A COORDINATED POLICY MODEL IN DISTRIBUTION SYSTEM WITH STOCHASTIC DEMAND

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Abstract: This paper addresses a coordinated inventory and ordering policies in two-level distribution system with stochastic demand composing of a single supplier and multiple retailers, in which the supplier (distribution center) takes base-stock policy and deals in multiple products. A mathematical model to maximize expected system profit is developed. Some properties of the model and existence of solutions of the model are analyzed. Then corresponding solution methods are proposed. Last, three cases on demand are studied and some future researches areas are identified. This work can provide insight for purchasing and inventory decisions of distribution system and supply chain management.

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1. Introduction

In downstream supply chain, distribution decisions is an important factor affecting performance of supply chain management. A number of researchers have developed models to study distribution system. Earlier researches with respect to distribution decisions focus on EOQ under deterministic demand or constant demand rate. For example see: Goyal (1976), [3], Monahan (1984), [10], Lal and Richard (1984), [6], Lee and Rosenblatt (1986), [7]. They assumed that the supplier had retailers’ complete information and had no backlog and stockout to the retailer or the buyer. The supplier induced the retailer or the buyer to increase order quantity at a time to reduce order frequency so that transaction costs such as production set-up cost, order processing cost and so forth can be lowered by various price discount. Kohli and Park (1989), see [4], Kohli and Park (1994), see [5] attempted to lower costs and increase profits by cooperative game theory and coordinated decisions respectively, but they did still consider only the situation that total demand is deterministic within some period. Fangruo Chen (2000), see [1] considered a distribution system with a central warehouse and multiple retailers, in which customer demand arrived at each retailers continuously at constant rate, a single product was dealt in and stockouts at the retailers were backlogged. Chia-Shin Chung et al (2001), see [2] addressed the inventory placement problem in a serial supply chain facing a stochastic demand for a single period. All stocking decisions were made before the demand occurs, all costs were proportional. There were no fixed costs and no backlog, all unsatisfied demands were lost, and proposed an algorithm to determine inventories in different stages to maximize the expected profit. Lee and Kim (2002), see [8] tried to find a production and distribution policy to minimize total cost by combining analytic method with simulation method. Thomas and Hackman (2003), see [11] analyzed a supply chain environment in which a distributor facing price-sensitive demand obtained fixed order frequency and deterministic order quantity by favorable prices. Moreover, there are much work concerning distribution or distribution system, but most of them address such situation in which only a single product is dealt in under deterministic demand or constant demand rate. Varieties of demands, diversification of product, disposal of excess demand and otherness of life cycle of product are not considered adequately.

This paper addresses a coordinated inventory and ordering policies in two-level distribution system with stochastic demand composing of a single supplier and multiple retailers, in which the supplier (distribution center) takes base-stock policy and deals in multiple products with short life cycle, replenishes
periodically at each stage within a period, and gains deterministic base stock
at the beginning of each stage within a period. A retailer deals in a sort of
product only. Retailer does not hold inventory or his inventory can be neglected.
Retailer decides a deterministic quantity ordering from the distribution center
at each stage within a period. The part of demand over the deterministic
quantity may be met by placing a second order from the distribution center
timely in this stage within this period; some backorder cost must be paid for
the backlog due to a second order. If demand exceeds the base stock in this stage
at distribution center, all unsatisfied demands are lost, but both distribution
center and retailer must pay some punishment cost. Owing to short life cycle,
surplus products both at distribution center and at retailer must be disposed by
lower price or even spending extra costs so that new products are dealt in next
stage. Demand only occurring at retailer follows some stochastic distribution.

2. Model

To determine the supplier’s base stock and the retailer’s deterministic order
quantity to maximize the expected total profit within a period, corresponding
mathematical model must be developed. For convenience, primary notations
used in this paper are summarized as follows:

\[ i = 1, 2, \ldots, N \] index the product variety and retailers;
\[ t = 1, 2, \ldots, T \] index stages in a period;
\[ \beta_i \] the increased proportion of price of product \( i \) placed a second order
within the same stage;
\[ b_{si}, Q_{si} \] the supplier’s base stock and the retailer’s deterministic order
quantity of product \( i \) at stage \( t \) respectively;
\[ s^b_{si}, s^1_{si}, s^2_{si} \] the supplier’s purchasing set-up cost and the retailer’s first and
second order set-up cost of product \( i \) within the same stage respectively;
\[ c_i, w_i, p_i \] the supplier’s purchasing unit price and wholesale unit price and
the retailer’s sale unit price of product \( i \) respectively;
\[ h_i, \theta^s_i, \theta^b_i \] inventory holding cost of product \( i \) at distribution center, unit
punishment cost of product \( i \) paid by the distribution center and the retailer
respectively;
\[ b_i \] unit backlogging cost of product paid to customers by the retailer;
\[ d^s_{si}, d^b_{si} \] unit value of product \( i \) at end of stage \( t \) at distribution center and
retailer respectively, which may be negative;
Thus as follows: three condition profit functions of distribution center and retailer respectively sale revenue and costs. According to demand within a stage, one can get at stage $i$ distribution function of product $t$.

$$\pi_{it}^s(x_{it} | x_{it} \leq Q_{it}) = (w_i - c_i)Q_{it} + (d_{it}^s - c_i)(bs_{it} - Q_{it}) - s_i^s - h_i(bs_{it} - Q_{it}),$$  \hspace{1cm} (1)

$$\pi_{it}^s(x_{it} | Q_{it} < x_{it} \leq bs_{it}) = (w_i - c_i)Q_{it} + ((1 + \beta_i)w_i - c_i)(x_{it} - Q_{it}) + (d_{it}^s - c_i)(bs_{it} - x_{it}) - s_i^s - h_i(bs_{it} - Q_{it}),$$  \hspace{1cm} (2)

$$\pi_{it}^s(x_{it} | x_{it} > bs_{it}) = (w_i - c_i)Q_{it} + ((1 + \beta_i)w_i - c_i)(bs_{it} - Q_{it}) - s_i^s$$  
$$- h_i(bs_{it} - Q_{it}) - \theta_i^b(x_{it} - bs_{it}),$$  \hspace{1cm} (3)

$$\pi_{it}^b(x_{it} | x_{it} \leq Q_{it}) = (p_i - w_i)x_{it} + (d_{it}^b - w_i)(Q_{it} - x_{it}) - s_i^{b1},$$  \hspace{1cm} (4)

$$\pi_{it}^b(x_{it} | Q_{it} < x_{it} \leq bs_{it}) = (p_i - w_i)Q_{it} + (p_i - (1 + \beta_i)w_i)(x_{it} - Q_{it})$$  
$$- s_i^{b1} - s_i^{b2} - b_i(x_{it} - Q_{it}),$$  \hspace{1cm} (5)

$$\pi_{it}^b(x_{it} | x_{it} > bs_{it}) = (p_i - w_i)Q_{it} + (p_i - (1 + \beta_i)w_i)(bs_{it} - Q_{it}) - s_i^{b1}$$  
$$- s_i^{b2} - b_i(bs_{it} - Q_{it}) - \theta_i^b(x_{it} - bs_{it}).$$  \hspace{1cm} (6)

Thus

$$\pi_{it} = \int_0^{Q_{it}} [\pi_{it}^b(x_{it} | x_{it} \leq Q_{it}) + \pi_{it}^s(x_{it} | x_{it} \leq Q_{it})]f_{it}(x_{it})dx_{it}$$  
$$+ \int_{bs_{it}}^{Q_{it}} [\pi_{it}^b(x_{it} | Q_{it} < x_{it} \leq bs_{it}) + \pi_{it}^s(x_{it} | Q_{it} < x_{it} \leq bs_{it})]f_{it}(x_{it})dx_{it}$$  
$$+ \int_{bs_{it}}^{\infty} [\pi_{it}^b(x_{it} | x_{it} > bs_{it}) + \pi_{it}^s(x_{it} | x_{it} > bs_{it})]f_{it}(x_{it})dx_{it}$$
\[ (p_i - w_i)Q_{it} + [(p_i - (1 + \beta_i)w_i - b_i)(bs_{it} - Q_{it}) - s_i^{b2}] [1 - F_{it}(Q_{it})] 
- s_i^{b1} - h_i(bs_{it} - Q_{it}) - (p_i - d_i^{b}) \int_0^{Q_{it}} (Q_{it} - x_{it}) f_{it}(x_{it}) dx_{it} 
- (p_i - (1 + \beta_i)w_i - b_i) \int_{bs_{it}}^{bs_{it}} (bs_{it} - x_{it}) f_{it}(x_{it}) dx_{it} - (\theta_i^b + \theta_i^s) 
\times \int_{bs_{it}}^{\infty} (x_{it} - bs_{it}) f_{it}(x_{it}) dx_{it} + (w_i - c_i)Q_{it} + (1 + \beta)w_i - c_i 
\times (bs_{it} - Q_{it})[1 - F_{it}(Q_{it})] - ((1 + \beta_i)w_i - c_i) \int_{Q_{it}}^{bs_{it}} (bs_{it} - x_{it}) f_{it}(x_{it}) dx_{it} 
+ (d_i^{s} - c_i)(bs_{it} - Q_{it}) F_{it}(bs_{it}) - s_i^s - (d_i^{s} - c_i) \int_{Q_{it}}^{bs_{it}} (x_{it} - Q_{it}) f_{it}(x_{it}) dx_{it} 
= (p_i - c_i)Q_{it} - s_i^s - s_i^{b1} + [(p_i - c_i - b_i)(bs_{it} - Q_{it}) - s_i^{b2}] [1 - F_{it}(Q_{it})] 
- h_i(bs_{it} - Q_{it}) - (p_i - d_i^{b}) \int_0^{Q_{it}} (Q_{it} - x_{it}) f_{it}(x_{it}) dx_{it} 
- (p_i - c_i - b_i) \int_{Q_{it}}^{bs_{it}} (bs_{it} - x_{it}) f_{it}(x_{it}) dx_{it} - (\theta_i^b + (d_i^{s} - c_i) 
\times (bs_{it} - Q_{it}) F_{it}(bs_{it}) + \theta_i^s) \int_{bs_{it}}^{\infty} (x_{it} - bs_{it}) f_{it}(x_{it}) dx_{it} 
- (d_i^{s} - c_i) \int_{Q_{it}}^{bs_{it}} (x_{it} - Q_{it}) f_{it}(x_{it}) dx_{it}. \quad (7) \]

Let
\[ \pi_t = \sum_{i=1}^N \pi_{it}, \quad \pi_i = \sum_{t=1}^T \pi_{it}, \quad \pi = \sum_{i=1}^T \sum_{i=1}^N \pi_{it} = \sum_{i=1}^N \sum_{t=1}^T \pi_{it} = \pi(BS, Q), \]
\[ U_N = \{1, 2, \cdots, N\}, \quad U_T = \{1, 2, \cdots, T\}, \]
where \(BS = (bs_{it})_{N \times T}; Q = (Q_{it})_{N \times T}\)

Hence, a mathematical model to maximize expected system profit can be developed as follows:

\[ (P) \text{ Max } \pi = \pi(BS, Q), \quad (8) \]
\[ \text{s.t. } \sum_{i=1}^N (c_i bs_{it} + s_i^s) \leq A(t), \quad (9) \]
\[ \sum_{i=1}^N \alpha_i bs_{it} \leq B(t), \quad (10) \]
3. Solutions

Obviously, obtaining solution of problem (P) is quite hard. Hence, one considers the following relaxed problem (RP) first:

\[(RP) \quad \text{Max } \pi = \pi(BS, Q), \quad \text{s.t. } bs_{it}, Q_{it} \in R. \quad (15)\]

At the same time, assume:

(A1) \(d_{it}^{l} = d_{it}^{u} \leq c_{i} < w_{i} < p_{i}, \quad p_{i} - c_{i} - h_{i} - b_{i} > 0, \forall i \in U_{N}, t \in U_{T}.\)

(A2) \(f_{it}(\cdot)\) continuously differentiable, \(F_{it}(\cdot)\) is an increasing concave function, and \(F_{it}(0) \approx 0, \forall i \in U_{N}, t \in U_{T}.\)

Distinctly, (A1) accords with almost all reality and many stochastic distributions such as uniform distribution, exponential distribution, normal distribution and so on, can satisfy (A2) under some conditions. For example, distribution function of normal distribution \(N(\mu, \sigma^{2})\) is a increasing concave function on \([\mu, +\infty),\) and if \(\mu\) is very great, while \(\sigma\) is very small (e.g. \(\mu \geq 3\sigma\)), then \(F_{it}(0) \approx 0.\) Thus, one can derive the following theorem.

**Theorem 1.** When (A1), (A2) are hold,

\[(BS^{*}, Q^{*}) = ((bs_{it}^{*})_{N \times T}, (Q_{it}^{*})_{N \times T})\]

is solution of (RP) if and only if

\[s_{i}^{l}b_{it}f_{it}(Q_{it}^{*}) - b_{i}F_{it}(Q_{it}^{*}) + h_{i} = 0, \quad \forall i \in U_{N}, \quad t \in U_{T} \quad (17)\]
and

\[ p_i + \theta_i^b + \theta_i^s - c_i - b_i - h_i - (p_i + \theta_i^b + \theta_i^s - b_i - d_i^s)F_{it}(bs_{it}) = 0, \]
\[ \forall i \in U_N, \ t \in U_T. \quad (18) \]

**Proof.** From (7), (8) and (A1), one has

\[ \frac{\partial \pi}{\partial Q_{it}} = s_i^b f_{it}(Q_{it}) - b_i F_{it}(Q_{it}) + h_i, \quad (19) \]
\[ \frac{\partial \pi}{\partial bs_{it}} = (p_i + \theta_i^b + \theta_i^s - c_i - b_i - h_i) - (p_i + \theta_i^b + \theta_i^s - b_i - d_i^s)F_{it}(bs_{it}). \quad (20) \]

Thus

\[ \frac{\partial^2 \pi}{\partial bs_{it} \partial Q_{jt}} = \frac{\partial^2 \pi}{\partial Q_{it} \partial bs_{lm}} = 0, \quad (21) \]

where \(i, j, l \in \{1, 2, \cdots, N\}, \ t, \tau, m \in \{1, 2, \cdots, T\}\) and \((i - j)^2 + (t - \tau)^2 \neq 0, (i - l)^2 + (t - m)^2 \neq 0,\)

\[ \frac{\partial^2 \pi}{\partial Q_{it}^2} = s_i^b f_i'(Q_{it}) - b_i f_i(Q_{it}) < 0, \quad (22) \]
\[ \frac{\partial^2 \pi}{\partial bs_{it}^2} = -(p_i + \theta_i^b + \theta_i^s - b_i - d_i^s)f_{it}(bs_{it}) < 0. \quad (23) \]

Hence, the objective function of problem (RP) is a concave function on feasible domain (a convex set), this completes the proof.

Furthermore, one can derive following theorem.

**Theorem 2.** When (A1) and (A2) hold, solution of original problem (P) is one and only one.

**Proof.** In fact, if the feasible domain of problem (P) is extended to a continuous bounded and closed space, then the objective function is obviously continuous on this space. Add concave of the objective function to this, one can gains that solution of original problem (P) is one and only one.

In practice, both \(U_{it}^{BS}\) and \(U_{it}^Q\) are often bounded discrete space, if potencies of them, \(N\) and \(T\) are small, one can solves problem (P) easily. Another situation can be dealt by using Theorem 1. If \((BS^*, Q^*) = (bs_{it}^*, Q_{it}^*)_{N \times T}\) determined by (17) and (18) of Theorem 1 is not in the feasible domain of problem (P), then problem (P) can be solved by precision searching algorithm or heuristic method.
4. Cases Study

For the situation that \((BS^*, Q^*) = ((bs^*_it)_N \times T, (Q^*_it)_N \times T)\) determined by (17) and (18) of Theorem 1 is in the feasible domain of problem (P), one analyzes three cases including uniform distribution, exponential distribution, normal distribution, and then presents detailed solution procedures.

4.1. Uniform Distribution

Let

\[
f_{it}(x_{it}) = \begin{cases} 
1/(b - a), & a \leq x_{it} \leq b, \\
0, & x_{it} < a \text{ or } x_{it} > b,
\end{cases}
\]

then

\[
F_{it}(x_{it}) = \begin{cases} 
0, & x_{it} \leq a, \\
(x_{it} - a)/(b - a), & a < x_{it} \leq b, \\
1, & x_{it} > b.
\end{cases}
\]

From (17) one has

\[
s_{i}^{b2}/(b - a) - b_i(\text{Q}_{it}^* - a)/(b - a) + h_i = 0. \tag{24}
\]

Thus

\[
\text{Q}_{it}^* = a + (h_i(b - a) + s_i^{b2})/b_i, \forall i \in U_N, t \in U_T. \tag{25}
\]

From (18) one has

\[
p_i + \theta_i^b + \theta_i^s - c_i - b_i - h_i - (p_i + \theta_i^b + \theta_i^s - b_i - d_i^s)(bs^*_it - a)/(b - a) = 0. \tag{26}
\]

Hence

\[
bs^*_it = a + (p_i + \theta_i^b + \theta_i^s - c_i - b_i - h_i)(b - a)/(p_i + \theta_i^b + \theta_i^s - b_i - d_i^s), \forall i \in U_N, t \in U_T. \tag{27}
\]

4.2. Exponential Distribution

Let

\[
f_{it}(x_{it}) = \begin{cases} 
\lambda_{it}e^{-\lambda_{it}x_{it}}, & x_{it} > 0, \\
0, & x_{it} \leq 0,
\end{cases}
\]

then

\[
F_{it}(x_{it}) = \begin{cases} 
1 - e^{-\lambda_{it}x_{it}}, & x_{it} > 0, \\
0, & x_{it} \leq 0.
\end{cases}
\]
From (17) one has
\[ s_i^{b2} \lambda_{it} e^{-\lambda_{it} Q_{it}^*} - b_i (1 - e^{-\lambda_{it} Q_{it}^*}) + h_i = 0. \]  
(28)

Thus
\[ Q_{it}^* = \left( 1/\lambda_{it} \right) \ln \left( \lambda_{it} s_i^{b2}/(h_i + b_i) \right), \quad \forall i \in U_N, t \in U_T. \]  
(29)

From (18) one has
\[ p_i + \theta_i^b + \theta_i^s - c_i - b_i - h_i = (p_i + \theta_i^b + \theta_i^s - b_i - d_{it}^s)(1 - e^{-\lambda_{it} Q_{it}^*}) = 0. \]  
(30)

Hence
\[ b_{it}^* = \left( 1/\lambda_{it} \right) \ln \left( (p_i + \theta_i^b + \theta_i^s - b_i - d_{it}^s)/(c_i + h_i - d_{it}^s) \right), \quad \forall i \in U_N, t \in U_T. \]  
(31)

4.3. Normal Distribution

Let
\[ f_{it}(x_{it}) = \begin{cases} 
(1/(\sigma \sqrt{2\pi})) e^{-(x_{it} - \mu)^2/2\sigma^2}, & x_{it} > 0, \\
0, & x_{it} \leq 0, 
\end{cases} \]
then
\[ F_{it}(x_{it}) = \begin{cases} 
(1/(\sigma \sqrt{2\pi})) \int_0^{x_{it}} e^{-(\tau_{it} - \mu)^2/2\sigma^2} d\tau_{it}, & x_{it} > 0, \\
0, & x_{it} \leq 0. 
\end{cases} \]

From (17) one has
\[ s_i^{b2} e^{-(Q_{it}^* - \mu)^2/2\sigma^2} - b_i \int_0^{Q_{it}^*} e^{-(x - \mu)^2/2\sigma^2} dx + h_i \sigma \sqrt{2\pi} = 0, \quad \forall i \in U_N, t \in U_T. \]  
(32)

Through this formula and by dint of function value table of standard normal distribution, one can obtain \( Q_{it}^* \) using one-dimensional searching algorithm.

From (18) one has
\[ p_i + \theta_i^b + \theta_i^s - c_i - b_i - h_i = (p_i + \theta_i^b + \theta_i^s - b_i - d_{it}^s) \times \left( 1/(\sigma \sqrt{2\pi}) \right) \int_0^{b_{it}^*} e^{-(x - \mu)^2/2\sigma^2} dx = 0. \]  
(33)

Hence
$$b_{st}^* = \mu + \sigma \Phi^{-1}\left((p_i + \theta_i^s + \theta_i^b - c_i - \theta_i^b - h_i)/(p_i + \theta_i^s + \theta_i^b - b_i - d_{it})\right),$$
$$\forall i \in U_N, t \in U_T.$$ (34)

5. Conclusions and Future Researches

This paper addresses a coordinated inventory and ordering policies in two-level distribution system with stochastic demand composing of a single supplier and multiple retailers, in which the supplier (distribution center) takes base-stock policy and deals in multiple products with short life cycle, replenish periodically at each stage within every period, and gains deterministic base stock at the beginning of each stage within a period. A retailer deals in a sort of product only. Retailer does not hold inventory or his inventory can be neglected. Retailer combines fixed order and instant order from distribution center in each stage within a period. Some backorder cost must be paid for the backlog due to a second order. If demand exceed the base stock in this stage at distribution center, all unsatisfied demands are lost, but both distribution center and retailer must pay some punishment cost. Owing to short life cycle, surplus products both at distribution center and at retailer must be disposed by lower price or even spending extra cost so that new products are dealt in the stage. This paper developed a mathematical model to maximize expected system profit and analyzed existence and uniqueness of solutions of the model, and then proposed corresponding solution methods. Last, three cases on demand are studied. The situation considered by this paper has universality. This work can provide insight for purchasing and inventory decisions of distribution system and supply chain management. In next work, we will consider uncertain lead time, combine production with transportation and so on.

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References


