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THE SOLVABILITY OF A GENERAL ENZYME MODEL

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Abstract: In this paper we investigate a mathematical model which is a generalization of the classical model of the enzyme reaction process. This includes a reaction- diffusion equation under initial and boundary conditions. Some existence conditions guaranteeing solvability are considered. It is shown existence of exponentially asymptotically stable solution to the problem under consideration. The case of existence of more than one solution is discussed as well.

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1. Introduction

In this paper we consider a mathematical model in the organic chemistry concerning dynamics of the enzyme processes. The enzyme reactions can be considered as a dynamical system. The chemical reaction starts with two substances: the free enzyme concentration E and substrate concentration S, resulting to the enzyme-substrate complex ES. We refer the reader to [9], where the enzyme process is described in detail. Here we only sketch the main reactions, following the same author: "The enzyme reaction process from the substrate S to the product P goes in two steps: the first represents a reversible binding to an enzyme, and the second disociates the complex ES into the product with release the enzyme E" (see, e.g. [5], [6], [10]). After setting $u = S$ the dynamics of the enzyme process can be described by the following initial and boundary value problem (IBVP):

$$\begin{array}{ll} (a) & u_t - \nabla \cdot (D\nabla u) = q(t,x,u), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ (b) & Bu = h(x), \quad t > 0, \quad x \in \partial\Omega, \\ (c) & u(0,x) = u_0(x), \quad x \in \Omega. \end{array} \tag{1}$$

Here (1)(b) is the boundary condition, (1)(c) - the initial condition, and Ω is bounded domain with sufficiently smooth boundary. The multiplier $D = D(x)$ is a sufficiently smooth function which is called the *diffusion coefficient* in the chemical diffusion processes, the term $\nabla \cdot (D\nabla u)$ in (1)(a), represents the rate of change due to the diffusion and $q(u)$ is the rate of change due to the reaction. The *reaction function* q is the density per unit volume and per unit time formed through the process of reaction or interaction (see, e.g. [9], [10]). In (1)(b) $B \equiv \alpha_0(x)\partial/\partial\nu + \beta_0(x)$ is referred to as the boundary operator, i.e.,

$$\alpha_0(x)\frac{\partial u}{\partial \nu} + \beta_0(x)u = h(x).$$

The reaction-diffusion dynamics takes place in a finite dimensional medium Ω. Furthermore, in the above stated boundary condition suppose that $\alpha_0 \geq 0$, $\beta_0 \geq 0$, and $\alpha_0 + \beta_0 > 0$ on $\partial\Omega$, and that α_0, β_0, h are Hölder continuous functions in $\bar{\Omega}$, $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$, and $\nu = \nu(x)$ is the normal vector pointing outward from $x \in \partial\Omega$.

The substrate concentration is governed by (1), (b), (c) and the reaction function q has the form $q(u) \equiv -\sigma u(1 + au)^{-1}$, where σ and a are positive constants.

If a competitive inhibitor (substrate inhibitor) is taken into account, then a special reaction rate can be described by $q(u) \equiv -\sigma u(1 + au + bu^2)^{-1}$ (σ, a, b are real constants).

The main goal of the present paper is to be studied the solvability of a general mathematical model in which the reaction function $q(u) \equiv -\sigma u/\gamma(u)$ ($\sigma > 0$), where the function $\gamma > 1$ is monotone nondecreasing in u. It turns out that the classical model stated above is a particular case, i.e. the function γ has a special form either $1 + au$ or $1 + au + bu^2$.

The motivation for this paper stems from the fact that the uniqueness of the solution of (1) with the general form of q is not always guaranteed. In Section 2 we introduce some known definitions and facts connected with the solvability. In Section 3 we establish some criteria guaranteeing existence and uniqueness, and also show another case when there exist many solutions.

2. Preliminaries

Let $T > 0$ be a fixed number, and let $\Omega \subset \mathbb{R}^3$ be a domain in the class $C^{2+\alpha}$. Denote $Q_T \equiv (0,T) \times \Omega$, $\Gamma_T \equiv (0,T) \times \partial\Omega$, $u_t = \partial u/\partial t, u_{x_i} = \partial u/\partial x_i, u_{x_i x_j} = \partial^2 u/\partial x_i \partial x_j$.

Consider the IBVP written in general form,

$$\begin{array}{lll} (a) & u_t - Lu = q(u) & \text{for } (t,x) \in Q_T, \\ (b) & Bu = h(x) & \text{on } \Gamma_T, \\ (c) & u(0,x) = u_0(x) & \text{in } \bar{\Omega}, \end{array} \tag{2}$$

where the reaction function $q(u) \equiv -\sigma u/\gamma(u)$, $a, \sigma \in \mathbb{R}$, $\gamma(u)$ is sufficiently smooth $\gamma(u) > 1$ $\forall u \in \mathbb{R}_+$, and the operator L,

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(x) u_{x_j}$$

is uniformly elliptic in the sense that the matrix $\{a_{ij}(x)\}$ is positive definite in $\bar{\Omega}$. Assume that the coefficients of L are Hölder continuous in $\bar{\Omega}$. The boundary operator B is defined by $Bu \equiv \partial u/\partial\nu + \beta_0(x)u$, where $\beta_0(x)$ is a nonnegative function in $C^{1+\theta}(\partial\Omega)$ $(0 < \theta < 1)$ and not identically zero on $\partial\Omega$; $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. The functions $u_0(x)$ and $h(x)$ are always assumed to be nonnegative and $u_0 \in C^2(\bar{\Omega})$, $h \in C^{1+\theta}(\partial\Omega)$ $(0 < \theta < 1)$.

The function $u \in C^{1,2}(Q_T)$ is called a solution of IBVP if it satisfies (2).

The function $u \in C^{1,2}(Q_T)$ is called an upper solution of IBVP if

$$\begin{array}{lll} (a) & u_t - Lu \geq q(u) & \text{for } (t,x) \in Q_T, \\ (b) & Bu \geq h(x) & \text{on } \Gamma_T \\ (c) & u(0,x) \geq u_0(x) & \text{in } \bar{\Omega}. \end{array}$$

A lower solution is defined analogously by reversing the above inequalities.

The function $u_s(x)$ is called *steady-state* solution of (2) if it is a solution of the elliptic boundary-value problem:

$$\begin{array}{ll} -Lu = f(u) & \text{in } \Omega, \\ Bu = h(x) & \text{on } \partial\Omega, \end{array} \tag{3}$$

where L and B are the same operators as in (2).

There are different mathematical models describing enzyme reaction processes which can be seen in [2], [5], [9], [10]. Numerical solving of (2) can be seen in [1], [8]. Mathematical models with delay can be seen in [7] and [9]. For the particular case when the effect of inhibition is taken into account it is known that $\gamma(u) \equiv 1 + au + bu^2$. It is shown in [9] that there exists at least one steady-state solution provided that either (i) $\sigma \leq \lambda_0/\bar{c}$ and arbitrary $h \geq 0$, $b > 0$ or (ii) $h < b^{-1/2}$, $\beta_0 = 1$, and arbitrary $\sigma > 0$, where $\bar{c} \equiv \sup\{(bu^2 - 1)(1 + au + bu^2)^{-2};\ u \geq 0\}$. Moreover, there exist positive constants $\bar{\sigma}$, h_0 such that the steady-state solution is also unique in the case $L = L_s$, $B = B_s$ and when (iii) $\sigma \geq \bar{\sigma}$, $h \geq h_0$ and any $b > 0$. Here, the pair $L = L_s$ and $B = B_s$ are said to be in self-adjoint form. For instance $L_s = \nabla \cdot (D(x)\nabla u)$ and $B_s u = D(x)\partial u/\partial\nu + \beta_0(x)u$, where ∇ is the gradient operator and $\partial/\partial\nu$ is the outward normal derivative on the boundary of Ω, and the function $D(x)$ is sufficiently smooth with respect to $x \in \bar{\Omega}$. It follows from Theorem 4.4 ([9], Chapter 5) that under any one of the conditions in (i), (ii) , and (iii) the unique steady-state solution of (2) is globally asymptotically stable with respect to nonnegative initial perturbations.

For the existence of more than one steady-state solution we refer the reader to [10].

Consider the linear problem

$$\begin{array}{ll} (L - c_0 + \lambda r)\Phi = 0 & \text{in } \Omega, \\ B\Phi = 0 & \text{on } \partial\Omega, \end{array} \tag{4}$$

where the functions $c_0 = c_0(x)$ and $r = r(x)$ belong to $C(\Omega)$.

Denote by λ_0 the principal eigenvalue of the eigenvalue problem (4) corresponding to $r = 1$ and $c_0 = 0$. It is known that $\lambda_0 > 0$ and the corresponding eigenfunction $\Phi_0(x)$ is positive in $\bar{\Omega}$. In what follows we assume that $\Phi_0(x)$ is normalized by $\max\{\Phi_0(x) :\ x \in \bar{\Omega}\} = 1$.

There exists a result in [9] (Chapter 5, Theorem 3.2) that a given steady-state solution $u_s(x)$ for the particular case of enzyme reaction model is asymptotically stable if $q_u(u_s) < \lambda_0$ and is unstable if $q_u(u_s) > \lambda_0$. Then the solutions of (3) belong to the class $C^{\alpha}(\bar{\Omega}) \cap C^2(\Omega)$.

The function $\tilde{u} \in C^{\alpha}(\bar{\Omega}) \cap C^2(\Omega)$ is called an upper solution of (3) if

$$\begin{aligned} -Lu &\geq q(u) \qquad \text{in } \Omega, \\ Bu &\geq h(x) \qquad \text{on } \partial\Omega. \end{aligned}$$

Similarly, the function $\hat{u} \in C^{\alpha}(\bar{\Omega}) \cap C^2(\Omega)$ is called a lower solution if it satisfies the reversed inequalities.

The steady-state solution u_s of (2) is said to be *stable* if given any constant $\varepsilon > 0$ there exists $\delta > 0$ such that:

(i) $|u(t,x) - u_s(x)| < \varepsilon$ in Q_T whenever $|u_0(x) - u_s(x)| < \delta$ in Ω, where $u(t,x)$ is the solution of (2).

If in addition,

(ii) $\lim\limits_{t\to\infty} |u(t,x) - u_s(x)| = 0$ in $\bar{\Omega}$ then u_s is said to be *asymptotically stable.*

The steady-state solution $u_s(x)$ is called *unstable* if it is not stable.

A steady-state solution is said to be *exponentially asymptotically stable* if both conditions (i) and (ii) hold and convergence in (ii) is in exponential order, i.e. this is the case when there exist positive constants ρ and α such that $|u(t,x) - u_s(x)| \leq \rho e^{-\alpha t}$ $(t > 0,\ x \in \bar{\Omega})$, whenever it holds at $t = 0$.

The set of initial functions $u_0(x)$ whose corresponding solutions $u(t,x)$ satisfy the inequality (i) and (ii) in the last definition stated above is called the *stability region* of u_s. If this is true for all initial functions then u_s is *globally asymptotically stable.*

It is known that if $q(0)$ and h are identically zero, and β_0 is not identically zero, and if there exists stability then we say that there is *stability of a uniform steady-state solution.*

If $q(0)$ and h are not both identically zero, and if there exists stability then we say that there is *stability of a nonuniform steady-state solution.*

Furthermore, we take into consideration the latter. The nonuniform steady-state solution will be called steady-state solution for simplicity.

A pair of upper and lower solutions of (2) is called ordered if $\tilde{u} \geq \hat{u}$ in the corresponding closed domain where they exist. For $\tilde{u}$ and $\hat{u}$ define a *sector* $\langle \hat{u}, \tilde{u} \rangle$, which contains all functions u of the corresponding functional class such that $\hat{u} \leq u \leq \tilde{u}$.

A sector $\langle v, w \rangle$ between two functions $v, w \in C(\bar{\Omega})$ with $v \leq w$ is called an *invariant set* of (2) if for any $u_0 \in \langle v, w \rangle$ the corresponding solution $u(t,x)$ of (2) remains in $\langle v, w \rangle$ for all $t > 0$.

Furthermore, introduce some assumptions that we should select according to the concrete case under consideration.

A1. $q(u)$ satisfies the one-sided Lipschitz condition

$$q(u_1) - q(u_2) \geq -\underline{c}(x)(u_1 - u_2) \quad \text{for} \quad \hat{u} \leq u_2 \leq u_1 \leq \tilde{u},$$

where $\underline{c}(x)$ is a bounded nonnegative function in Ω.

A2. $q(0) \geq 0$ and $h(x) > 0$ $(x \in \partial\Omega)$.

Since only positive solutions of the model under consideration are of interest we state an existence and uniqueness result for those solutions.

Theorem 1. (see [9]) *Let $\tilde{u}$ be a positive upper solution of (3) and let A1 and A2 hold with $\hat{u} = 0$. Then (3) has at least one solution u_s in $\langle 0, \tilde{u}\rangle$ which is either positive or zero. It is positive when either $q(0)$ or $h(x)$ is not identically zero. Moreover, u_s is the unique solution in $\langle 0, \tilde{u}\rangle$ if β_0 is not identically zero, and the inequality $q(u_1) \leq q(u_2)$ for $\hat{u} \leq u_2 \leq u_1 \leq \tilde{u}$ holds with $\hat{u} = 0$.*

The following theorem gives a sufficient condition for the stability of u_s for arbitrary $q(u)$.

Theorem 2 (see [9]) *Let u_s be a given steady-state solution of (2). If there exist constants ρ_i, α_i with $\rho_i \geq 0$, $0 \leq \alpha_i \leq \lambda_0$ $(i = 1, 2)$, such that*

$$\begin{array}{ll} (a) & q(u_s + \eta) - q(u_s) \leq (\lambda_0 - \alpha_1)\eta \quad \text{for} \quad 0 \leq \eta \leq \rho_1 \quad (x \in \Omega) \\ (b) & q(u_s) - q(u_s - \eta) \leq (\lambda_0 - \alpha_2)\eta \quad \text{for} \quad 0 \leq \eta \leq \rho_2 \quad (x \in \Omega) \end{array} \tag{5}$$

then problem (2) has a unique solution u such that

$$u_s(x) - \rho_2 e^{-\alpha_2 t}\Phi_0(x) \leq u(t, x) + \rho_1 e^{-\alpha_1 t}\Phi_0(x) \quad (t > 0, \ x \in \bar{\Omega}) \tag{6}$$

whenever it holds at $t = 0$.

By the following conditions we define the rate of $\gamma(z)$.

A3. $q(z) \equiv -\sigma z/\gamma(z)$, where $\sigma > 0$, $\gamma(z) > 1$ is continuous and nonincreasing for $z \in \mathbb{R}_+$.

Suppose for our purposes that there is at least one steady-state solution $u_s = u_s(x)$ $(x \in \overline{\Omega})$ of (2) delivered with the standard supremum norm $\|\cdot\|_\infty$.

A4. There exist $\alpha_1, \ \rho_1 \geq 0$, such that

$$\frac{\sigma}{\lambda_0 - \alpha_1} \leq \frac{\gamma(u_s)\eta}{\|u_s\|_\infty}, \tag{7}$$

where $\lambda_0 > \alpha_1$, $0 \leq \eta \leq \rho_1$.

A5. There exist $\alpha_2,\ \rho_2 \geq 0$, such that

$$\frac{\sigma}{\lambda_0 - \alpha_2} \leq \frac{\gamma(u_s - \eta)\eta}{\|u_s\|_\infty}, \tag{8}$$

where $\lambda_0 > \alpha_2$, $0 \leq \eta \leq \rho_2$.

A6. Assume that

$$\|u_s\|_\infty \leq \frac{\eta(\lambda_0 - \alpha)}{\sigma},$$

where $\lambda_0 > \alpha \geq 0$.

3. Main Result

We consider a general chemical model with the reaction function

$$q(u) \equiv -\sigma u/\gamma(u),$$

where the function $\gamma(u) > 1$ is sufficiently smooth and positive for all $u \in \mathbb{R}_+$.

3.1. Existence and Uniqueness

Let $u_s(x)$ be a unique and positive steady-state solution of (2). Then we state:

Proposition 1. *Let the assumptions A2-A4 be satisfied. Then the relation (5) (a) holds with* $0 \leq \alpha_1 < \lambda_0$.

Proof. The proof follows from

$$(\lambda_0 - \alpha_1)\eta \geq \frac{\sigma\|u_s\|_\infty}{\gamma(u_s)} \geq \frac{\sigma u_s}{\gamma(u_s)} \geq -\frac{\sigma(u_s + \eta)}{\gamma(u_s + \eta)} + \frac{\sigma u_s}{\gamma(u_s)}\ .$$

Proposition 2. *Let the assumptions A2, A3, A5 be satisfied. Then the relation (5) (b) holds with* $0 \leq \alpha_2 < \lambda_0$.

Proof. The proof follows by the same arguments used in the proof of Proposition 1, i.e.

$$\eta(\lambda_0 - \alpha_2) \geq \frac{\sigma\|u_s\|_\infty}{\gamma(u_s - \eta)} \geq \frac{\sigma u_s}{\gamma(u_s - \eta)} \geq \frac{\sigma(u_s - \eta)}{\gamma(u_s - \eta)} \geq -\frac{\sigma u_s}{\gamma(u_s)} + \frac{\sigma(u_s - \eta)}{\gamma(u_s - \eta)}\ .$$

Proposition 3. *Let A3 and A6 be satisfied. Then both A4 and A5 hold true as well.*

Proof. The proof follows from the inequality

$$\|u_s\|_\infty \leq \frac{\eta(\lambda_0 - \alpha)}{\sigma} \leq \frac{\eta\gamma(u_s)(\lambda_0 - \alpha)}{\sigma},$$

where $\alpha = \alpha_i$.

Our observation is that replacing the assumptions A2-A4 and A2, A3, A5 in Proposition 1 and 2, respectively, by A2, A3, A6 and having in mind the Proposition 3 then may unite both Proposition 1 and 2 and get the following statement:

Proposition 3´. *Let A2, A3, A6 be satisfied. Then both (5)(a) and (5)(b) hold.*

The following statement provide us with the existence and uniqueness of the solution of (3).

Theorem 3. *Suppose that:*

1) $\hat{u} = 0$, $\tilde{u} > 0$ are the lower and upper solutions, respectively, and A1-A3 hold.

2) β_0 is not identically zero.

Then there exists a unique positive solution of (3) in $\langle 0, \tilde{u}\rangle$.

Proof. The required conditions 1) and 2) assure validity of Theorem 1. Since A3 holds it follows that $q(u_1) \leq q(u_2)$ for $\hat{u} = 0 \leq u_2 \leq u_1 \leq \tilde{u}$ which leads to uniqueness of the solution of (3). □

Until now we have proved the existence of a unique positive solution of (3), which on the other hand appears as a steady-state solution for (2).

Theorem 4. *Let $u_s(x)$ be a steady-state solution of (2) such that A2, A3, A6 hold. Then the problem (2) has a unique solution $u(t, x)$ satisfying (6).*

Proof. The result follows from Proposition 3´ and also Theorem 2. Thus it follows from (6) that

$$|u_s(x) - u(t,x)| \leq \rho_2 e^{-\alpha_2 t}\Phi(x) + \rho_1 e^{-\alpha_1 t}\Phi(x) \leq \ \leq \rho e^{-\alpha t} \qquad (t > 0,\ x \in \bar{\Omega}),$$

where $\rho = \max\{\rho_1, \rho_2\}$, $\sup\limits_{\bar{\Omega}} \Phi_0(x) = 1$, $\alpha = \min\{\alpha_1, \alpha_2\}$. □

The last statement shows that the steady-state solution u_s is exponentially asymptotically stable.

3.2. Nonuniqueness of the Solutions

Consider the result in [9] (Theorem 6.1), which is a sufficient condition for the existence of many solutions to (2). Thus it guarantees existence of infinitely many solutions to the problem (2) provided that the following conditions hold:

(i) $q(0) = h(x) = u_0(x) = 0$;

(ii) q is Hölder continuous;

(iii) q satisfies the one-side Lipschitz condition, that is,

$$q(u_1) - q(u_2) \geq -\underline{c}(u_1 - u_2) \quad \forall\ \hat{u} \leq u_2 \leq u_1 \leq \tilde{u}, \tag{9}$$

where the functions $\tilde{u}$, $\hat{u}$ are ordered upper and lower solutions, respectively, with $\tilde{u} \geq \hat{u}$ in $\overline{Q}_T$;

(iv) there exist constants $\sigma_0 \in \mathbb{R}_+$, $\sigma_1 > 0$, $0 < \mu < 1$ such that

$$q(u) \geq -\sigma_0 u + \sigma_1 u^{\mu} \quad u > 0, \tag{10}$$

then for any $T < T^*$ the problem (2) admits infinitely many solutions. The number T^* is determined by the ordinary differential equation with initial data,

$$z' = q(z), \quad z(0) = z_0 > 0. \tag{11}$$

There exists from the continuity of q a number $T^* \leq \infty$ such that the problem (11) has at least one positive solution $z = z(t)$ in $[0, T]$ for any $T < T^*$.

Note. Following the theory in [9] the number $\sigma_0 \in \mathbb{R}$, however we accept $\sigma_0 \in \mathbb{R}_+$ for the physical clearness.

Furthermore, assume that the scalar function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the condition: **A7.** (a) $\gamma(z) > 1$, and $z\gamma(z)$ is Lipschitz function for $z > 0$;

(b) $\gamma(z)$ is monotone increasing for $z > 0$.

Lemma 1. *Let A7 be satisfied. Then the condition (ii) hold, i.e. the function q is Hölder continuous.*

Proof. Consider the following equalities which stem from H6:

$$\begin{aligned}\left|\frac{-\sigma u}{\gamma(u)} - \frac{-\sigma v}{\gamma(v)}\right| \cdot \frac{1}{|u-v|^{\alpha}} &= \frac{\sigma}{|u-v|^{\alpha}} \frac{|\gamma(u)v - \gamma(v)u|}{\gamma(u)\gamma(v)} \\ &\leq \frac{\sigma}{|u-v|^{\alpha}} \frac{|\gamma(u)u - \gamma(v)v|}{1} \leq \frac{\sigma}{|u-v|^{\alpha}} L\,|u-v|\end{aligned}$$

$$= \sigma L |u-v|^{1-\alpha} \leq \sigma L |\tilde{u}-\hat{u}|^{1-\alpha} < +\infty,$$

where $L > 0$ is the Lipschitz constant. Thus we prove the Hölder continuity.

Lemma 2. *Let A7 be satisfied. Then the function q satisfies (iii), i.e. the inequality (9) hold.*

Proof.

$$q(u) - q(v) = \frac{-\sigma u}{\gamma(u)} - \frac{-\sigma v}{\gamma(v)} \geq \frac{\sigma[v\gamma(v) - u\gamma(u)]}{\gamma(u)\gamma(v)}$$
$$\geq \frac{-\sigma L(u-v)}{\gamma^2(u)} \geq -\sigma L(u-v), \quad \hat{u} \leq v < u \leq \tilde{u}.$$

Thus the statement is proved.

Next we consider an upper solution of (2) satisfying $\tilde{u} < [(\sigma_0 - \sigma)/\sigma_1]^{\frac{1}{\mu-1}}$ $(0 < \mu < 1)$. Obviously, it is upper solution provided that the condition (i) hold.

Lemma 3. *Let $\gamma(z) > 1$, and the upper solution of (2) be bounded,*

$$\tilde{u} < \left(\frac{\sigma_1}{\sigma_0 - \sigma}\right)^{\frac{1}{1-\mu}} \quad (0 < \mu < 1).$$

Then the condition (iv) hold.

Proof. From the inequalities

$$u \leq \tilde{u} < \left(\frac{\sigma_1}{\sigma_0 - \sigma}\right)^{\frac{1}{1-\mu}} \quad \text{and} \quad -\sigma u < -\sigma_0 u + \sigma_1 u^{\mu}$$

obtain

$$\frac{-\sigma u}{\gamma(u)} > -\sigma u > -\sigma_0 u + \sigma_1 u^{\mu}.$$

by which the proof is completed. □

From Lemma 1 and 2 obtain the final result:

Theorem 5. *Let the condition A7 be satisfied, and $\tilde{u}$ be the same as in Lemma 3. Then the problem (2) has infinitely many solutions.*

The proof follows from the above stated propositions and the condition A7.

As a conclusion we emphasize that the function $\gamma(u)$ can be taken for instance in the quadratic form $\gamma(u) = 1 + au + bu^2$, where $a, b \in \mathbb{R}$. It turns out that last function satisfies all statements stated above.

In the present paper we discussed the existence of classical solutions of the problem under consideration. We thank deeply Prof. I. Dobrev, Prof. V. Beshkov and Prof. M. Krasteva for their kind and warm encouragement, fruitful notes and several discussions.

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