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CONTINUOUS y-FUNCTION HYBRID METHODS FOR DIRECT SOLUTION OF DIFFERENTIAL EQUATIONS

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Abstract: The aim of this study is to produce consistent two-step y-function hybrid numerical methods for a direct solution of general second-order differential equations. The basis function is interpolated at both grid and off-grid points and its associated differential system are collocated at all grid points. The method developed is continuous, consistent and symmetric. The results show that the proposed method is better when compared with existing methods.

AMS Subject Classification: 65L05, 65L06
Key Words: collocation and interpolation, two-point hybrid method, continuous, predictor-corrector mode, y-function, symmetric

1. Introduction

In this paper, a y-function hybrid continuous method of order 6 is proposed for direct solution of general second order initial value problem of ordinary differential equation of the form

$$y'' = f(t, y, y'); \quad y(t_0) = y_0, \quad y'(t_0) = y_1 \tag{1}$$

It is known that, this type of problem (1) may or may not be solved in the closed form especially the non-linear type, and even when it can be solved in the closed form the problems are conventionally reduced to systems of first

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order equations and any methods for first order equations are adopted to solve them. [Fatunla (1988); Lambert (1973); Awoyemi and Kayode (2002); Spiegel (1973); Brugnano and Trigiante (1998); Omolehin, Ibiejugba, Alabi and Evans (2003)].

This approach has been identified to have many setbacks [Awoyemi, 1999; Awoyemi and Kayode, (2002); (2005)]. This has led many authors to attempt at solving problem (1) directly without reduction to systems of first order equations. Brown (1977) and Lambert (1991) proposed multi-derivative methods to solve equation (1) directly. Omolehin et al (2003) presented a two step implicit method which is of order four to solve (1). To improve upon the output of the result generated by the method of Omolehin et al (2003), Kayode and Awoyemi (2005) proposed a 5-step methods for a direct solution to (1), using power series method. All these methods considered the solution of (1) at selected grid points without any consideration at the off-grid points. Recently, attention has been focused at incorporating off-grid points into the solution of (1). Yahaya and Badmus (2009) developed a class of hybrid block methods for solving (1) but the methods are of low order of accuracy. Kayode (2011), Kayode and Adeyeye (2011) also developed a class of one-point and two-point hybrid implicit methods respectively with order of accuracy higher than that of Yahaya and Badmus (2009). However, these authors introduced the hybrid points at $f-$function . This increases the computational efforts and contributes to lower accuracy of the methods.

In this presentation, efforts are made to develop implicit hybrid continuous methods in which the hybrid points are at the y-function for solving (1) directly. The method was designed to have reduced function evaluations as a result of the location of the hybrid point at y-function. The method is consistent, symmetric and of smaller error constant, and can effectively handle mildly stiff linear and non-linear problems.

2. Derivation of the Method

We proposed a polynomial function

$$y(x) = \sum_{j=0}^{3k+1} a_j x^j \tag{2}$$

as an approximate solution to (1).

Using the second derivatives of (2) in equation (1) gives a differential system

$$\sum_{j=2}^{3k+1} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{3}$$

Collocating (3) at all grid points x_{n+c}, $c = 0, 1, 2$ and interpolating (2) at points x_{n+i}, $i = 0, 1, r, s$ where $r \in (0, 1)$ and $s \in (1, 2)$; giving rise to a system of $c+i$ equations

$$\mathbf{AX} = \mathbf{B} \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & x_{n+r}^4 & x_{n+r}^5 & x_{n+r}^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+s} & x_{n+s}^2 & x_{n+s}^3 & x_{n+s}^4 & x_{n+s}^5 & x_{n+s}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 12x_{n+2}^3 & 30x_{n+2}^4 \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}^T,$$

$$\mathbf{B} = \begin{bmatrix} y_n & y_{n+r} & y_{n+1} & y_{n+s} & f_n & f_{n+1} & f_{n+2} \end{bmatrix}^T$$

The values of a_j's in (4) were determined and substituted into the approximate equation (2) to have the required continuous coefficients hybrid methods

$$y_k(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \{\tau_1(x) y_{n+r} + \tau_2(x) y_{n+s}\} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{5}$$

Applying the transformation $t = \frac{1}{h}(x - x_{n+k-1})$, $t \in (0, 1]$ in Kayode [2011] to (5), its continuous coefficients α_j's, β_j's, τ_1 and τ_2 are obtained as

$$\alpha_0(t) = \frac{A_0}{D_0}, \quad \alpha_1(t) = \frac{A_1}{D_1}, \quad \tau_1(t) = \frac{A_2}{D_2}, \quad \tau_2(t) = \frac{A_3}{D_3}$$
$$\beta_0(t) = \frac{B_0}{6D_4}, \quad \beta_1(t) = \frac{B_1}{3D_4}, \quad \beta_2(t) = \frac{B_2}{6D_4}$$

where

$$A_0 = (2t - 1)(s - 2t)(r - 2t)$$

$$\left\{\begin{array}{l} t^3\left(48r^2s+48rs^2-192rs+128+48r^3+48s^3-192r^2\right. \\ \left.-192s^2+128r+128s\right)+t^2\left(24r^3s+24rs^3-96r^3-96s^3\right. \\ +364r^2+364s^2-196r-196s-192r^2s-192rs^2+428rs \\ \left.+24r^2s^2-260\right)+t\left(12+12r^2s^3-228rs-96r^2s^2-48r^3s\right. \\ -48rs^3+214rs^2+214r^2s-10r-10s+32r^3+32s^3 \\ \left.-98r^2-98s^2+120\right)+\left(16r^3+16s^3-65r^2-65s^2-5rs\right. \\ +6r^3s^3+91r^2s^2-24r^2s^3-24r^3s^2+60r+60s-49r^2s \\ \left.-49rs^2+16r^3s+16rs^3\right) \end{array}\right\}$$

$$A_1 = 2t(s-2t)(r-2t)$$

$$\left\{\begin{array}{l} t^3\left(48r^2s+48rs^2-240rs-240s^2-240r^2+48r^3+48s^3\right. \\ \left.+320r+320s\right)+t^2\left(24r^3s+24rs^3-240r^2s-240rs^2\right. \\ +24r^2s^2+740rs+580r^2+580s^2-120r^3-120s^3-720r \\ \left.-720s\right)+t\left(12r^3s^2+12r^2s^3-720rs-120r^2s^2-60r^3s\right. \\ -60rs^3+370r^2s+370rs^2+400r+400s+80r^3+80s^3 \\ \left.-360r^2-360s^2\right)+\left(145r^2s^2-30r^2s^3-30r^3s^2-180r^2s\right. \\ \left.-180rs^2+40r^3s+40rs^3+200rs+6r^3s^3\right) \end{array}\right\}$$

$$A_2 = 2t(2t-1)(s-2t)$$

$$\left\{\begin{array}{l} t^3\left(128+48s^3-192s^2+128s\right)+t^2\left(-260-96s^3+364s^2\right) \\ +t\left(-10s+32s^3-98s^2+120\right)+\left(16s^3-65s^2+60s\right) \end{array}\right\}$$

$$A_3 = 2t(2t-1)(r-2t)$$

$$\left\{\begin{array}{l} t^3\left(128+48r^3-192r^2+128r\right)+t^2\left(-260-96r^3+364r^2\right. \\ \left.-196r\right)+t\left(-10r+32r^3-98r^2+120\right)+\left(16r^3-65r^2+60r\right) \end{array}\right\}$$

$$B_0 = t(2t-1)(s-2t)(r-2t)$$

$$\left\{\begin{array}{l} t^2\left(-60r^2s-60rs^2+384+12r^2s^{+}292rs+84r^2+84s^2\right. \\ \left.-368r-368s\right)+t\left(-30r^2s^2-678rs+146r^2s+146rs^2\right. \\ \left.+782r+782s-184r^2-184s^2-780\right)+\left(96r^2+96s^2\right. \\ \left.+391rs+21r^2s^2-92r^2s-92rs^2-390r-390s-360\right) \end{array}\right\}$$

$$B_1 = t(2t-1)(s-2t)(r-2t)$$

$$\left\{\begin{array}{l} t^2\left(-36r^2s-36rs^2+12r^2s^2+76rs-36r^2-36s^2+112r\right.\\ \left.+112s\right)+t\left(-18r^2s^2-48rs+38r^2s+38rs^2-160r\right.\\ \left.-160s+56r^2+56s^2\right)+\left(-80rs-9r^2s^2+28r^2s+28rs^2\right) \end{array}\right\}$$

$$B_2=t(2t-1)(s-2t)(r-2t)$$
$$\left\{\begin{array}{l} t^2\left(-12r^2s-12rs^2+4rs+12r^2s^2-12r^2-12s^2+16r\right.\\ \left.+16s\right)+t\left(-6r^2s^2+6rs+2r^2s+2rs^2-10r-10s+8r^2\right.\\ \left.+8s^2\right)+\left(-5rs-3r^2s^2+4r^2s+4rs^2\right) \end{array}\right\}$$

$$D_0=rs\left\{\begin{array}{l} 60r+60s-49r^2s-49rs^2+16r^3s+16rs^3+91r^2s^2-24r^2s^3\\ -24r^3s^2+6r^3s^3-5rs-65r^2-65s^2+16r^3+16s^3 \end{array}\right\}$$

$$D_1=(s-1)(r-1)$$
$$\left\{\begin{array}{l} 60r+60s-49r^2s-49rs^2+16r^3s+16rs^3+91r^2s^2-24r^2s^3\\ -24r^3s^2+6r^3s^3-5rs-65r^2-65s^2+16r^3+16s^3 \end{array}\right\}$$

$$D_2=r(r-1)(r-s)$$
$$\left\{\begin{array}{l} 60r+60s-49r^2s-49rs^2+16r^3s+16rs^3+91r^2s^2-24r^2s^3\\ -24r^3s^2+6r^3s^3-5rs-65r^2-65s^2+16r^3+16s^3 \end{array}\right\}$$

$$D_3=s(s-1)(r-s)$$
$$\left\{\begin{array}{l} 60r+60s-49r^2s-49rs^2+16r^3s+16rs^3+91r^2s^2-24r^2s^3\\ -24r^3s^2+6r^3s^3-5rs-65r^2-65s^2+16r^3+16s^3 \end{array}\right\}$$

$$D_4=\left\{\begin{array}{l} 60r+60s-49r^2s-49rs^2+16r^3s+16rs^3+91r^2s^2-24r^2s^3\\ -24r^3s^2+6r^3s^3-5rs-65r^2-65s^2+16r^3+16s^3 \end{array}\right\}$$

The first derivatives of the coefficients of (5), (noting that $\mathrm{d}t=\dfrac{1}{h}\,\mathrm{d}x$ in the transformation equation), are obtained as:

$$\alpha_0'(t)=\frac{A_0'}{hD_0},\quad \alpha_1'(t)=\frac{A_1'}{hD_1},\quad \tau_1'(t)=\frac{A_2'}{hD_2},\quad \tau_2'(t)=\frac{A_3'}{hD_3}$$
$$\beta_0'(t)=\frac{B_0'}{6D_4},\quad \beta_1'(t)=\frac{B_1'}{3D_4},\quad \beta_2'(t)=\frac{B_2'}{6D_4},\ \text{where}$$

$$A_0' = 2\left\{\begin{array}{l} t^5\left(1152r^2s+1152rs^2-4608rs+3072+1152r^3+1152s^3-4608r^2\right.\\ \left.-4608s^2+3072r+3072s\right)+t^4\left(-480r^3s-480rs^3-480r^2s^2-6480\right.\\ -480r^4-480s^4-480r^3-480s^3+7920r^2+7920s^2-6480r-6480s\\ \left.-480r^2s-480rs^2+7920rs\right)+t^3\left(-2400r^2s-2400rs^2+960r^2s^2+4000\right.\\ +960r^4+960s^4-2400r^3-2400s^3-2400r^2-2400s^2+4000r+4000s\\ \left.+960r^3s+960rs^3-2400rs\right)+t^2\left(-720rs-480r^2s^2-720-480r^4\right.\\ -480s^4+1680r^3+1680s^3-720r^2-720s^2-720r-720s-480r^3s\\ \left.-480rs^3+1680r^2s+1680rs^2\right)+\left(6r^4s^4-24r^3s^4-24r^4s^3+91r^3s^3\right.\\ +16r^2s^4+16r^4s^2+11r^2s^2-49r^2s^3-49r^3s^2+16r^4s+16rs^4+60rs\\ \left.+60r^2+60s^2-65r^3-65s^3+16r^4+16s^4\right) \end{array}\right\}$$

$$A_1' = 2\left\{\begin{array}{l} t^5\left(1152r^2s+1152rs^2-5760rs+1152r^3+1152s^3-5760s^2-5760r^2\right.\\ \left.+7680r+7680s\right)+t^4\left(-480r^3s-480rs^3+8400rs-480r^2s^2-480r^4\right.\\ \left.-480s^4-8400r^2-8400s^2-14400r-14400s\right)+t^3\left(9600r^3s+960rs^3\right.\\ -3360r^2s-3360rs^2+960r^2s^2+960r^4+960s^4-3360r^3-3360s^3\\ \left.+6400r+6400s\right)+t^2\left(-480r^3s-480rs^3+2160r^2s+2160rs^2-2400rs\right.\\ \left.-480r^2s^2-480r^4-480s^4+2160r^3+2160s^3-2400r^2-2400s^2\right)\\ +\left(6r^4s^4-30r^3s^3-30r^3s^4+145r^3s^3+40r^2s^4+40r^4s^2+200r^2s^2\right.\\ \left.-180r^3s^2-180r^2s^3\right) \end{array}\right\}$$

$$A_2' = 2\left\{\begin{array}{l} t^5\left(-3072-1152s^3+4608s^2-3072s\right)+t^4\left(6480+480s^4+480s^3-7920s^2\right.\\ \left.+6480s\right)+t^3\left(-4000-960s^4+2400s^3-4000s\right)+t^2\left(720+480s^4-1680s^3\right.\\ \left.+720s^2+720s\right)+\left(-60s^2+65s^3-16s^4\right) \end{array}\right\}$$

$$A_3' = 2\left\{\begin{array}{l} t^5\left(-3072-1152r^3+4608r^2-3072r\right)+t^4\left(6480+480r^4+480r^3-7920r^2\right.\\ \left.+6480r\right)+t^3\left(-4000-960r^4+2400r^3+2400r^2-4000r\right)+t^2\left(720+480r^4\right.\\ \left.-1680r^3+720r^2+720r\right)+\left(-60r^2+65r^3-16r^4\right) \end{array}\right\}$$

$$B_0' = \left\{\begin{array}{l} t^5\left(-2880r^2s-2880rs^2+14016rs+576r^2s^2+18432+4032s^2+4032r^2\right.\\ \left.-17664r-17664s\right)+t^4\left(1200r^3s+1200rs^3-480r^2s-480rs^2-18240rs\right.\\ -240r^3s^2-240r^2s^3+960r^2s^2-38880-1680r^2-1680s^2-1680r^3-1680s^3\\ \left.+30960r+30960s\right)+t^3\left(-2144r^3s-2144rs^3+8576r^2s+8576rs^2-4480rs\right.\\ +96r^3s^3+96r^2s^3+96r^3s^2-2144r^2s^2+2400+3616r^3+3616s^3-9440r^2\\ \left.-9440s^2-9440r-9440s\right)+t^2\left(624r^3s+624rs^3-4116r^2s-4116rs^2+8220rs\right.\\ -216r^3s^3+624r^3s^2+624r^2s^3-1236r^2s^2-4320-2256s^3-2256r^3+8220r^2\\ \left.+8220s^2-4320r-4320s\right)+t\left(-120rs+144r^3s^3-576r^3s^2-576r^2s^3\right.\\ +2184r^2s^2+384r^3s+384rs^3-1176r^2s-1176rs^2+1440r+1440s+384r^3\\ \left.+384s^3-1560r^2-1560s^2\right)+\left(-360rs-21r^3s^3-391r^2s^2+92r^2s^3+92r^3s^2\right.\\ \left.+390r^2s+390rs^2-96r^3s-96rs^3\right) \end{array}\right\}$$

$$B_1' = \left\{\begin{array}{l} t^5\left(-1728r^2s-1728rs^2+3648rs+576r^2s^2-1728r^2-1728s^2\right.\\ \left.+5736r+5736s\right)+t^4\left(720r^3s+720rs^3+1440r^2s+1440rs^2-7920rs\right.\\ -240r^3s^2-240r^2s^3+480r^2s^2+720r^3+720s^3+720r^2+720s^2\\ \left.-8640r-8640s\right)+t^3\left(-1184r^3s-1184rs^3+1376r^2s+1376rs^2-5120rs\right.\\ +96r^3s^3+96r^2s^3+96r^3s^2-1184r^2s^2-1184r^3-1184s^3+2560r^2\\ \left.+2560s^2+2560r+2560s\right)+t^2\left(336r^3s+336rs^3-624r^2s-624rs^2\right.\\ -960rs-144r^3s^3+336r^3s^2+336r^2s^3-624r^2s^2+336r^3+336s^3-960r^2\\ \left.-960s^2\right)+\left(9r^3s^3+80r^2s^2-28r^2s^3-28r^3s^2\right) \end{array}\right\}$$

$$B_2' = \left\{ \begin{array}{l} t^5\left(-576r^2s - 576rs^2 + 192rs + 576r^2s^2 - 576r^2 - 576s^2\right. \\ \left.+768r + 768s\right) + t^4\left(240r^3s + 240rs^3 + 480r^2s + 480rs^2 - 480rs\right. \\ -240r^3s^2 - 240r^2s^3 + 240r^3 + 240s^3 + 240r^2 + 240s^2 \\ \left.-720r - 720s\right) + t^3\left(-224r^3s - 224rs^3 - 64r^2s - 64rs^2 + 320rs\right. \\ +96r^3s^3 + 96r^2s^3 + 96r^3s^2 - 224r^2s^2 - 224r^3 - 224s^3 + 160r^2 \\ \left.+160s^2 + 160r + 160s\right) + t^2\left(48r^3s + 48rs^3 - 12r^2s - 12rs^2 - 60rs\right. \\ -72r^3s^3 + 48r^3s^2 + 48r^2s^3 - 124r^2s^2 + 48r^3 + 48s^3 - 60r^2 \\ \left.-60s^2\right) + \left(3r^3s^3 - 4r^3s^2 - 4r^2s^3 + 5r^2s^2\right) \end{array} \right\}$$

We obtained the required 2−Step hybrid methods by evaluating (5) at $t = 1$ as

$$y_{n+2} = -\frac{1}{D_0}\alpha_0 y_0 - \frac{1}{D_2}\tau_1 y_{n+r} + \frac{1}{D_2}\alpha_1 y_{n+1} + \frac{1}{D_3}\tau_2 y_{n+s} + \frac{h^2}{6D_4}(\beta_0 f_n - 2\beta_1 f_{n+1} + \beta_2 f_{n+2}) \quad (6)$$

where

$$\alpha_0 = -(s-2)(r-2)\left\{21(r^s + rs^2) - 8(r^3 + rs^3) - 12(r^3s^2 + r^2s^3) + 3rs +19r^2s^2 + 6r^3s^3 - 18(r+s) + 9(r^2+s^2) - 12\right\}$$

$$\alpha_1 = 2(s-2)(r-2)\left\{6r^3s^3 + 49r^2s^2 - 20rs + 4(r^3s + rs^3) - 18(r^3s^2 + r^2s^3) +8(r^3+s^3) - 20(r^2+s^2)\right\}$$

$$\tau_1 = -2(s-2)(4s^3 + 9s^2 + 178s - 12)$$

$$\tau_2 = 2(r-2)(4r^3 + 9r^2 + 178r - 12)$$

$$\beta_0 = (s-2)(r-2) \left\{-6(rs^2 + r^2s) + 3r^2s^2 + 5rs - 4(r^2 - s^2) + 760(r-s) - 36\right\}$$

$$\beta_1 = -(s-2)(r-2)\left\{30(rs^2 + r^2s) + 20(r^2+s^2) - 48(r-s) - 15r^2s^2 - 52rs\right\}$$

$$\beta_2 = -(s-2)(r-2)\left\{-6(rs^2 + r^2s) - 4(r^2+s^2) + 6(r-s) + 3r^2s^2 + 5rs\right\}$$

$$D_0 = rs\left\{60(r+s) - 49(rs^2 + r^2s) + 16(r^3s + rs^3) - 24(r^2s^3 + r^3s^2) +16(r^3+s^3) - 65(r^2+s^2) - 5rs + 91r^2s^2 + 6r^3s^3\right\}$$

$$D_1 = r(r-1)(r-s)\left\{60(r+s) - 49(rs^2 + r^2s) + 16(r^3s + rs^3) -24(r^2s^3 + r^3s^2) + 16(r^3+s^3) - 65(r^2+s^2) - 5rs + 91r^2s^2 + 6r^3s^3\right\}$$

$$D_2 = (s-1)(r-1)\left\{60(r+s) - 49(rs^2 + r^2s) + 16(r^3s + rs^3) -24(r^2s^3 + r^3s^2) + 16(r^3+s^3) - 65(r^2+s^2) - 5rs + 91r^2s^2 + 6r^3s^3\right\}$$

$$D_3 = s(s-1)(r-s)\left\{60(r+s) - 49(rs^2 + r^2s) + 16(r^3s + rs^3) -24(r^2s^3 + r^3s^2) + 16(r^3+s^3) - 65(r^2+s^2) - 5rs + 91r^2s^2 + 6r^3s^3\right\}$$

$$D_4 = 60(r+s) - 49(rs^2 + r^2s) + 16(r^3s + rs^3) - 24(r^2s^3 + r^3s^2) + 16(r^3+s^3)$$

$$- 65(r^2 + s^2) - 5rs + 91r^2s^2 + 6r^3s^3$$

The first derivative of (6) is

$$y'_{n+2} = -\frac{1}{hD_0}\alpha'_0 y_0 - \frac{1}{hD_1}\tau'_1 y_{n+r} + \frac{1}{hD_2}\alpha'_1 y_{n+1} + \frac{1}{hD_3}\tau'_3 y_{n+s} + \frac{h}{6D_4}(\beta'_0 f_n - 2\beta'_1 f_{n+1} + \beta'_2 f_{n+2}) \quad (7)$$

where

$$\alpha'_0 = -2\{16(rs^4 + r^4s) - 49(rs^3 + r^3s) - 53(rs^2 + r^2s) + 252rs + 11r^2s^2 + 91r^3s^3 + 6r^4s^4 + 16(r^4 + s^4) - 113(r^3 + s^3) + 252(r^2 + s^2) - 128(r + s) - 24(r^4s^3 + r^3s^4) + 16(r^4s^2 + r^2s^4) - 49(r^3s^2 - r^2s^3) - 128\}$$

$$\alpha'_1 = 2\{240rs - 48(rs^2 + r^2s) - 30(r^4s^3 + r^3s^4) - 180(r^3s^2 + r^2s^3) + 40(r^2s^4 + r^4s^2) - 48(r^3 + s^3) + 240(r^2 + s^2) - 320(r - s) + 6r^4s^4 + 145r^3s^3 + 200r^2s^2\}$$

$$\tau'_1 = -2(128 - 16s^4 + 2417s^3 - 252s^2 + 128s)$$

$$\tau'_2 = 2(128 - 16s^4 + 2417s^3 - 252s^2 + 128s)$$

$$\beta'_0 = 314(rs^2 + r^2s) - 32(rs^3 + r^3s) - 964rs - 51r^2s^2 + 3r^3s^3 - 4(r^3s^2 + r^2s^3) + 66(r^3 + s^3) - 428(r^2 + s^2) + 976(r + s) - 768$$

$$\beta'_1 = 464(rs^2 + r^2s) - 128(rs^3 + r^3s) + 164(r^3s^2 + r^2s^3) - 128(r^3 + s^3) + 592(r^2 + s^2) - 704(r + s) - 112rs - 672r^2s^2 - 399r^3s^3$$

$$\beta'_2 = 64(rs^3 + r^3s) - 172(rs^2 + r^2s) - 100(r^2s^3 + r^3s^2) + 64(r^3 + s^3) + 916(r^2 + s^2) + 208(r + s) - 28rs + 345r^2s^2 + 27r^3s^3$$

In order to test the properties of the continuous methods (6), we chose the values of r and s in the mid-subintervals $[0, 1]$ and $[1, 2]$, to have a particular discrete hybrid method

$$y_{n+2} = \frac{32}{3}y_{n+\frac{3}{2}} - \frac{58}{3}y_{n+1} + \frac{32}{3}y_{n+\frac{1}{2}} - y_n + \frac{h^2}{36}(f_{n+2} - 62f_{n+1} + f_n) \quad (8)$$

$$p = 6,\ c_{p+2} = -2.9624 \times 10^{-5}$$

and its first derivative as

$$y'_{n+2} = \frac{1}{225h}\left\{6064y_{n+\frac{3}{2}} - 12287y_{n+1} + 6832y_{n+\frac{1}{2}} - 609y_n\right\} + \frac{h}{5400}(1027f_{n+2} - 24626f_{n+1} + 379f_n) \quad (9)$$

3. Implementation of the Method

Implementation of the discrete method (8) obtained from (6) for problem (1) requires the generation of some starting values. This is obtained in Predictor-Corrector mode of the same order of accuracy. The following symmetric explicit predictor scheme and its derivative, of the same order with the corrector scheme, are obtained using the same procedure in Section 2 for y_{n+2} and y'_{n+2}.

$$y_{n+2} = -16y_{n+\frac{3}{2}} + 34y_{n+1} - 16y_{n+\frac{1}{2}} - y_n + \frac{h^2}{3}\left(2f_{n+\frac{1}{2}} + 11f_{n+1} + 2f_{n+\frac{3}{2}}\right) \quad (10)$$

$$p = 6,\ c_{p+2} = 2.3768 \times 10^{-5}$$

$$y'_{n+2} = -\frac{3176}{21h}y_{n+\frac{3}{2}} + \frac{6245}{21h}y_{n+1} - \frac{2920}{21h}y_{n+\frac{1}{2}} - \frac{149}{21h}y_n + \frac{h}{630}\left(18f_n - 3394f_{n+\frac{1}{2}} - 19531f_{n+1} - 2818f_{n+\frac{3}{2}}\right) \quad (11)$$

Other explicit schemes were also generated to evaluate other starting values and Taylor's series was used to evaluate the values for y_{n+i}, $i = \frac{1}{2}, 1, \frac{3}{2}$, as

$$y_{n+i} = y_n + \frac{h}{2}y'_n + \frac{\left(\frac{h}{2}\right)^2}{2!}f_n + \frac{\left(\frac{h}{2}\right)^3}{3!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + \ldots + O(h^6) \quad (12)$$

and

$$y'_{n+i} = y'_n + \frac{h}{2}f_n + \frac{h^2}{2!}\left\{\frac{\partial f_n}{\partial x_n} + y'\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + \ldots + O(h^6) \quad (13)$$

3.1. Numerical Examples

The developed method is applied to solve linear and non-linear second order initial value problems and the results are shown in Tables 1-3.

Problem 1.

$$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0, \quad y(1) = 1 = y'(1),\ h = \frac{1}{320}.$$

The exact solution is

$$y(x) = \frac{5x^3 - 2}{3x^4}.$$

Absolute errors $|y_{\text{exact}} - y_{\text{computed}}|$ obtained with the method for problem 1 is compared with that of Kayode (2010) 5-step Adams Moulton-type method of same order of accuracy.

x	y_{exact}	y_{computed}	Errors in Kayode, (2010) $k=5,\ h=\frac{1}{320}$	Errors in new scheme (8), $k=2,\ h=\frac{1}{320}$
1.0094	1.0089449950888376	1.008944898477578	9.6400$e-$07	9.661126$e-$08
1.0125	1.0117410181679884	1.0117409239106636	3.6750$e-$06	9.425732$e-$08
1.0156	1.0144475426864139	1.0144474507153338	5.9320$e-$06	9.197108$e-$08
1.0188	1.0170664942356724	1.0170664044851787	6.2160$e-$06	8.975049$e-$08
1.0219	1.0195997547562876	1.0195996671626961	7.4430$e-$06	8.759359$e-$08
1.0250	1.0220491636294318	1.0220490781309679	7.7370$e-$06	8.549846$e-$08
1.0281	1.0244165187384027	1.0244164352751302	4.3530$e-$06	8.346327$e-$08
1.0313	1.0267035775008062	1.0267034960145822	1.1610$e-$06	8.148622$e-$08

Table 1: Result of Problem 1

x	y_{exact}	y_{computed}	Errors in Badmus &, Yahaya (2009)	Errors in new scheme (8)
0.1	1.0500417292784914	1.0500417264723956	5.891000$e-$06	2.806096$e-$09
0.2	1.1003353477310756	1.1003353320450284	8.239900$e-$05	1.568605$e-$08
0.3	1.1511404359364668	1.1511403957190882	3.464210$e-$04	4.021738$e-$08
0.4	1.2027325540540821	1.2027324751796589	7.521010$e-$04	7.887442$e-$08
0.5	1.2554128118829952	1.2554126761094033	1.380283$e-$03	1.357736$e-$07

Table 2: Result of Problem 2

Problem 2.

$$y'' = x(y')^2,\ y(0) = 1,\ y'(0) = \frac{1}{2},\ h = \frac{1}{30}.$$

The exact solution is

$$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right).$$

Absolute errors $|y_{\text{exact}} - y_{\text{computed}}|$ obtained with the method for problem 2 is compared with that of Badmus and Yahaya (2009) block method of same order of accuracy.

Problem 3.

$$y_1'' = -y_2' + \cos x,\ y_1(0) = -1,\ y_1'(0) = -1.$$

The exact solution is

$$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right),$$

$$y_2'' = y_1' + \sin x,\ y_2(0) = 1,\ y_2'(0) = 0,\ x \in [0,\ 4\pi],\ h = 0.01.$$

x	y_{1_exact}	y_{2_exact}	$y_{1\_computed}$	$y_{2\_computed}$	Errors in y_1	Errors in y_2	Time
0.1	0.9950041652	0.0998334166	0.9950040976	0.0998356629	6.766228e–08	2.246316e–06	0.0893
0.2	0.9800665778	0.1986693307	0.9800660253	0.1986779815	5.525014e–07	8.650784e–06	0.0935
0.3	0.9553364891	0.2955202066	0.9553346385	0.2955392544	1.850616e–06	1.904784e–05	0.0976
0.4	0.9210609940	0.3894183423	0.9210566678	0.3894515823	4.326158e–06	3.324003e–05	0.1017
0.5	0.8775825618	0.4794255386	0.8775742627	0.4794765054	8.299176e–06	5.096683e–05	0.1058

Table 3: Result of Problem 2

4. Conclusion

In this paper, we have demonstrated an easier application of linear 2−step method to solve general second order ordinary differential equations directly without reducing it to first order ordinary differential equations. The hybrid points in the methods are obtained at the y-function as against the usual f−function. This reduces the problem of additional function evaluations at the hybrid points. The accuracy of the derived discrete method from the continuous methods was tested with two linear and non linear problems as well as a system of second order problem. The results, when compared with existing methods (Kayode, 2010 and Badmus and Yahaya, 2009), performed better than the existing methods in terms of accuracy.

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