SOLVING SYSTEM OF NONLINEAR EQUATIONS USING FAMILY OF JARRATT METHODS

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Abstract: The aim of this paper is generalization fourth order Jarratt formula iterative methods for solving system of nonlinear equations (SNLE) of n-dimension with n-variables. We present three algorithms for solving (SNLE). We prove that these algorithms have convergence. Several numerical examples are tested of the new iterative methods. These new algorithms may be viewed as an extensions and generalizations of the existing methods for solving the system of nonlinear equations.

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Key Words: system of non-linear equations (SNLE), Jarratt iterative methods

1. Introduction

Recently, several iterative methods have been used to solve nonlinear equations.

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[1], [2], [3], [4], [5], [6], [7] and they are several researchers used and developed Jarratt’s iterative method [8] for solving nonlinear equations in one dimensional equations.


Finally, Hafiz ([19], [20], [21]) modified some iterative schemes to get new algorithms for solving system of nonlinear equations. It is also remarked that methods in which there are two evaluations of the first-order derivative and one of evaluation of the function (with the fourth-order convergence) are called as optimal Jarratt-type methods in literature. For further reading on this topic, one may consult [22], [23].

2. Iterative Methods

Newton’s iterative method is the best known method for finding a real or complex root $x$ of the nonlinear equations $f(x) = 0$ which is given by [24]

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad (1)$$
This method converges quadratically in some neighborhood of \( x \). It is also well-known that for any function \( H \), with \( H(0) = 1 \), \( H'(0) = 1/2, |H''(0)| < \infty \), the iterative method can be written as \([14]\).

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} H(t(x_i))
\]

(2)

Where

\[
t(x_i) = \frac{f(x_i) f''(x_i)}{[f'(x_i)]^2},
\]

(3)

Is iterative method of order 3.

Now we obtain the optimal fourth-order methods for the Newton iteration function \([14]\) can be written as.

\[
y_i = x_i - \frac{2}{3} \frac{f(x_i)}{f'(x_i)}
\]

(4)

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} H(t(x_i))
\]

(5)

Where

\[
t(x_i) = \frac{3}{2} \frac{f'(x_i) - f'(y_i)}{f'(x_i)}
\]

(6)

If we take \( H(t) = 1 + \frac{1}{2} \left( \frac{t}{1-t} \right) \) then eq (5), eq (6) lead the well-known Jarratt’s forth order method \([13], [14], [25]\)

\[
x_{i+1} = x_i - \left[ 1 - \frac{3}{2} \frac{f'(y_i) - f'(x_i)}{3f'(y_i) - f'(x_i)} \right] \frac{f(x_i)}{f'(x_i)}
\]

(7)

Where

\[
y_i = x_i - \frac{2}{3} \frac{f(x_i)}{f'(x_i)}
\]

(8)

If we take another \( H(t) = 1 + \frac{9}{6-4t} - \frac{9}{6-2t} \) in eq (5), eq (6) leads to another optimal fourth-order Jarratt’s method \([14], [25]\)

\[
x_{i+1} = x_i - w_1(x_i) - \frac{2}{3} w_2(x_i) + \frac{3f(x_i)}{f'(x_i) + f'(z_i)}
\]

(9)

Where

\[
w_1(x_i) = \frac{f(x_i)}{f'(x_i)}, \quad w_2(x_i) = \frac{f(x_i)}{f'(z_i)}, \quad z_i = x_i - \frac{2}{3} w_1(x_i)
\]

(10)
If we take another \( H(t) = 1 + \frac{t}{2} + \frac{t^2}{2} \) in eq (5), eq (6) leads to another optimal fourth-order Jarratt’s method \([14], [25]\)

\[
x_{i+1} = x_i - \left[ 1 + \frac{3}{4} \frac{f'(x_i) - f'(y_i)}{f'(x_i)} + \frac{9}{8} \left( \frac{f'(x_i) - f'(y_i)}{f'(x_i)} \right)^2 \right] \frac{f(x_i)}{f'(x_i)}
\]

(11)

Where

\[
y_i = x_i - \frac{2 f(x_i)}{3 f'(x_i)}
\]

(12)

Another optimal fourth-order Jarratt’s method \([14], [25]\)

Recently, an efficient fourth-order technique in which we have two evaluations of the first derivative and one evaluation of the function had been presented by Khattri and Abbasbandy in \([12]\) as comes next.

\[
x_{i+1} = x_i - \frac{f(x)}{f'(x_i)} \left[ I + \sum_{j=1}^{4} \alpha_i \left( \frac{f'(y_i)}{f'(x_i)} \right)^j \right]
\]

(13)

where \( \alpha_i \in \mathbb{R} \ni \alpha_1 = \frac{21}{8}, \alpha_2 = -\frac{9}{2}, \alpha_3 = \frac{15}{8}, \alpha_4 = 0 \), this formula can be written as \([12]\)

\[
y_i = x_i - \frac{2 f(x_i)}{3 f'(x_i)}
\]

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \left[ 1 + \left( \frac{21}{8} \right) \left( \frac{f'(y_i)}{f'(x_i)} \right)^1 + \left( -\frac{9}{2} \right) \left( \frac{f'(y_i)}{f'(x_i)} \right)^2 \right.
\]

\[
+ \left. \left( \frac{15}{8} \right) \left( \frac{f'(y_i)}{f'(x_i)} \right)^3 \right]
\]

(14)

3. System of Nonlinear Equations (SNLE)

The general form of a system of non-linear equations is

\[
f_1(x_1, x_2, \cdots, x_n) = 0,
f_2(x_1, x_2, \cdots, x_n) = 0,
\]

\[
\vdots
\]

\[
f_n(x_1, x_2, \cdots, x_n) = 0,
\]

(15)
where each function $f_i$ can be thought of as mapping a vector $x = (x_1, x_2, \ldots, x_n)$ of the $n$-dimensional space $\mathbb{R}^n$, into the real line $\mathbb{R}$. The system can alternatively be represented by defining a functional $F$, mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ by:

$$F(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n))^T$$

Using vector notation to represent the variables $x_1, x_2, \ldots, x_n$, a system (1) can be written as the form:

$$F(x) = 0 \quad (16)$$

The functions $f_1, f_2, \ldots, f_n$ are called the coordinate functions of $F$ \[24\]


**Definition 1.** Suppose that $x$ be the simple zero of sufficiently differentiable functions and consider the numerical solution of the system of equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth mapping that has continuous second order partial derivatives on a convex open set $D$, and that has a locally unique root $x$ in $D$, $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T$, $x = (x_1, x_2, \ldots, x_n)^T$ and $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a nonlinear function.

For solving system of nonlinear equations on n-dimensional and n-variables we develop fourth order iteration method in eq (7),equ (8) can be written as:

**Algorithm 1.** For a given $x_0$, compute the approximate solution $x_{i+1}$ by iterative method schemes in eq (7),equ (8) we get:

$$y_i = x_i - \frac{2}{3}J^{-1}(x_i)F(x_i),$$

$$x_{i+1} = x_i - \left[I - \frac{3}{2} (3J(y_i) - J(x_i))^{-1} (J(y_i) - J(x_i))\right] J^{-1}(x_i)F(x_i) \quad (17)$$

**Algorithm 2.** For a given $x_0$, compute the approximate solution $x_{i+1}$ by iterative scheme method of eq (9),equ (10) can be written as:

$$y_i = x_i - \frac{2}{3}J^{-1}(x_i)F(x_i),$$

$$x_{i+1} = x_i - [J(x_i)^{-1} + \frac{3}{2} J(y_i)^{-1}] F(x_i) + 3 \left[J(x_i) + J(y_i)\right]^{-1} F(x_i) \quad (18)$$

**Algorithm 3.** For a given $x_0$, compute the approximate solution $x_{i+1}$ by iterative scheme in equation eq (14) can be written as:

$$y_i = x_i - \frac{2}{3}J^{-1}(x_i)F(x_i),$$
\[ x_{i+1} = x_i - \left[ I + \frac{21}{8} J^{-1}(x_i) J(y_i) - \frac{9}{2} \left( J^{-1}(x_i) J(y_i) \right)^2 + \frac{15}{8} \left( J^{-1}(x_i) J(y_i) \right)^3 \right] J^{-1}(x_i) F(x_i) \] (19)

5. Convergence Analysis

In this section, we consider the convergence of our algorithm using the Taylor’s series technique.

**Theorem 1.** Let \( x^* \) be a sample zero of sufficient differentiable function \( F : \subseteq \mathbb{R}^n \to \mathbb{R}^n \) for an open interval. If \( x_0 \) is sufficiently close to \( x^* \), then the two step method defined by our algorithm (3) has convergence is at least of order 3.

**Proof.** Consider to

\[ y_i = x_i - \frac{2}{3} J^{-1}(x_i) F(x_i), \]

\[ x_{i+1} = x_i - \left[ I + \frac{21}{8} J^{-1}(x_i) J(y_i) - \frac{9}{2} \left( J^{-1}(x_i) J(y_i) \right)^2 + \frac{15}{8} \left( J^{-1}(x_i) J(y_i) \right)^3 \right] J^{-1}(x_i) F(x_i) \] (20)

where \( w_n = \frac{1}{2} (x_n + y_n) \). Let \( x^* \) be a simple zero of \( F \). Since \( F \) is sufficiently differentiable, by expanding \( F(x_n) \) and \( F'(x_n) \) about \( x^* \), we get

\[ F(x_n) = F(x^*) + F'(x^*)(x_n - x^*) + \frac{F^{(2)}(x^*) (x_n - x^*)^2}{2!} + \frac{F^{(3)}(x^*) (x_n - x^*)^3}{3!} + \cdots, \]

But \( F(x^*) = 0 \) then

\[ F(x_n) = F'(x^*) [E_n + C_2 E_n^2 + C_3 E_n^3 + O(||E_n^4||)], \] (21)

and

\[ F'(x_n) = F'(x^*) [I + 2C_2 E_n + 3C_3 E_n^2 + O(||E_n^3||)], \] (22)

where \( C_k = \frac{1}{k!} \frac{F^{(k)}(x^*)}{F'(x^*)} \), \( k = 2, 3, \ldots \) and \( E_n = x_n - x^* \).

Now from (21) and (22), we have (see [26])

\[ \frac{F(x_n)}{F'(x_n)} = E_n - C_2 E_n^2 + 2(C_2^2 - C_3) E_n^3 + O(||E_n^4||), \] (23)

From (20) and (23), we get

\[ y_n = x^* + \frac{1}{3} E_n + \frac{2}{3} \left( C_2 E_n^2 + 2(C_3 - C_2^2) E_n^3 \right) + O(||E_n^4||), \] (24)
From (17), we get,
\[ F(y_n) = F'(x^*)[(y_n - x^*) + C_2(y_n - x^*)^2 + C_3(y_n - x^*)^3 + O(||E_n^4||)] \]
and
\[ F'(y_n) = F'(x^*)[I + 2C_2(y_n - x^*) + 3C_3(y_n - x^*)^2 + O(||E_n^3||)] \]
\[ = F'(x^*)[I + \frac{2}{3}C_2E_n + \frac{4}{3}C_2^2E_n^2 + O(||E_n^3||)]. \]

In general, according to [26]:
\[ [I + M_2E_n + M_3E_n^2 + O_3]^{-1} = [I - M_2E_n + (M_2^2 - M_3)E_n^2 + O_3] \]
\[ J^{-1}(x_i) = F'(x^*)[I + 2C_2E_n + 3C_3E_n^2]^{-1}, \]
Set
\[ M_2 = 2C_2, \quad M_3 = 3C_3 \]
Then we will have
\[ J^{-1}(x_i)F'(y_n) = [I - \frac{4}{3}C_2E_n + (4C_2^2 - \frac{8}{3}C_3)E_n^2 + O(||E_n^3||)] \] (26)

Finally, From (22)
\[ E_{n+1} = (\frac{85}{9}C_2^3 - C_2C_3 + \frac{1}{9}C_4)E_n^4 + O(||E_n^5||) \] which shows that Algorithm (1) is at least a third order convergent method, the required result. Since asymptotic convergence of Newton method is \(c_2\) and from Theorem (1), we deduce that the convergence rate of our algorithm is better than the Newton’s method. And the cubic convergent method is vastly superior to the linear and the quadratically convergent methods [27]

6. Numerical Examples

For comparisons, we have used the third-order Hafiz and Bahgat method (HBM) [16] and Darvishi (DAM) [14] algorithms, respectively.
\[ x_{n+1} = x_n - 12 [J(x_n) + 10J(w_n) + J(y_n)]^{-1} F(x_n), \]
\[ x_{n+1} = x_n - 2 [F'(x_n) + F'(y_n)]^{-1} F(x_n), \]
where
\[ y_n = x_n - J^{-1}(x_n)F(x_n). \]
We present some examples to illustrate the efficiency of our proposed methods. Here, numerical results are performed by Maple 15 with 200 digits but only 14 digits are displayed. In Tables 1-5 we list the results obtained in Algorithms (1-3), which we called, Khirallah and Hafiz Methods (KHM1, KHM2, KHM3), respectively and compare them with Newton-Raphson method (NM), Hafiz and Bahgat method (HBM) [16], and Darvishi (DAM). The following stopping criteria is used for computer programs:

\[ ||x^{(n+1)} - x^{(n)}|| + ||F(x^{(n)})|| < 10^{-15} \]

and the computational order of convergence (COC) can be approximated using the following formula:

\[
\text{COC} \approx \frac{\ln \left( \frac{||x^{(n+1)} - x^{(n)}||}{||x^{(n)} - x^{(n-1)}||} \right)}{\ln \left( \frac{||x^{(n)} - x^{(n-1)}||}{||x^{(n-1)} - x^{(n-2)}||} \right)}
\]

Table 2 shows the number of iterations and the computational order of convergence (COC). \[ ||x^{(n+1)} - x^{(n)}|| \] and the norm of the function \[ F(x^{(n)}) \] is also shown in Table 2 for various methods.

### 6.1. Small Systems of Nonlinear Equations

**Example 1.** In a case of one dimension, consider the following nonlinear functions [12],

\[ f_1(x) = xe^{x^2} - \sin^2 x + 3 \cos x + 5, \text{ with } x_0 = -3 \text{ and } f_2(x) = e^{x^2 + 7x - 30} - 1 \]

with \( x_0 = 4 \).

**Example 2.** In a case two dimension, consider the following systems of nonlinear functions [28],

\[
F_3 (x) = \begin{cases} 
  f_1(x, y) = x^2 - 10x + y^2 + 8 = 0 \\
  f_2(x, y) = xy^2 + x - 10y + 8 = 0 
\end{cases}, \quad (x_0, y_0) = (2, 2).
\]

\[
F_4 (x) = \begin{cases} 
  f_1(x, y) = x^4y - xy + 2x - y - 1 = 0 \\
  f_2(x, y) = ye^{-x} + x - y - e^{-1} = 0 
\end{cases}, \quad (x_0, y_0) = (0.8, 0.8).
\]

**Example 3.** In a case three dimension, consider the following systems of nonlinear functions [28],

\[
F_5 (x) = \begin{cases} 
  f_1(x, y, z) = 15x + y^2 - 4z - 13 = 0 \\
  f_2(x, y, z) = x^2 + 10y - e^{-z} - 11 = 0 \\
  f_3(x, y, z) = y^3 - 25z + 22 = 0 \\
  f_4(x, y, z) = 3x - \cos(yz) - 0.5 = 0 
\end{cases}, \quad X_0 = (10, 6, -5).
\]

\[
F_6 (x) = \begin{cases} 
  f_2(x, y, z) = x^2 - 81(y + 0.1)^2 + \sin z + 1.06 = 0 \\
  f_3(x, y, z) = e^{-xy} + 20z + \frac{10\pi - 2}{3} = 0 
\end{cases}, \quad X_0 = (1.1, 1.1, 1.1).
\]
### Table 1: Number of iterations for Example 1

<table>
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<tr>
<th></th>
<th>F&amp;M</th>
<th>IT</th>
<th>COC</th>
<th>$|x^{(n+1)}-x^{(n)}|$</th>
<th>$|F(x^{(n)})|$</th>
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### Table 2: Number of iterations for Example 2

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<th>$|F(x^{(n)})|$</th>
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6.2. Large Systems of Nonlinear Equations

In this subsection, we test HPM with some sparse systems with $m$ unknown variables. In examples (4-6), we compare the NR method with the proposed HPM method focusing on iteration numbers [15].

**Example 4.** Consider the following system of nonlinear equations:

$$F_7: f_i = e^{x_i} - 1, \quad i = 1, 2, ..., m.$$ 

The exact solution of this system is $X^* = [0, 0, ..., 0]^T$. To solve this system, we set $x_0 = [0.5, 0.5, ..., 0.5]^T$ as an initial value. Table 4 is shown the result.

**Example 5.** Consider the following system of nonlinear equations:

$$F_8: f_i = x_i^2 - \cos(x_i - 1), \quad i = 1, 2, ..., m.$$ 

One of the exact solutions of this system is $X^* = [1, 1, ..., 1]^T$. To solve this system, we set $x_0 = [2, 2, ..., 2]^T$ as an initial value. The results are presented in Table 4.

**Example 6.** Consider the following system of nonlinear equations [13]:

$$F_9: f_i = \cos x_i - 1, \quad i = 1, 2, ..., m.$$
One of the exact solutions of this system is $x^* = [0,0,\ldots,0]^T$. To solve this system, we set $x_0 = [2,2,\ldots,2]^T$ as an initial guess. The results are presented in Table 4.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$F_7$</th>
<th>$F_8$</th>
<th>$F_9$</th>
<th>$F_7$</th>
<th>$F_8$</th>
<th>$F_9$</th>
<th>$F_7$</th>
<th>$F_8$</th>
<th>$F_9$</th>
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<tr>
<td>$\varepsilon = 10^{-15}$</td>
<td>$m = 50$</td>
<td>$m = 75$</td>
<td>$m = 100$</td>
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<td>6</td>
<td>7</td>
<td>53</td>
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<td>4</td>
<td>5</td>
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</tbody>
</table>

Table 4: Number of iterations for Examples 4-6

**Example 7.** Consider the nonlinear boundary value problem [17]

$$y'' = -(y'^2 - y + \ln x), \ 1 \leq x \leq 2, \ y(1) = 0, \ y(2) = \ln 2,$$

whose exact solution is $y = \ln x$. We consider the following partition of the interval:

$$x_0 = 1, \ x_n = 2, \ x_j = x_0 + jh, \ h = \frac{1}{m}, \ j = 1, 2, \cdots, m-1$$

Let us define now

$$y_0 = y(x_0) = 0, \ y_m = \ln 2, \ y_i = f(x_i), \ i = 1, 2, \cdots, m-1.$$

If we discretize the problem by using the second order finite differences method defined by the numerical formulas

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2h}, \ i = 1, 2, \cdots, m-1,$$

$$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \ i = 1, 2, \cdots, m-1,$$

then, we obtain a $(m \times 1) \times (m \times 1)$ system of nonlinear equations $F_{10}$:

$$4y_2 + y_2^2 + 4y_1(h^2 - 2) - 4h^2 \ln x_2 = 0,$$

$$4(y_{i+1} + y_{i-1}) + (y_{i+1} - y_{i-1})^2 + 4y_i(h^2 - 2) - 4h^2 \ln x_{i+1} = 0,$$

$$4(2 + y_{m-2}) + (\ln 2 - y_{m-2})^2 + 4y_{m-1}(h^2 - 2) - 4h^2 \ln x_m = 0,$$

$$4\ln 2 + y_{m-2}^2 + 4y_{m-1}(h^2 - 2) - 4h^2 \ln x_m = 0.$$
We take $X_0$ with $y_k^{(0)} = \ln(\frac{k}{2})$, $k = 1, 2, \cdots, m - 1$, as a starting point. In particular, we solve this problem for $m = 50, 75$ and $100$. The numerical results for the above system of nonlinear equations are presented in Table 5. The number of iterations of methods HBM, KHM1, KHM2 and KHM3 are equal but method KHM2, . . . KHM3 has advantage of they are free from second derivatives, over methods NM and HBM, because the cost of computing second derivatives is very high, see Table 5.

In Table 1 – Table 5, we list the results obtained by modified iterations methods. As we see from this tables, it is clear that, in most cases, the result obtained by DAM, HBM, KHM1,. . . KHM3 are equivalent and they very superior to that obtained NM.

### 7. Conclusions

In this paper, we presented three new algorithms for solving the system of nonlinear equations by generalizing and applying fourth order Jarratt iterative methods and used these algorithms for the first time for solving initial value problem. These methods have the same efficiency as the other fourth-order methods in the literature. We conclude from the numerical examples that
the proposed methods have at least equal performance as compared with the other methods of the same order. Moreover, our proposed methods provide highly accurate results in a less number of iterations as compared with Newton-Raphson method.

References


