into $\dot{y} = 0$ gives $x - x^2 = 0$ which yields two fixed points $(0, 0)$ and $(1, 1)$.

By linearizing the system we find the Jacobian

$$J = \begin{pmatrix} -1 & 1 \\ -2x & 1 \end{pmatrix}.$$

At $(0, 0)$, the eigenvalues are ± 1 , making it a saddle point. At $(1, 1)$, the eigenvalues are $\pm i$, making it a center.

The homoclinic orbit begin and ends at the saddle point $(0, 0)$, where the energy $H(0, 0) = 0$.

Since the orbit lies on the level $H(x, y) = 0$, we solve $y = x \pm x\sqrt{1 - \frac{2}{3}x}$.

Substituting this expression for y back into $\dot{x} = y - x$ we have $\frac{dx}{dt} = \pm x\sqrt{1 - \frac{2}{3}x}$.

Choosing the branch for the loop ($x > 0$) and integrating using the substitution $\frac{3}{2}\text{sech}^2(u)$, we find $u = \frac{t}{2}$.

This yields the $x(t)$ coordinate. The $y(t)$ coordinate is then found by calculating $y = x + \dot{x}$.

Thus the theorem is proven.

The homoclinic orbit is shown in Fig. 1.

System (3) is of the form

$$\begin{cases} \frac{dx}{dt} &= f_1(x, y) \\ \frac{dy}{dt} &= f_2(x, y) - \epsilon g_2(x, y, t). \end{cases} \quad (5)$$

We prove the existence of homoclinic orbits using Melnikov's method.

We note that the Melnikov function [20] corresponding to model (5) is of the form

$$M(t_0) = \int_{-\infty}^{\infty} (y_h - x_h)y_h \left(\alpha + x_h + \sum_{j=1}^N a_j \cos(j\omega(t + t_0)) \right) dt \quad (6)$$

where the homoclinic orbit $\{x_h, y_h\}$ is given by (4).

$$\begin{aligned}
M(t_0) := & \frac{3}{140} e^{-\frac{3}{2} i \omega} \left(28 a_1 + 48 e^{\frac{3}{2} i \omega} + 28 a_1 e^{\frac{3}{2} i \omega} + 56 e^{\frac{3}{2} i \omega} a + 70 i a_1 \omega - 70 i a_1 e^{\frac{3}{2} i \omega} \omega - 56 a_1 \omega^2 - 56 a_1 e^{\frac{3}{2} i \omega} \omega^2 - 70 i a_1 \omega^3 + 70 i a_1 e^{\frac{3}{2} i \omega} \omega^3 + 28 a_1 \omega^4 + 28 a_1 e^{\frac{3}{2} i \omega} \omega^4 + \right. \\
& 7 a_1 \omega (1 + \omega^2) (2 i - 5 \omega - 2 i \omega^2 + e^{\frac{3}{2} i \omega} (2 i + 5 \omega - 2 i \omega^2)) \text{PolyGamma}\left[0, 1 - \frac{i \omega}{2}\right] + 7 a_1 \omega (1 + \omega^2) (-2 i + 5 \omega + 2 i \omega^2 + e^{\frac{3}{2} i \omega} (-2 + 5 i \omega + 2 \omega^2)) \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i \omega}{2}\right] - \\
& 14 i a_1 \omega \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] - 14 i a_1 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 35 a_1 \omega^2 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] - 35 a_1 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 35 a_1 \omega^4 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] - \\
& 35 a_1 e^{\frac{3}{2} i \omega} \omega^4 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 14 i a_1 \omega^5 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 14 i a_1 e^{\frac{3}{2} i \omega} \omega^5 \text{PolyGamma}\left[0, 1 + \frac{i \omega}{2}\right] + 14 i a_1 \omega \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] + 14 i a_1 e^{\frac{3}{2} i \omega} \omega \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] - \\
& 35 a_1 \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] + 35 a_1 e^{\frac{3}{2} i \omega} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] - 35 a_1 \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] + 35 a_1 e^{\frac{3}{2} i \omega} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] - 14 i a_1 \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] \\
& \left. - 14 i a_1 e^{\frac{3}{2} i \omega} \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i \omega}{2}\right] \right) = 0
\end{aligned}$$

Figure 2. The answer to $M(t_0) = 0$ utilizing Proposition 1 of CAS Mathematica

Remarks:

1. For any arbitrary N , the reader can define the Melnikov criterion for the occurrence of chaos in the dynamical system under consideration.
2. The verification of the conditions: $M(t_0) = 0$ and $\frac{dM(t_0)}{dt_0} \neq 0$ implies the knowledge of all zeros of specific nonlinear equations.

This requires the use of specialized numerical algorithms for simultaneously finding all zeros (simple or multiple) of nonlinear equations in confidential intervals (see for example [21]).

3. The planned by us WEB-based application [22] has such a specialized module implemented in the CAS Mathematica. An interesting approach for the qualitative graphical representation of the describing function in nonlinear systems stability analysis, the reader can find in [23]. The effect of the perturbed factor $\sum_{j=1}^N a_j \sin(j\omega t)$ on the dynamics of some differential models is investigated in [24, 25].

The following is valid:

Proposition 1. The solutions of the equation in Fig. 2 are the roots of the Melnikov function $M(t_0)$ if $N = 1$, $\alpha > 0$.

Example 1.a. For given $N = 1$, $\alpha = 0.01$, $\omega = 0.19$, $a_1 = 1.1$, the Melnikov function $M(t_0)$ is depicted in Figure 3.

We see that the roots (in confidential interval) are simple. The stable and unstable manifolds intersect transversely, and thus differential system exhibits a chaotic behavior in the sense of Smale-Birkhoff [26]–[28] for small enough values of ϵ .

Example 1.b. For given $N = 1$, $\alpha = 0.2$, $\omega = 0.19$, $a_1 = 1$, the Melnikov function $M(t_0)$ is depicted in Figure 4. We see that $M(t_0)$ has multiple roots with multiplicity two.

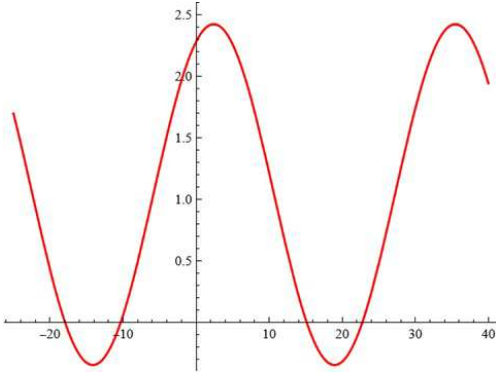


Figure 3. The Melnikov function $M(t_0)$
(Example 1.a)

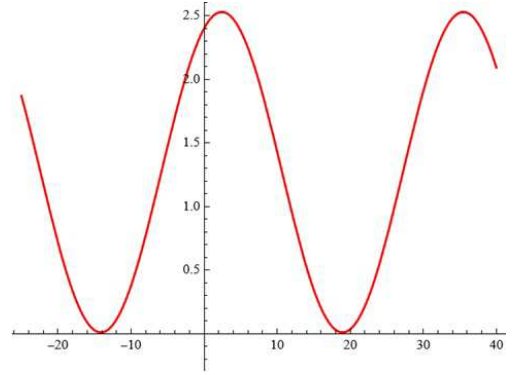


Figure 4. The Melnikov function $M(t_0)$
(Example 1.b)

The following is valid

Proposition 2. The solutions of the equation in Fig. 5 are the roots of the Melnikov function $M(t_0)$ if $N = 3$, $\alpha > 0$.

$$\begin{aligned}
 M(t_0) = & \frac{3}{140} e^{-3i\omega t_0} \left[28a_3 + 28a_2 e^{i\omega t_0} + 28a_1 e^{2i\omega t_0} + 48e^{3i\omega t_0} + 28a_1 e^{4i\omega t_0} + 28a_2 e^{5i\omega t_0} + 28a_3 e^{6i\omega t_0} + 56e^{7i\omega t_0} + 210i a_3 \omega + 140i a_2 \omega e^{i\omega t_0} + 70i a_1 \omega e^{2i\omega t_0} - 70i a_1 \omega e^{4i\omega t_0} \right. \\
 & - 140i a_2 \omega e^{5i\omega t_0} - 210i a_3 \omega e^{6i\omega t_0} - 504a_3 \omega^2 - 224a_2 e^{i\omega t_0} \omega^2 - 56a_1 e^{2i\omega t_0} \omega^2 - 56a_1 e^{4i\omega t_0} \omega^2 - 224a_2 e^{5i\omega t_0} \omega^2 - 504a_3 e^{6i\omega t_0} \omega^2 - 1890i a_3 \omega^3 - 560i a_2 \omega e^{i\omega t_0} \omega^3 - 70i a_1 \omega e^{2i\omega t_0} \omega^3 \\
 & + 70i a_1 \omega e^{4i\omega t_0} \omega^3 + 560i a_2 \omega e^{5i\omega t_0} \omega^3 + 1890i a_3 \omega e^{6i\omega t_0} \omega^3 + 2268a_3 \omega^4 + 448a_2 e^{i\omega t_0} \omega^4 + 28a_1 e^{2i\omega t_0} \omega^4 + 28a_1 e^{4i\omega t_0} \omega^4 + 448a_2 e^{5i\omega t_0} \omega^4 + 2268a_3 e^{6i\omega t_0} \omega^4 + 7a_1 e^{2i\omega t_0} \omega (1 + \omega^2) (2i - 5\omega - 2i\omega^2 \\
 & + e^{2i\omega t_0} (2i + 5\omega - 2i\omega^2)) \text{PolyGamma}\left[0, 1 - \frac{i\omega}{2}\right] + 7a_1 e^{2i\omega t_0} \omega (1 + \omega^2) (-2i + 5\omega + 2i\omega^2 + i e^{2i\omega t_0} (-2 + 5i\omega + 2\omega^2)) \text{PolyGamma}\left[0, \frac{3}{2} - \frac{i\omega}{2}\right] - 14i a_1 e^{2i\omega t_0} \omega \text{PolyGamma}\left[0, 1 + \frac{i\omega}{2}\right] - \\
 & 14i a_1 e^{4i\omega t_0} \omega \text{PolyGamma}\left[0, 1 + \frac{i\omega}{2}\right] + 35a_1 e^{2i\omega t_0} \omega^2 \text{PolyGamma}\left[0, 1 + \frac{i\omega}{2}\right] - 35a_1 e^{4i\omega t_0} \omega^2 \text{PolyGamma}\left[0, 1 + \frac{i\omega}{2}\right] + 35a_1 e^{2i\omega t_0} \omega^4 \text{PolyGamma}\left[0, 1 + \frac{i\omega}{2}\right] - 35a_1 e^{4i\omega t_0} \omega^4 \text{PolyGamma}\left[0, 1 + \frac{i\omega}{2}\right] + \\
 & 14i a_1 e^{2i\omega t_0} \omega^5 \text{PolyGamma}\left[0, 1 + \frac{i\omega}{2}\right] + 14i a_1 e^{4i\omega t_0} \omega^5 \text{PolyGamma}\left[0, 1 + \frac{i\omega}{2}\right] + 14i a_1 e^{2i\omega t_0} \omega \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i\omega}{2}\right] + 14i a_1 e^{4i\omega t_0} \omega \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i\omega}{2}\right] - 35a_1 e^{2i\omega t_0} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i\omega}{2}\right] \\
 & + 35a_1 e^{4i\omega t_0} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i\omega}{2}\right] - 35a_1 e^{2i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i\omega}{2}\right] + 35a_1 e^{4i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i\omega}{2}\right] - 14i a_1 e^{2i\omega t_0} \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i\omega}{2}\right] - 14i a_1 e^{4i\omega t_0} \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{i\omega}{2}\right] \\
 & + 28i a_2 e^{i\omega t_0} \omega \text{PolyGamma}\left[0, 1 - i\omega\right] + 28i a_2 e^{3i\omega t_0} \omega \text{PolyGamma}\left[0, 1 - i\omega\right] - 140a_2 e^{i\omega t_0} \omega^2 \text{PolyGamma}\left[0, 1 - i\omega\right] + 140a_2 e^{3i\omega t_0} \omega^2 \text{PolyGamma}\left[0, 1 - i\omega\right] - 560a_2 e^{i\omega t_0} \omega^4 \text{PolyGamma}\left[0, 1 - i\omega\right] + \\
 & 560a_2 e^{3i\omega t_0} \omega^4 \text{PolyGamma}\left[0, 1 - i\omega\right] - 448i a_2 e^{i\omega t_0} \omega^5 \text{PolyGamma}\left[0, 1 - i\omega\right] - 448i a_2 e^{3i\omega t_0} \omega^5 \text{PolyGamma}\left[0, 1 - i\omega\right] - 28i a_2 e^{i\omega t_0} \omega \text{PolyGamma}\left[0, \frac{3}{2} - i\omega\right] - 28i a_2 e^{3i\omega t_0} \omega \text{PolyGamma}\left[0, \frac{3}{2} - i\omega\right] + \\
 & 140a_2 e^{i\omega t_0} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} - i\omega\right] - 140a_2 e^{3i\omega t_0} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} - i\omega\right] + 560a_2 e^{i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} - i\omega\right] - 560a_2 e^{3i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} - i\omega\right] + 448i a_2 e^{i\omega t_0} \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} - i\omega\right] + \\
 & 448i a_2 e^{3i\omega t_0} \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} - i\omega\right] - 28i a_2 e^{i\omega t_0} \omega \text{PolyGamma}\left[0, 1 + i\omega\right] - 28i a_2 e^{3i\omega t_0} \omega \text{PolyGamma}\left[0, 1 + i\omega\right] + 140a_2 e^{i\omega t_0} \omega^2 \text{PolyGamma}\left[0, 1 + i\omega\right] - 140a_2 e^{3i\omega t_0} \omega^2 \text{PolyGamma}\left[0, 1 + i\omega\right] + \\
 & 560a_2 e^{i\omega t_0} \omega^4 \text{PolyGamma}\left[0, 1 + i\omega\right] - 560a_2 e^{3i\omega t_0} \omega^4 \text{PolyGamma}\left[0, 1 + i\omega\right] + 448i a_2 e^{i\omega t_0} \omega^5 \text{PolyGamma}\left[0, 1 + i\omega\right] + 448i a_2 e^{3i\omega t_0} \omega^5 \text{PolyGamma}\left[0, 1 + i\omega\right] + 28i a_2 e^{i\omega t_0} \omega \text{PolyGamma}\left[0, \frac{3}{2} + i\omega\right] + \\
 & 28i a_2 e^{3i\omega t_0} \omega \text{PolyGamma}\left[0, \frac{3}{2} + i\omega\right] - 140a_2 e^{i\omega t_0} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + i\omega\right] + 140a_2 e^{3i\omega t_0} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + i\omega\right] - 560a_2 e^{i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + i\omega\right] + 560a_2 e^{3i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + i\omega\right] - \\
 & 448i a_2 e^{i\omega t_0} \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} + i\omega\right] - 448i a_2 e^{3i\omega t_0} \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} + i\omega\right] + 42i a_3 \omega^3 \text{PolyGamma}\left[0, 1 - \frac{3i\omega}{2}\right] + 42i a_3 e^{6i\omega t_0} \omega^3 \text{PolyGamma}\left[0, 1 - \frac{3i\omega}{2}\right] - 315a_3 \omega^2 \text{PolyGamma}\left[0, 1 - \frac{3i\omega}{2}\right] + \\
 & 315a_3 e^{6i\omega t_0} \omega^2 \text{PolyGamma}\left[0, 1 - \frac{3i\omega}{2}\right] - 2835a_3 \omega^4 \text{PolyGamma}\left[0, 1 - \frac{3i\omega}{2}\right] + 2835a_3 e^{6i\omega t_0} \omega^4 \text{PolyGamma}\left[0, 1 - \frac{3i\omega}{2}\right] - 3402i a_3 \omega \text{PolyGamma}\left[0, 1 - \frac{3i\omega}{2}\right] - 3402i a_3 e^{6i\omega t_0} \omega \text{PolyGamma}\left[0, 1 - \frac{3i\omega}{2}\right] - \\
 & 42i a_3 \omega \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3i\omega}{2}\right] - 42i a_3 e^{6i\omega t_0} \omega \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3i\omega}{2}\right] + 315a_3 \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3i\omega}{2}\right] - 315a_3 e^{6i\omega t_0} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3i\omega}{2}\right] - 2835a_3 \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3i\omega}{2}\right] - 2835a_3 e^{6i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3i\omega}{2}\right] - \\
 & 2835a_3 e^{6i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3i\omega}{2}\right] + 3402i a_3 \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} - \frac{3i\omega}{2}\right] - 2835a_3 \omega^5 \text{PolyGamma}\left[0, 1 + \frac{3i\omega}{2}\right] - 2835a_3 e^{6i\omega t_0} \omega^5 \text{PolyGamma}\left[0, 1 + \frac{3i\omega}{2}\right] + 3402i a_3 \omega^5 \text{PolyGamma}\left[0, 1 + \frac{3i\omega}{2}\right] + \\
 & 315a_3 \omega^2 \text{PolyGamma}\left[0, 1 + \frac{3i\omega}{2}\right] - 315a_3 e^{6i\omega t_0} \omega^2 \text{PolyGamma}\left[0, 1 + \frac{3i\omega}{2}\right] + 42i a_3 \omega \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3i\omega}{2}\right] + 42i a_3 e^{6i\omega t_0} \omega \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3i\omega}{2}\right] - 315a_3 \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3i\omega}{2}\right] - 315a_3 e^{6i\omega t_0} \omega^2 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3i\omega}{2}\right] - \\
 & 2835a_3 \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3i\omega}{2}\right] - 2835a_3 e^{6i\omega t_0} \omega^4 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3i\omega}{2}\right] - 3402i a_3 \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3i\omega}{2}\right] - 3402i a_3 e^{6i\omega t_0} \omega^5 \text{PolyGamma}\left[0, \frac{3}{2} + \frac{3i\omega}{2}\right] = 0
 \end{aligned}$$

Figure 5. The answer to $M(t_0) = 0$ utilizing Proposition 2 of CAS Mathematica

Example 2. For given $N = 3$, $\alpha = 1.8$, $\omega = 0.4$, $a_1 = 2.1$, $a_2 = 0.1$, $a_3 = 2.3$ the Melnikov function $M(t_0)$ is depicted in Figure 6.

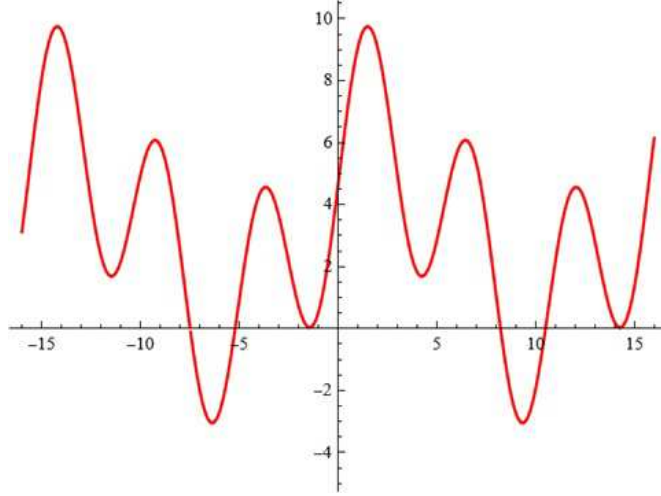


Figure 6. The Melnikov function $M(t_0)$ (Example 2)

We see that $M(t_0)$ has single and multiple roots (in confidential interval) of multiplicity two and thus we have a tangent contact between the stable and unstable manifolds.

This is an indicator for a complex (bifurcational) behavior.

Remarks:

1. From Propositions 1 and Propositions 2, we see that the Melnikov function is expressed by *PolyGamma* function, where *PolyGamma* $[n, z]$ gives the n^{th} derivative of the *digamma function*, $\psi^{(n)}(z)$, i.e., $\psi^{(0)}(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

For more details, see [29, 30].

2. The explicit representation of $M(t_0)$ (especially for high values of the parameter N) is time-consuming.

In addition, the user needs to specify reasonable restrictions on some of the parameters of the studied differential model when calculating such integrals in the intended modules implemented in the existing software environments for scientific calculations.

One possibility in this direction is the restriction $|Im(\omega)| < \frac{2}{N}$.

As we have already noted in our other articles, the user does not have to be a professional mathematician, which requires upgrading the existing modules.

Figures 1 and 4 illustrates the calculation of $M(t_0)$ using CAS Mathematica.

We include this figure to illustrate the difficulties that the user encounters.

The above-mentioned WEB application [22] was used significantly to generate the equation $M(t_0) = 0$ and to analyze its zeros (simple, or multiple).

2.2. SOME SIMULATIONS

Example 3. For given $N = 3$, $\alpha = 0.9$, $\epsilon = 0.05$, $\omega = 1.05$, $a_1 = 1.4$, $a_2 = 1.2$, $a_3 = 1.1$, the system's simulations (3) for $x_0 = 0.1$, $y_0 = 0.15$ are depicted on Figure 7.

Example 4. For given $N = 9$, $\alpha = 0.9$, $\epsilon = 0.001$, $\omega = 0.11$, $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.19$, $a_4 = 0.3$, $a_5 = 0.1$, $a_6 = 0.1$, $a_7 = 0.1$, $a_8 = 0.2$, $a_9 = 0.1$ the system's simulations (3) for $x_0 = 0.2$, $y_0 = 0.26$ are depicted on Figure 8.

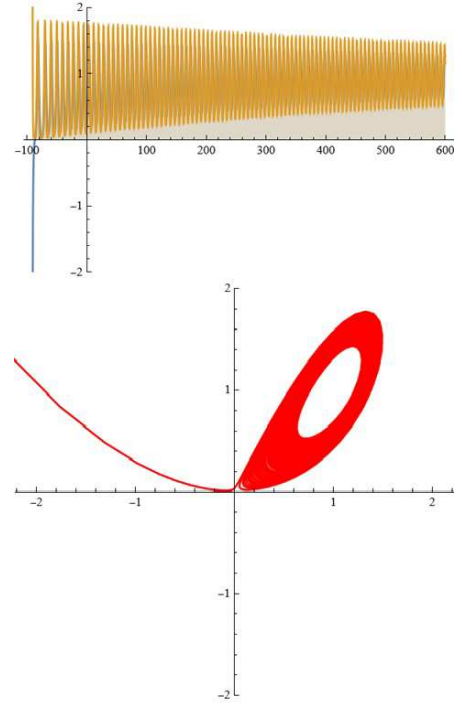
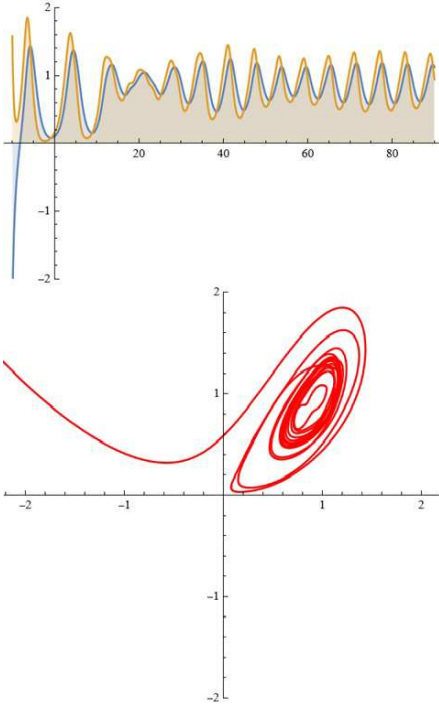


Figure 7. (a) solutions of the system (3); (b) phase space (Example 3)

Figure 8. (a) solutions of the system (3); (b) phase space (Example 4)

Figure 8 illustrates coexisting attractors in considered differential model.

Challenges for Learners

We offer an efficient learning strategy that emphasizes learning and challenges our master's and doctoral students to consider the triangle of enigmatics, creativity, and acmeology. After providing curriculum modules we will set the following tasks for self-learning.

Task 1. (self-learning). Study the behavior of the model (3) at fixed values: $N = 6$, $\alpha = 0.9$, $a_1 = 5.8$, $a_2 = 0.2$, $a_3 = 0.9$, $a_4 = 1.5$, $a_5 = 0.1$ and $a_6 = 0.4$.

- a) For what values of the parameters ϵ and ω is the phase portrait depicted in Fig. 9 b obtained;

(Answer: when performing the task correctly, you should get the following approximate values: $\epsilon = 0.06$, $\omega = 0.11$)

- b) show the coexisting attractors in considered differential model;
- c) draw the corresponding conclusions.

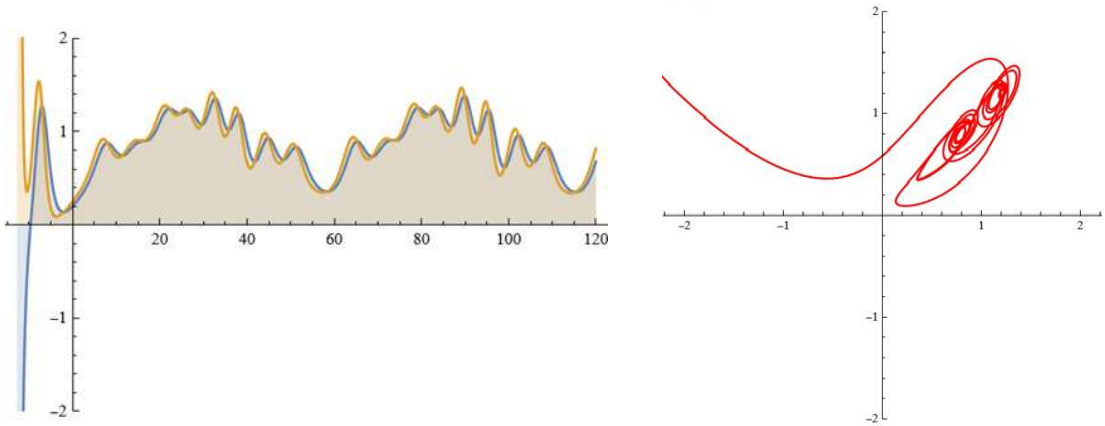


Figure 9. a) x and y – components of the solution of system (3);
b) phase space (Task 1 (self-learning))

Our conclusion as teachers is that with the proposed methodology, students adapt relatively well and touch upon the elegant theory of Melnikov for the possible occurrence of chaos in the dynamical system under consideration.

Task 2 (self-learning with increased difficulty). We define the hypothetical normalized antenna factor as follows: $M^*(\theta) = \frac{1}{D}|M(K \cos \theta + k_2)|$ where θ is the azimuth angle;

$K = k_1 d$; $k_1 = \frac{2\pi}{\lambda}$; λ is the wave length; d is the distance between emitters; k_2 is the phase difference [31]–[33].

Let $N = 8$, $\alpha = 0.39$, $\omega = 0.16$, $K = 4.5$, $k_2 = -1.38$.

- a) For what values of the remaining parameters of our model is the Melnikov function $M(t_0)$ depicted in Fig. 10 a and corresponding Melnikov antenna factor depicted in Fig. 10 b.

(Answer: when performing the task correctly, you should get the following approximate values: $a_1 = 1.02$, $a_2 = 1.8$, $a_3 = 1.9$, $a_4 = 1.65$, $a_5 = 1.95$, $a_6 = 1.9$, $a_7 = 4.2$, $a_8 = 4.2$).

- b) draw the corresponding conclusions.

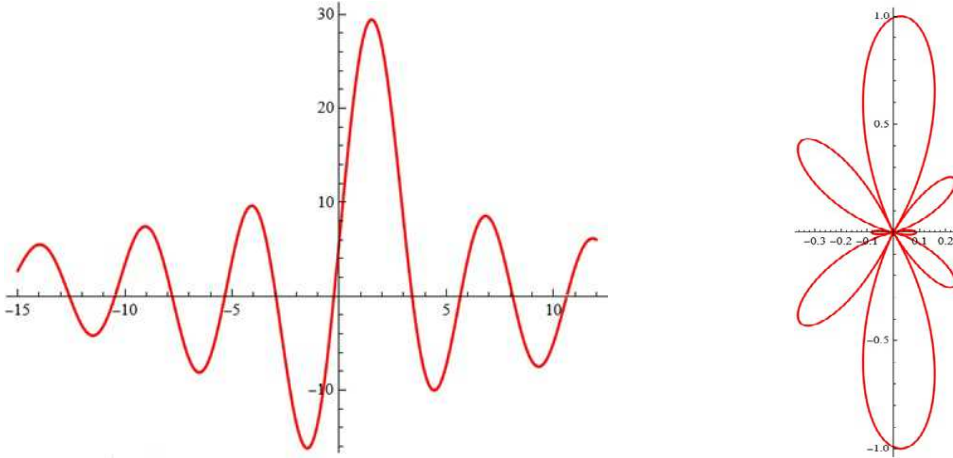


Figure 10. a) Melnikov function $M(t_0)$ (Task 3);
b) Melnikov antenna factor (Task 2)

3. CONCLUDING

In this article, we study prediction of chaos in a modified differential system with a specific potential, containing N number of free parameters, which could be of interest to specialists working in the field of epidemiology.

We pay special attention to the Melnikov function, corresponding to the proposed new model.

With a specially developed software product, we generate the Melnikov equation $M(t) = 0$ and examine all its zeros.

This opens up an opportunity for the researcher to correctly understand and formulate the classical Melnikov criterion for the possible occurrence of chaos in the dynamical system.

Several simulations are composed. We demonstrate also some specialized modules for investigating the dynamics of the proposed model.

We find a specific unfolding which produces limit cycles and homoclinic bifurcations. These findings provide an analytic understanding of the numerical observations in the SIQR model.

We also conduct some numerical calculations which confirm our analytical results.

We plan to consider other interesting differential systems with specific potentials in our future publications that are relevant to modified models of the SIR/SIRD/SEIR/GSEIR type [34]–[49].

We envisage future research on modified SIQR-type models [3, 4] in light of the considerations in this article.

The study of model in terms of bifurcations, attractors, Lyapunov exponents, attractor merging, amplitude control of hyperchaos and saturation in the Hausdorff sense [50] will be the subject of our future explorations.

Application of Impulsive SIRQ Models for the Development of Forecasting and Cyberattack Mitigation Scenarios can be found in [51]. The model extends classical epidemic frameworks by combining the continuous-time dynamics of malicious object spreading with discrete control actions corresponding to mass updates, node isolation, and access control policies.

Future research in this important scientific direction can be successfully expanded on the basis of other epidemiological models of the SIR/SIRD/SEIR/GSEIR/SIRS/SIQR type. This also applies to future developments in the field of debugging theory [52].

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