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INFINITE DIMENSIONAL HEISENBERG GROUP ALGEBRA AND FIELD–THEORETIC STRICT DEFORMATION QUANTIZATION

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Abstract: For arbitrary dimensional pre–symplectic test function spaces (E, σ) the discrete Heisenberg group C*–algebra is introduced and investigated. The (partially) regular and partially decomposable representations of the latter are analyzed. The Heisenberg group C*–algebra is shown to be *–isomorphic to a global C*–algebra generated by a continuous field of C*–Weyl algebras (incorporating simultaneously all values of the Planck parameter $\hbar \in \mathbb{R}$). The Heisenberg group (algebra) approach is compared with the method of strict and continuous deformation quantization. A related quantization scheme, using the (possibly infinite dimensional) Heisenberg Lie algebra is outlined, where a correspondence between the quantum mechanical $\hbar$–sectors and certain leaves of the classical phase space shows up.

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1. Introduction

Projective representations of groups arise in many fields of physics: not only in all branches of quantum theory [45], [3] but also in classical theory (as, e.g., in the representation theory of the Galilei group [27]). It is well known that for fixed multiplier the projective representations of a locally compact group E are in 1:1–correspondence with the representations of the twisted group Banach, resp. C*–algebra. The theory of twisted group algebras is classical [26], [17], [14], [35] and has been especially initiated by the mathematical foundations of quantum mechanics [29], [30]. There are intimate connections to C*–algebraic systems [36], [35] and C*–algebraic structure theory [22], [34], [18]. Twisted group C*–algebras acquire also increasing importance in the developments of Poisson manifolds and momentum maps [24], [32] and in the theory of strict (C*–algebraic) deformation quantization [38], [37], [39], [24], and in many other deformation strategies (cf., e.g., [42] and references therein).

A projective representation of a group E defines a representation in the usual sense of the so–called central extension of E by an Abelian group R, where these notions are connected with each other as follows: For each element z of a 2–cocycle group $\mathrm{Z}^2(E,R)$ the central extension group E^z is defined to be the Cartesian product $R \times E$ equipped with the group operation

$$(s,f) \circ (t,g) := (s+t+z(f,g), f \circ g), \;\; s,t \in R, \;\; f,g \in E.$$

Multipliers (also called 'antisymmetric bicharacters') on E are given by $(f,g) \mapsto \chi(z(f,g))$ and involve, beside a 2–cocycle, also a character χ on the Abelian group R. These are used to construct a twisted group algebra of the given group E. For a *locally compact* group E and a *compact* extension group R the connection between the group algebra of the central extension group E^z and the twisted group algebras of E is well established and described, e.g., in [17]. One can now use either the group algebra of E^z or a twisted group algebra of E to study the projective representations of the group E.

One purpose of the present investigation is to generalize these notions to the case of arbitrary groups, using a strategy which has been developed for the field quantization in terms of C*–Weyl algebras. Recall that Weyl's form of the canonical commutation relations, simply called *Weyl relations*, may be written

$$W^\hbar(f)W^\hbar(g) = \exp\{-\frac{i}{2}\hbar\sigma(f,g)\}W^\hbar(f+g)\,, \quad W^\hbar(f)^* = W^\hbar(-f)\,, \quad (1.1)$$

where $E \ni f,g$ is a real vector space, which determines the degrees of freedom of a physical system in a basis independent manner. In usual quantum mechanics E is the finite dimensional phase space with the (non–degenerate) symplectic form $\sigma(f,g)$ and the Weyl operators $W^\hbar(f)$ are unitary operators in a Hilbert space $\mathcal{H}$. Due to the Stone–von Neumann uniqueness theorem all irreducible representations of the Weyl relations for a fixed, finite dimensional E are unitary equivalent. Starting from the additive vector group E, which is locally compact in the unique vector space topology, the Weyl relations may then be viewed as a projective unitary representation $E \ni f \mapsto W^\hbar(f) \in U(\mathcal{H})$.

In the field theory two types of generalizations arise: The phase space is infinite dimensional, and thus not locally compact in a vector space topology, and the canonical structure may require (degenerate) pre–symplectic forms. The degrees of freedom are then described by a test function space E, usually realized by smooth functions on a space (or space–time) region. The kind of functions, which constitute E, is connected with a certain locally convex vector space topology. The dual space E', in this topology, is the (flat) phase space for the field theory. Already the realization of a Poisson algebra in terms of differentiable functions on E' meets considerable mathematical difficulties (not dealt with in Section 5). This is one of the reasons why the present authors have suggested an algebraization not only of the quantized but also of the classical field theory. To consider the Weyl relations (1.1) for the quantized theory algebraically, without a Hilbert space specification, is in accordance with the philosophy of usual algebraic quantum field theory. Even for a degenerate $\sigma(f,g)$, natural assumptions led us in [7] to a unique C*–norm for the linear hull of Weyl elements, and completion in the norm topology produced the C*–Weyl algebra. From these and related investigations in the literature (cf., e.g., [31]) one infers that not the locally convex vector space topology of the function representation, but the discrete topology of E, taken as an additive group, is relevant for the construction of the observable algebra. In group theoretical terms, the algebraic Weyl relations (1.1) constitute a twisting of the additive group composition in E. That is, the algebraic strategy amounts to first constructing an abstract, twisted C*–group algebra $C^*(E,\Sigma_\hbar)$ corresponding to the multiplier

$$\Sigma_\hbar(f,g) := \exp\{-\frac{i}{2}\hbar\sigma(f,g)\}\,, \quad \forall f,g \in E\,,$$

and then to choosing one of the many inequivalent Hilbert space representations in a second step. Note that the indicated multiplier is the composition of the 2–cocycle $z := \frac{1}{2}\sigma \in \mathrm{Z}^2(E,\mathbb{R})$ with the continuous character $\mathbb{R} \ni s \mapsto \exp\{-i\hbar s\}$, where the fixed parameter $\hbar \in \mathbb{R}$ assumes the value of the Planck constant (divided by 2π) in physical applications.

If one prefers, on the other hand, as the first step in deforming the additive group E the central extension E^z, corresponding to the above 2–cocycle $z :=$

$\frac{1}{2}\sigma$, then one arrives at the Heisenberg group $\mathrm{HG}(E,\sigma)$. We introduce $\mathrm{HG}(E,\sigma)$ as the Cartesian product $\mathbb{R}\times E$ — with elements (s,f), where $s\in\mathbb{R}$ and $f\in E$ — endowed with the group operation

$$(s,f)\circ(t,g)=(s+t+\tfrac{1}{2}\sigma(f,g),f+g)\,,\quad \forall s,t\in\mathbb{R}\,,\quad \forall f,g\in E\,. \tag{1.2}$$

As a mere group, without a differentiable (Lie group) structure, this definition presents obviously no difficulties, even not for infinite dimensional E. We shall consider $\mathrm{HG}(E,\sigma)$ as a discrete topological group. In contrast to the above mentioned twisted C*–group algebra $C^*(E,\Sigma_\hbar)$ of E, there appears an additional real parameter in the Heisenberg group elements.

This feature shares the Heisenberg group with the procedure of geometric pre–quantization, where in the finite dimensional case the phase space is supplemented by the one–dimensional torus group (for the connection between geometric pre–quantization and the Heisenberg group cf., e.g., [4]). And one finds in the literature an alternative definition of the Heisenberg group, namely as the central extension of E by the compact torus group $\mathbb{T}$ by means of the above multiplier $\Sigma_\hbar$. For example, the relation between such kind of Heisenberg group with the Weyl algebra plays an essential role in [21] (for σ non–degenerate).

Our choice (1.2) of the Heisenberg group product incorporates only the 2–cocycle z and is independent from the Planck parameter. So, in contrast to, e.g., the investigations in [17] and in [21], our chosen extension group $\mathbb{R}$ is non–compact, but considered discrete, and requires completely different techniques from those in the cited works. It is plausible, that certain results depending on the existence of a normalized Haar measure cannot be derived in the present situation. The (untwisted) group algebra of the discrete Heisenberg group $\mathrm{HG}(E,\sigma)$ is constructed here by original operator algebraic methods. It is introduced as an abstract C*–algebra $C^*(\mathrm{HG}(E,\sigma))$, which is generated by the family $\{H(s,f)\,|\,s\in\mathbb{R},f\in E\}$, all being different from zero and satisfying the product relations

$$H(s,f)H(t,g):=H((s,f)\circ(t,g))=H(s+t+\tfrac{1}{2}\sigma(f,g),f+g)\,. \tag{1.3}$$

More precisely, $C^*(\mathrm{HG}(E,\sigma))$ is defined in Section 4 as the norm completion of the linear hull of the $H(s,f)$, $(s,f)\in\mathrm{HG}(E,\sigma)$, with respect to the C*–norm from equation (2.4). The *–operation is $H(s,f)^*:=H(-s,-f)$. The commutator $[H(s,f),H(t,g)]$ can always be formed algebraically, but is, of course, different from the Lie product of the Heisenberg Lie algebra, which we mention only in Subsection 5.3.

That the C*–Heisenberg group algebra, which adds to the product operation of the Heisenberg group the linear structure and *–operation, has any relationship to a quantized observable algebra is not completely obvious. We demonstrate, however, that there is a natural *–isomorphism from $C^*(\mathrm{HG}(E,\sigma))$ onto the C*–algebra $C^*_{\mathrm{WF}}(E,\sigma)$, constituted by a continuous field of C*–Weyl algebras $\{\mathcal{W}(E,\hbar\sigma)\}_{\hbar\in\mathbb{R}}$ (where we appeal to a continuous field of C*–algebras in the sense of Dixmier [16]). The latter has been introduced and investigated in [7], [8], [20], where strong continuity properties in the dependence on $\hbar$ have been proved. In this manner we obtain the concise connection to the program of *continuous deformation quantization*, in which the Planck parameter varies within a set containing 0 as accumulation point. The connection to *strict deformation quantization* is gained by fixing $\hbar$.

Let us recall that 'strict deformation quantization' (cf. [39], [40], [41], [25], [24]), is a C*–algebraic variant of 'deformation quantization', where in the original ansatz [5], [15] the commutative, point–wise product of phase space functions is deformed into a non–commutative multiplication. The formal expansions into powers of $\hbar$, which are typical for 'deformation quantization', are avoided in the 'strict' variant. Whereas the cited works treat the finite dimensional case, we concentrate here on the infinite dimensional formalism. Twisted convolutions in the infinite dimensional case are mentioned, e.g., in [22], [18], [21], and the strict deformation quantization of special cases of an infinite dimensional test function space have been discussed in [39], [40], [44]. The notion of 'continuous deformation quantization' is borrowed from [24], but we modify the definition a little in view of our applications to the continuous field of C*–Weyl algebras.

To arrive at a physical interpretation of the formalism one has, of course, to specify (E,σ). There are test function spaces which consist of smooth sections of vector bundles, see [9]. That means, that a whole bundle section determines one element $f\in E$. Since the domain of f may be a rather arbitrary space (or space–time) manifold, geometric structures may well be incorporated into the (classical and quantized) theory, in spite of the linearity of E.

Let us finally give a short overview on the subsequent sections of the present analysis. In Subsection 2.1 we recapitulate and generalize the previous construction of [7], treating now the twisted group C*–algebra of an arbitrary discrete group G. Introduction of the C*–norm is, of course, also possible for a group C*–algebra (with trivial twisting). This method of generalizing the corresponding notions for locally compact groups seems of interest for its own. For $G=E$ this leads to the Weyl algebra $\mathcal{W}(E,\hbar\sigma)$ and also to the group C*–algebra $C^*(\mathrm{HG}(E,\sigma))$ of the (possibly infinite dimensional) discrete Heisenberg group.

In Subsection 2.2 we compile the basic notions of strict deformation quantization (cf. [39], [40], [41], [25], [24], and references therein) and recall the definition of a continuous field of C*–algebras, adapted to our intentions. This leads in Section 3 to a specific continuous field of the C*–Weyl algebras $\{\mathcal{W}(E,\hbar\sigma)\}_{\hbar\in\mathbb{R}}$ and to the global C*–algebra $C^*_{\mathrm{WF}}(E,\sigma)$.

In Section 4 the Heisenberg group and the Heisenberg group C*–algebra are introduced. In Subsection 4.1 (partially regular) factor representations of the Heisenberg group algebra are reduced to those of a Weyl algebra. The main result in Subsection 4.2 describes a *–isomorphism from $C^*(\mathrm{HG}(E,\sigma))$ onto $C^*_{\mathrm{WF}}(E,\sigma)$, justifying our specific choice of the Heisenberg group product and providing the decisive link to quantization. In Subsection 4.3 partially decomposable representations, leading to a specific $\hbar$–spectrum are analyzed, employing results of the preceding subsection.

The classical case $\sigma\equiv 0$ obtains a special treatment in Subsection 5.1. A selection of abstract Poisson algebras is presented, appropriate to serve as the pre–image of quantization mappings. In spite of being subsets of the classical C*–algebra (for $\hbar$=0), the norm topology is not appropriate for a Poisson algebra and various topologies are discussed.

Whereas in the bulk of the present paper mathematical technalities predominate we return in Subsection 5.2 to interpretational questions and outline a Heisenberg group (algebra) approach to quantization. We conclude that also in the infinite dimensional case a pertinent Heisenberg group algebra may characterize the quantum observables abstractly. In order to relate the discussion more distinctly with known results of the finite dimensional case, we outline in the last Subsection 5.3 notions for a rather general Heisenberg Lie algebra and its Hilbert space, resp. phase space realizations.

2. Preliminary Notions and Results

If not specified otherwise every (bi–)linear map is understood to be $\mathbb{C}$-(bi-) linear. By the linear hull $\mathrm{LH}\{M\}$ we mean all (finite) complex linear combinations of the elements of the set M. We consider exclusively *–algebras over the complex field $\mathbb{C}$ with associative, but possibly non–commutative product and the (antilinear) *–operation, resp. involution of adjoining. A linear functional ω on a *–algebra $\mathcal{A}$ with unit element 1 is called a state, if it is *positive*, i.e. $\langle\omega;A^*A\rangle\geq 0$ for all $A\in\mathcal{A}$ ($\langle .;.\rangle$ means the duality bracket) and if it is normalized, i.e. $\langle\omega;1\rangle=1$. A representation $(\Pi,\mathcal{H}_\Pi)$ of the *–algebra $\mathcal{A}$ is a *–homomorphism Π from $\mathcal{A}$ into the C*–algebra $\mathcal{L}(\mathcal{H}_\Pi)$ of all *bounded* operators of a complex Hilbert space $\mathcal{H}_\Pi$. $(\Pi,\mathcal{H}_\Pi)$ is called non–degenerate, if $\Pi(\mathcal{A})\mathcal{H}_\Pi$ is dense in $\mathcal{H}_\Pi$, or equivalently, if $\Pi(1)=1_\Pi$ (provided $1\in\mathcal{A}$). A C*–

norm $\|.\|$ on a *–algebra $\mathcal{A}$ is an algebra norm which satisfies the C*–property $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$.

2.1. Twisted Group C*–Algebras

Throughout the present subsection let G be a group with group operation $\circ$ and neutral element e equipped with the discrete topology. Since it is sufficient for our subsequent applications we use only multipliers, which are antisymmetric bicharacters Σ on G satisfying especially $\Sigma(x,x) = 1\ \forall x \in G$ (for these notions, cf., e.g., [23]).

The twisted group Banach–*–algebra with respect to the multiplier Σ may be constructed in the standard way (cf., e.g., [17], [36], [35], [18]) as follows: Since G is discrete, the Haar measure is the counting measure, and the Banach space of absolutely continuous measures coincides with the sequence space $\mathrm{l}^1(G)$ of summable sequences over G, i.e., the elements of $\mathrm{l}^1(G)$ are the functions $A : G \to \mathbb{C}$, $y \mapsto A[y]$ satisfying $\|A\|_1 := \sum_{y\in G} |A[y]| < \infty$. For each $x \in G$ a specific element $V(x)$ of $\mathrm{l}^1(G)$ is given by the Kronecker delta function $V(x) : G \to \mathbb{C}$ satisfying $V(x)[y] = 1$ for $y = x$ and $V(x)[y] = 0$ for $y \neq x$. The twisted product and the *–operation on $\mathrm{l}^1(G)$ are defined by linear and $\|.\|_1$–continuous extension of

$$V(x)V(y) = \Sigma(x,y)V(x\circ y)\,, \quad V(x)^* = V(x^{-1})\,, \quad \forall x,y \in G\,, \tag{2.1}$$

which makes $\mathrm{l}^1(G)$ to a Banach–*–algebra denoted by $(\mathrm{l}^1(G),\Sigma)$. Its identity is given by $V(e)$, and every $V(x)$ is unitary.

The *twisted group C*–algebra* $C^*(G,\Sigma)$ of our discrete group G with respect to the multiplier Σ is the enveloping C*–algebra of the twisted group Banach–*–algebra $(\mathrm{l}^1(G),\Sigma)$.

In [7] the present authors have developped an alternative, more direct construction of the twisted group C*–algebra for a given, discrete vector group. It avoids the non–constructive formulation of an enveloping C*–algebra for the twisted group Banach–*–algebra. Let us, in the remaining part of this subsection, refer the essential features of our approach. Since the arguments are completely parallel to those for the special case in [7], where G equals the test function space E of a Weyl system, we omit the proofs.

By equation (2.1) every polynomial of the $V(x)$ reduces to a linear combination of the $V(x)$, which implies that the linear hull

$$\Delta(G,\Sigma) := \mathrm{LH}\{V(x) \mid x \in G\} \tag{2.2}$$

forms a *–algebra, which is $\|.\|_1$–dense in $(\mathrm{l}^1(G),\Sigma)$, the twisted Banach–

*–algebra. Note that an arbitrary element of $\Delta(G,\Sigma)$ has the form $A = \sum_{k=1}^{n} z_k V(x_k)$ with $n \in \mathbb{N}$, $z_k \in \mathbb{C}$, and different x_k's from G. Thus, the function $A : G \to \mathbb{C}$ satisfies $A[y] = z_k$, for y being equal to some x_k, and $A[y] = 0$ elsewhere, and it holds $\|A\|_1 = \sum_{k=1}^{n} |z_k|$.

It is obvious that every Σ*–projective unitary representation* $(\pi, \mathcal{H}_\pi)$ of the group G, consisting of the unitaries $\pi(x) \in \mathcal{L}(\mathcal{H}_\pi)$, $x \in G$, which satisfy the relations (2.1), extends to a non–degenerate representation $(\Pi, \mathcal{H}_\pi)$ of the *–algebra $\Delta(G,\Sigma)$ by means of the prescription $\Pi(V(x)) := \pi(x)$ for all $x \in G$, and conversely. So the Σ–projective unitary representations of G are in 1:1–correspondence with the non–degenerate representations of the *–algebra $\Delta(G,\Sigma)$.

A function $C : G \to \mathbb{C}$ is called Σ*–positive–definite*, if it satisfies

$$\sum_{j,k=1}^{n} \overline{z_j} z_k \overline{\Sigma(x_j, x_k)} C(x_j^{-1} \circ x_k) \geq 0$$

for each $n \in \mathbb{N}$, all $z_k \in \mathbb{C}$, and all $x_k \in G$. For trivial $\Sigma \equiv 1$ this coincides with a usual positive–definite function, which, by the way, is called 'normalized' if $C(e) = 1$.

The convex set $\mathcal{C}(G,\Sigma)$ of Σ–positive–definite, normalized functions on G are in 1:1–correspondence with the states on $\Delta(G,\Sigma)$: For $C \in \mathcal{C}(G,\Sigma)$ the associated state ω_C is given by $\langle \omega_C; \sum_k z_k V(x_k)\rangle = \sum_k z_k C(x_k)$. For example, the Σ–positive–definite function $C_{\mathrm{tr}} \in \mathcal{C}(G,\Sigma)$, given by $C_{\mathrm{tr}}(e) := 1$ and $C_{\mathrm{tr}}(x) := 0$ for $x \neq e$, is in correspondence with the *tracial* state ω_{tr} on $\Delta(G,\Sigma)$, which fulfills for arbitrary $A = \sum_{k=1}^{n} z_k V(x_k) \in \Delta(G,\Sigma)$ with different x_k's the relation

$$\langle \omega_{\mathrm{tr}}; A^* A \rangle = \sum_{y \in G} |A[y]|^2 = \sum_{k=1}^{n} |z_k|^2 =: \|A\|_2^2\,. \tag{2.3}$$

We write $\mathcal{C}(G)$ for $\mathcal{C}(G,\Sigma)$ in the case of the trivial bicharacter $\Sigma \equiv 1$.

Proposition 2.1. (C*–Norm) *The mapping*

$$\Delta(G,\Sigma) \ni A \longmapsto \|A\| := \sup\{\sqrt{\langle \omega_C; A^* A\rangle} \mid C \in \mathcal{C}(G,\Sigma)\} \tag{2.4}$$

defines a C–norm on the *–algebra $\Delta(G,\Sigma)$, which satisfies*

$$\|A\| = \sup\{\|\Pi(A)\| \mid \Pi \text{ representation of } \Delta(G,\Sigma)\}\,, \tag{2.5}$$

and thus is equal to the norm of the enveloping C–algebra. Furthermore it holds $\|A\|_2 \leq \|A\| \leq \|A\|_1$ for all $A \in \Delta(G,\Sigma)$.*

The norm $\|.\|$ is the unique C–norm on $\Delta(G,\Sigma)$, such that every represen-*

tation Π *of* $\Delta(G,\Sigma)$ *is* $\|.\|$*–continuous. Moreover,* $\|.\|$ *is the unique* C^**–norm on* $\Delta(G,\Sigma)$ *with* $\|A\|_2 \leq \|A\|$ *for all* $A \in \Delta(G,\Sigma)$*.*

The $\|.\|$–completion of $\Delta(G,\Sigma)$ is just the above mentioned twisted group C*–algebra

$$C^*(G,\Sigma) = \overline{\Delta(G,\Sigma)}\,. \tag{2.6}$$

By the previous result every representation and every state extends $\|.\|$–continuously from the *–algebra $\Delta(G,\Sigma)$ to $C^*(G,\Sigma)$. It is remarkable that the so extended tracial state $\omega_{\rm tr}$ turns out to be faithful on $C^*(G,\Sigma)$.

In the picture where the $V(x)$ are Kronecker delta functions $V(x) : G \to \mathbb{C}$ the completion of $\Delta(G,\Sigma)$ with respect to the norm $\|.\|_2$ coincides with the sequence Hilbert space $\mathrm{l}^2(G)$ equipped with the inner product $(A|B)_2 = \langle \omega_{\rm tr}; A^*B\rangle$. With the estimations $\|A\|_2 \leq \|A\| \leq \|A\|_1$ $\forall A \in \Delta(G,\Sigma)$ one obtains canonical mappings from $\overline{\Delta(G,\Sigma)}^1$ into $\overline{\Delta(G,\Sigma)}$ and from $\overline{\Delta(G,\Sigma)}$ into $\overline{\Delta(G,\Sigma)}^2$. The mentioned faithfulness of the tracial state on $\overline{\Delta(G,\Sigma)}$ now ensures that these mappings are injective, and thus we have the inclusions

$$\overline{\Delta(G,\Sigma)}^1 = \mathrm{l}^1(G) \subseteq C^*(G,\Sigma) = \overline{\Delta(G,\Sigma)} \subseteq \overline{\Delta(G,\Sigma)}^2 = \mathrm{l}^2(G)\,, \tag{2.7}$$

which are proper for non–finite G (in which case these norms are not equivalent). The subsequent theorem follows straightforwardly from the results in [7]

Theorem 2.1. $C^*(G,\Sigma)$*, as introduced in equation* (2.6)*, is the unique* C^**–algebra (up to *–isomorphisms) which is generated by non–zero elements* $V(x)$*,* $x \in G$*, satisfying the projective relations from equation* (2.1)*, such that every* Σ*–projective unitary representation* $(\pi,\mathcal{H}_\pi)$ *of* G *arises from a (unique) representation* $(\Pi,\mathcal{H}_\pi)$ *of the* C^**–algebra* $C^*(G,\Sigma)$ *with* $\pi(x) = \Pi(V(x))$ *for all* $x \in G$*. Especially, the* $V(x)$*,* $x \in G$*, are linearly independent.*

The mapping $C \mapsto \omega_C$ *is an affine homeomorphism from* $\mathcal{C}(G,\Sigma)$ *onto the state space* $\mathcal{S}(C^*(G,\Sigma))$ *of* $C^*(G,\Sigma)$*, with the topology of pointwise convergence on* $\mathcal{C}(G,\Sigma)$ *and the weak*–topology on* $\mathcal{S}(C^*(G,\Sigma))$*.*

Furthermore, the twisted group C^**–algebra* $C^*(G,\Sigma)$ *is simple, if and only if* Σ *is non–degenerate (i.e., for each* $e \neq x \in G$ *there exists a* $y \in G$ *with* $\Sigma(x,y) \neq 1$*).*

$C \in \mathcal{C}(G,\Sigma)$ is called the *characteristic function* of the $\omega_C \in \mathcal{S}(C^*(G,\Sigma))$, since $C(x) = \langle \omega_C; V(x)\rangle$ for all $x \in G$.

Let G_0 be a subgoup of G, the restriction of Σ from G to G_0 is also denoted by Σ. Then we may perform the same construction for $\Delta(G_0,\Sigma)$. $\Delta(G_0,\Sigma)$ may be regarded as a sub–*–algebra of $\Delta(G,\Sigma)$ by identifying the elements $V(x) \in \Delta(G_0,\Sigma)$ with the associated elements $V(x) \in \Delta(G,\Sigma)$ for each $x \in G_0$.

Lemma 2.2. *Let G_0 be a subgroup of G. If $C \in \mathcal{C}(G_0, \Sigma)$ is extended trivially to G by putting $F(x) := C(x)$ for $x \in G_0$ and $F(x) := 0$ elsewhere, then $F \in \mathcal{C}(G, \Sigma)$. Moreover, the restriction $F \mapsto F|_{G_0}$ from G to G_0 is a surjective affine map from $\mathcal{C}(G, \Sigma)$ onto $\mathcal{C}(G_0, \Sigma)$.*

As a consequence of the above Lemma one may easily show that the C*–norm on the *–algebra $\Delta(G_0, \Sigma)$ leads to the same norm as the C*–norm arising by restriction of the norm on $\Delta(G, \Sigma)$, implying that

$$C^*(G_0, \Sigma) = \overline{\Delta(G_0, \Sigma)} \subseteq \overline{\Delta(G, \Sigma)} = C^*(G, \Sigma)$$

is a sub–C*–algebra. If the inclusion $G_0 \subset G$ is proper, then $\|A - V(x)\| \geq 1$ for all $A \in C^*(G_0, \Sigma)$ and $x \notin G_0$, yielding $C^*(G_0, \Sigma) = C^*(G, \Sigma)$, if and only if $G_0 = G$.

2.2. Strict Deformation Quantization, Continuous Field of C*–Algebras

A Poisson algebra $(\mathcal{P}, \{.,.\})$ consists of a commutative complex *–algebra $\mathcal{P}$ equipped with a Poisson bracket $\{.,.\}$ (a bilinear mapping $\{.,.\} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, which is anticommutative, real, fulfills the Jacobi identity and Leibniz rule). In order to operate within a C*–algebraic frame it is assumed that $\mathcal{P}$ is a sub–*–algebra of a commutative C*–algebra, which will be denoted by $\mathcal{A}^0$, that is by $\hbar = 0$, in the context of quantization.

In physics Poisson algebras describe classical Hamiltonian systems or canonical field systems. They commonly arise from a Poisson tensor on a differentiable manifold P [1], [2], [28], [11], [32], [43], [6]. A quantum field system at a value $\hbar \neq 0$ of a varying Planck parameter is given in terms of a C*–algebra $\mathcal{A}^\hbar$, with C*–norm written as $\|.\|_\hbar$, for which the $\hbar$–scaled commutator is defined as

$$[A, B]_\hbar := \frac{i}{\hbar}(AB - BA)\,, \quad \forall A, B \in \mathcal{A}^\hbar\,. \tag{2.8}$$

For simplicity we let $\hbar$ vary in all of $\mathbb{R}$ (not merely assuming an accumulation point at $\hbar = 0$). According to Rieffel [37], [39] and Landsman [24], a precise form of a 'quantization map' from $\mathcal{P}$ into $\mathcal{A}^\hbar$ is characterized by the following relations in the classical correspondence limit $\hbar \to 0$.

Definition 2.3. (Strict Deformation Quantization) A strict deformation quantization $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar \in \mathbb{R}}$ of the Poisson algebra $(\mathcal{P}, \{.,.\})$ consists for each $\hbar \in \mathbb{R}$ of a linear, injective, *–preserving map $Q_\hbar : \mathcal{P} \to \mathcal{A}^\hbar$ (called quantization map), where Q_0 is the identical embedding and $Q_\hbar(\mathcal{P})$ is a sub–*–algebra of $\mathcal{A}^\hbar$, such that for all $A, B \in \mathcal{P}$ one has:

(a) (Dirac's Condition) $\lim_{\hbar\to 0} \|[Q_\hbar(A), Q_\hbar(B)]_\hbar - Q_\hbar(\{A,B\})\|_\hbar = 0$.

(b) (von Neumann's Condition) $\lim_{\hbar\to 0} \|Q_\hbar(A)Q_\hbar(B) - Q_\hbar(AB)\|_\hbar = 0$.

(c) (Rieffel's Condition) $\mathbb{R} \ni \hbar \mapsto \|Q_\hbar(A)\|_\hbar$ is continuous.

We follow Dixmier [16, Chapter 10] for the notion of a continuous field of C*–algebras, here adapted to our quantization context with topological space $\mathbb{R}$. We denote by $\prod_{\hbar\in\mathbb{R}} \mathcal{A}^\hbar$ the Cartesian product resp. bundle of the C*–algebras $\mathcal{A}^\hbar$. The elements K of $\prod_{\hbar\in\mathbb{R}} \mathcal{A}^\hbar$ are sections $\hbar \mapsto K(\hbar) \in \mathcal{A}^\hbar$, for which we also write $K = [\hbar \mapsto K(\hbar)] \in \prod_{\hbar\in\mathbb{R}} \mathcal{A}^\hbar$. If the *–algebraic operations (scalar multiplication, sum, product, *–operation) are taken pointwise, then $\prod_{\hbar\in\mathbb{R}} \mathcal{A}^\hbar$ becomes a *–algebra. The point evaluation *–homomorphism $\alpha_\hbar$ is defined by $\alpha_\hbar(K) := K(\hbar)$.

Definition 2.4. (Continuous Field of C*–Algebras) A continuous field of C*–algebras $(\{\mathcal{A}^\hbar\}_{\hbar\in\mathbb{R}}, \mathcal{K})$ consists of a sub–*–algebra $\mathcal{K}$ of $\prod_{\hbar\in\mathbb{R}} \mathcal{A}^\hbar$ such that the following conditions are valid:

(a) $\mathbb{R} \ni \hbar \mapsto \|K(\hbar)\|_\hbar$ is continuous for all $K \in \mathcal{K}$.

(b) For each $\hbar \in \mathbb{R}$ it holds $\{K(\hbar) \mid K \in \mathcal{K}\} = \mathcal{A}^\hbar$.

(c) $\mathcal{K}$ is locally complete, that is: $K \in \prod_{\hbar\in\mathbb{R}} \mathcal{A}^\hbar$ is an element of $\mathcal{K}$, if for each $\hbar_0$ and each $\varepsilon > 0$ there exists an $H \in \mathcal{K}$ and a neighborhood U_0 of $\hbar_0$ so that $\|K(\hbar) - H(\hbar)\|_\hbar < \varepsilon \ \forall \hbar \in U_0$.

Note, if $K \in \mathcal{K}$ and $u : \mathbb{R} \to \mathbb{C}$ is continuous, then $[\hbar \mapsto u(\hbar)K(\hbar)] \in \mathcal{K}$.

In order to get a C*–algebra we take the bounded, continuous sections $\mathcal{K}_\mathrm{b}$, that are those sections $K \in \mathcal{K}$ for which the continuous mapping $\mathbb{R} \ni \hbar \mapsto \|K(\hbar)\|_\hbar$ is bounded. $\mathcal{K}_\mathrm{b}$ is a C*–algebra with respect to the C*–norm

$$\|K\|_{\sup} := \sup_{\hbar\in\mathbb{R}} \|K(\hbar)\|_\hbar \,. \tag{2.9}$$

The following definition is a slight generalization of [24, Definition II.1. 2.5].

Definition 2.5. (Continuous Quantization) The symbol $(\{\mathcal{A}^\hbar\}_{\hbar\in\mathbb{R}}, \mathcal{K}; Q)$ denotes a continuous quantization of the Poisson algebra $(\mathcal{P}, \{.,.\})$. It is given in terms of a linear, *–preserving map $Q : \mathcal{P} \to \mathcal{K}$ (called 'global quantization map') so that the following conditions are valid:

(a) $\mathcal{P} \subseteq \mathcal{A}^0$, and $\alpha_0(Q(A)) = A$ for all $A \in \mathcal{P}$.

(b) Dirac's condition is fulfilled for the quantization mappings $Q_\hbar := \alpha_\hbar \circ Q$, where $\hbar \in \mathbb{R}$.

Proposition 2.6. *Let $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar\in\mathbb{R}}$ be a strict quantization of the Poisson algebra $(\mathcal{P}, \{.,.\})$ so that the *–algebraic span of $Q_\hbar(\mathcal{P})$ is dense in $\mathcal{A}^\hbar$ for each $\hbar \in \mathbb{R}$ (richness condition). Then the following assertions are equivalent:*

(i) $\mathbb{R} \ni \hbar \mapsto \|P(Q_\hbar(A_1), \ldots, Q_\hbar(A_m))\|_\hbar$ *is continuous for all polynomial P in the $Q_\hbar(A_k)$, $A_k \in \mathcal{P}$, of arbitrary degrees $m \in \mathbb{N}$.*

(ii) There exists a continuous quantization $(\{\mathcal{A}^\hbar\}_{\hbar\in\mathbb{R}}, \mathcal{K}; Q)$ of $(\mathcal{P}, \{.,.\})$ satisfying $Q_\hbar = \alpha_\hbar \circ Q$ for every $\hbar \in \mathbb{R}$.

A more detailed discussion of the connections between these different notions of quantization is elaborated in [20].

Since only the $\hbar \to 0$–properties are specified, a strict quantization of a Poisson algebra is highly non–unique. Rieffel's continuity property allows for an elegant notion of equivalent strict quantizations (cf. [24]).

Definition 2.7. (Equivalent Quantizations) Let be given two strict quantizations $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar\in\mathbb{R}}$ and $(\breve{\mathcal{A}}^\hbar, \breve{Q}_\hbar)_{\hbar\in\mathbb{R}}$ of the same Poisson algebra $\mathcal{P}$. They are called equivalent, if $\mathcal{A}^\hbar = \breve{\mathcal{A}}^\hbar$ for all $\hbar \in \mathbb{R}$, and if

$$\mathbb{R} \ni \hbar \mapsto \|Q_\hbar(A) - \breve{Q}_\hbar(A)\|_\hbar$$

is continuous for every $A \in \mathcal{P}$.

Since $Q_0(A) = A = \breve{Q}_0(A)$ in virtue of Definition 2.3, the norm difference $\|Q_\hbar(A) - \breve{Q}_\hbar(A)\|_\hbar$ for two equivalent quantizations vanishes for $\hbar \to 0$.

3. Continuous Field of C*–Weyl Algebras

For the remainder of the present investigation we suppose (E, σ) to be an arbitrary *pre–symplectic space*, that is, E is a real vector space of arbitrary dimensions equipped with an $\mathbb{R}$–bilinear mapping $\sigma : E \times E \to \mathbb{R}$, $(f, g) \mapsto \sigma(f, g)$, which is antisymmetric, i.e., $\sigma(f, g) = -\sigma(g, f)\ \forall f, g \in E$. We do not assume that σ be non–degenerate (non–degeneracy of σ means that $\sigma(f, g) = 0\ \forall f \in E$ implies $g = 0$). By $\widehat{\mathbb{R}}$ we denote the set of all (also non–continuous) characters on the real line $\mathbb{R}$, treated as an additive group.

The construction of subsection 2.1 applies to the following situation. E is considered as a commutative group with respect to the addition $\circ := +$ for the group operation. For each $\chi \in \widehat{\mathbb{R}}$ we choose an antisymmetric bicharacter (multiplier) Σ_χ on E as

$$\Sigma_\chi(f,g) := \chi(\tfrac{1}{2}\sigma(f,g)), \quad \forall f,g \in E. \tag{3.1}$$

The associated twisted group C*–algebra $C^*(E,\Sigma_\chi)$ is denoted by $\mathcal{W}(E,\chi\sigma)$ and called (generalized) *Weyl algebra*. It is generated by the non–zero elements $W^\chi(f)$, $f \in E$, where equation (2.1) goes over to the Weyl relations

$$W^\chi(f)W^\chi(g) = \chi(\tfrac{1}{2}\sigma(f,g))W^\chi(f+g), \quad W^\chi(f)^* = W^\chi(-f), \quad \forall f,g \in E. \tag{3.2}$$

Σ_χ is non–degenerate, if and only if σ is non–degenerate and $\chi \neq 1$, and thus by Theorem 2.1 $\mathcal{W}(E,\chi\sigma)$ is simple, if and only if σ is non–degenerate and $\chi \neq 1$.

If we are concerned with a continuous character $\chi_\hbar(s) = \exp\{-i\hbar s\}$ for all $s \in \mathbb{R}$ we briefly use the index $\hbar \in \mathbb{R}$ instead of $\chi_\hbar$. Clearly, for every $\hbar \in \mathbb{R}$ the twisted group C*–algebra $\mathcal{W}(E,\hbar\sigma)$ is just the familiar Weyl algebra over the pre–symplectic space $(E,\hbar\sigma)$ from [31], [7], where (in contrast to the above abbreviation $\chi\sigma$) $\hbar\sigma$ means the pointwise multiplication of the $\mathbb{R}$–bilinear form $\sigma(f,g)$ with the real scalar $\hbar$, and where the Weyl relations (3.2) specialize to

$$W^\hbar(f)W^\hbar(g) = \exp\{-\tfrac{i}{2}\hbar\sigma(f,g)\}W^\hbar(f+g), \; W^\hbar(f)^* = W^\hbar(-f), \quad \forall f,g \in E.$$

From our previous work [8], we have the following

Theorem 3.1. (Continuous Field of Weyl Algebras) *For a pre–symplectic space (E,σ) there exists a uniquely given, continuous field of C*–algebras $(\{\mathcal{W}(E,\hbar\sigma)\}_{\hbar\in\mathbb{R}},\mathcal{K})$ such that for each $f \in E$ the section $[\hbar \mapsto W^\hbar(f)]$ is an element of $\mathcal{K}$.*

It is immediately checked that the linear space

$$\Delta_{\mathrm{WF}}(E,\sigma) := \mathrm{LH}\{\,[\hbar \mapsto \exp\{-i\hbar s\}W^\hbar(f)] \mid s \in \mathbb{R},\; f \in E\} \tag{3.3}$$

is a sub–*–algebra of the C*–algebra $\mathcal{K}_{\mathrm{b}}$ of the bounded continuous sections (cf. subsection 2.2). The $\|.\|_{\sup}$–norm closure of $\Delta_{\mathrm{WF}}(E,\sigma)$ will be denoted by

$$C^*_{\mathrm{WF}}(E,\sigma) := \overline{\Delta_{\mathrm{WF}}(E,\sigma)}^{\sup}. \tag{3.4}$$

(the C*–norm $\|.\|_{\sup}$ has been defined in equation (2.9)). $C^*_{\mathrm{WF}}(E,\sigma)$ is a proper sub–C*–algebra of $\mathcal{K}_{\mathrm{b}}$. Since the Weyl relations imply

$$W^\hbar(f)^*W^\hbar(g)^*W^\hbar(f)W^\hbar(g) = \exp\{-i\hbar\sigma(f,g)\}W^\hbar(0)$$

(note that $W^\hbar(0)$ is the identity of $\mathcal{W}(E,\hbar\sigma)$), we conclude that for $\sigma \neq 0$ the *–algebra $\Delta_{\mathrm{WF}}(E,\sigma)$ is *–algebraically generated by the continuous sections $[\hbar \mapsto W^\hbar(f)]$ alone.

Lemma 3.2. *The following assertions are valid:*

*(a) For each $\hbar \in \mathbb{R}$ the evaluation map $\alpha_\hbar$ — defined by $\alpha_\hbar(K) = K(\hbar)$ for $K \in \prod_{\hbar\in\mathbb{R}} \mathcal{W}(E,\hbar\sigma)$ — is a non–injective *–homomorphism from $C^*_{\rm WF}(E,\sigma)$ onto the C*–Weyl algebra $\mathcal{W}(E,\hbar\sigma)$.*

*(b) The generating elements $[\hbar \mapsto \exp\{-i\hbar s\}W^\hbar(f)]$, with $(s,f) \in \mathbb{R}\times E$, of the *–algebra $\Delta_{WF}(E,\sigma)$ are linearly independent.*

Proof. Part (a). Since $\alpha_\hbar(\Delta_{\rm WF}(E,\sigma)) = \Delta(E,\hbar\sigma)$, we have a dense image. $\alpha_\hbar$ being 'onto' now follows from the fact that the image of a *–homomorphism from a C*–algebra into another one is closed. The non–injectivity of $\alpha_{\hbar_0}$ follows from

$$\begin{aligned}\alpha_{\hbar_0}([\hbar \mapsto \exp\{-i\hbar s\}W^\hbar(f)]) &= \exp\{-i\hbar_0 s\}W^{\hbar_0}(f)\\ &= \alpha_{\hbar_0}(\exp\{-i\hbar_0 s\}[\hbar \mapsto W^\hbar(f)])\,,\end{aligned}$$

but $[\hbar \mapsto \exp\{-i\hbar s\}W^\hbar(f)] \neq \exp\{-i\hbar_0 s\}[\hbar \mapsto W^\hbar(f)]$. Part (b) is proved in [8]. □

Both of the C*–algebras $C^*_{\rm WF}(E,\sigma)$ and $\mathcal{K}_{\rm b}$ possess the identity $[\hbar \mapsto W^\hbar(0)]$, and each $[\hbar \mapsto \exp\{-i\hbar s\}W^\hbar(f)]$ is unitary for every $(s,f) \in \mathbb{R}\times E$. In [8] and [20] it is shown how typical elements of $C^*_{\rm WF}(E,\sigma)$ look like: Let $\|K\|_1 := \sum_k |z_k| < \infty$ for coefficients $z_k \in \mathbb{C}$, then the section $K := \sum_{k=1}^{\infty} z_k[\hbar \mapsto \exp\{-i\hbar s_k\}W^\hbar(f_k)]$ is an element of $C^*_{\rm WF}(E,\sigma)$ for all tuples $(s_k,f_k) \in \mathbb{R}\times E$, where $z_k \in \mathbb{C}$. Moreover, if $K \in C^*_{\rm WF}(E,\sigma)$ and if $u : \mathbb{R} \to \mathbb{C}$ is a continuous almost periodic function, then $[\hbar \mapsto u(\hbar)K(\hbar)] \in C^*_{\rm WF}(E,\sigma)$. Let us close this section with the following chain of inclusions

$$\Delta_{\rm WF}(E,\sigma) \subseteq \overline{\Delta_{\rm WF}(E,\sigma)}^1 \subseteq C^*_{\rm WF}(E,\sigma) \subseteq \mathcal{K}_{\rm b} \subseteq \mathcal{K}\,, \tag{3.5}$$

which have the meaning of being sub–*–algebras.

4. Heisenberg Group

Let us apply the techniques of subsection 2.1 to the Heisenberg group algebra.

Definition 4.1. The *Heisenberg group* $\mathrm{HG}(E,\sigma)$ over an arbitrary pre–symplectic space (E,σ) is defined as the central extension of the additive group E by the additive group $\mathbb{R}$ with respect to the 2–cocycle $\sigma/2 \in \mathrm{Z}^2(E,\mathbb{R})$(e.g., [17], [24, Section II.2.1]). That is, $\mathrm{HG}(E,\sigma)$ is given as the Cartesian product

$\mathbb{R} \times E$ — with elements (s, f), where $s \in \mathbb{R}$ and $f \in E$ — endowed with the group operation

$$(s,f) \circ (t,g) = (s+t+\frac{1}{2}\sigma(f,g), f+g)\,, \quad \forall s,t \in \mathbb{R}\,, \quad \forall f,g \in E\,. \tag{4.1}$$

If not stated otherwise, the groups E and $\mathrm{HG}(E, \sigma)$ are equipped with the (locally compact) discrete topology.

The Heisenberg group $\mathrm{HG}(E, \sigma)$ is non–commutative, if and only if $\sigma \neq 0$. We have $(s, f)^{-1} = (-s, -f)$, and the neutral element is given by $(0, 0)$. The center of $\mathrm{HG}(E, \sigma)$ contains always $\{(s, 0) \,|\, s \in \mathbb{R}\}$, but may be larger if σ is degenerate.

4.1. Group C*–Algebra and Factor Representations

The discrete (untwisted) group C*–algebra $C^*(\mathrm{HG}(E, \sigma))$ of the Heisenberg group (the latter incorporating already the twisting in the addition of the scalars) is obtained from our general formula equation (2.6) if we insert there $\mathrm{HG}(E, \sigma)$ for G and the trivial multiplier 1 for Σ. Thus, it is generated by the family of non–zero elements $\{H(\kappa) \,|\, \kappa \equiv (s, f) \in \mathrm{HG}(E, \sigma)\}$ which satisfy the relations

$$H(s,f)H(t,g) := H((s,f) \circ (t,g)) = H(s+t+\frac{1}{2}\sigma(f,g), f+g)\,, \tag{4.2}$$

for all $(s, f), (t, g) \in \mathbb{R} \times E$. The *–operation is $H(s, f)^* := (-s, -f)$. The C*–algebra $C^*(\mathrm{HG}(E, \sigma))$, as the norm completion of the linear hull of the $\{H(s, f) \,|\, (s, f) \in \mathrm{HG}(E, \sigma)\}$, with respect to the C*–norm from equation (2.4) is unique in the sense of Theorem 2.1.

Especially from the applications in quantum field theory one knows that the test function space E should in the general, algebraic context be handled as a discrete topological group. A locally convex vector space topology, characterizing the function realization of E, is mostly used in connection with special states and representations. This is different for the scalar parameter s in the Heisenberg group elements (s, f), with respect to which continuity is more basic. For illustration consider a factor representation

$$\Pi : C^*(\mathrm{HG}(E,\sigma)) \longrightarrow \mathcal{B}(\mathcal{H}), \tag{4.3}$$

by bounded operators in some Hilbert space $\mathcal{H}$. The center of $\Pi(C^*(\mathrm{HG}(E, \sigma)))$ being trivial, $\Pi(H(s, 0)) = \chi(s)1$, where 1 is the unit operator in $\mathcal{H}$ and χ is a character on $\mathbb{R}$. For all of the intended applications χ should be continuous.

If the index space $\mathbb{R} \times E$ for the Heisenberg group elements is equipped with the non–discrete locally compact product topology, arising from the usual topology on $\mathbb{R}$ combined with the discrete topology on E, we write $\mathrm{HG}(E,\sigma)_{\mathrm{pr}}$ (the index 'pr' appealing to 'product' as well as to 'partially regular').

A representation Π of the Heisenberg group C*–algebra $C^*(\mathrm{HG}(E,\sigma))$ is called *partially regular*, if $\mathbb{R} \ni s \mapsto \Pi(H(s,f)) =: H_\Pi(s,f)$ is continuous for every $f \in E$ (it is sufficient to require this for some $f \in E$). Let us apply a general result of the theory of locally compact groups, which tells us, that various ways may lead to a continuous $\mathbb{R} \ni s \mapsto H_\Pi(s,f)$, where the latter are always unitary operators in virtue of the *–preservation of the representation map Π. It is well known that there are biunivocal correspondences (preserving irreducibility and factor property) between the non–degenerate partially regular representations Π of $C^*(\mathrm{HG}(E,\sigma))$, the strongly continuous unitary representations π of $\mathrm{HG}(E,\sigma)_{\mathrm{pr}}$, and the non–degenerate representations of the group C*–algebra $C^*(\mathrm{HG}(E,\sigma)_{\mathrm{pr}})$ (for the second 1:1–correspondence, see, e.g., [19, §22], [36]; and for a more detailed discussion of such group resp. group C*–algebra representations, also in the projective resp. twisted case, consult [18]).

We are now going to formulate a first connection between the Heisenberg group C*–algebra $C^*(\mathrm{HG}(E,\sigma))$ and the C*–Weyl algebras $\mathcal{W}(E,\chi\sigma)$ from Section 3 for arbitrary characters χ, acting on the real line $\mathbb{R}$.

Proposition 4.2. *The following assertions are valid:*

(a) *For each $\chi \in \widehat{\mathbb{R}}$ there exists a unique *–homomorphism*

$$\theta_\chi : C^*(\mathrm{HG}(E,\sigma)) \to \mathcal{W}(E,\chi\sigma)\,,$$
$$\text{such that} \quad \theta_\chi(H(s,f)) = \chi(s) W^\chi(f)\,, \tag{4.4}$$

for all $s \in \mathbb{R}$ and all $f \in E$ (the $W^\chi(f)$ generate $\mathcal{W}(E,\chi\sigma)$ and satisfy equation (3.2)). The map θ_χ is surjective, but not injective.

(b) *The Heisenberg group C*–algebra $C^*(\mathrm{HG}(E,\sigma))$ is not separable for $\dim E > 0$.*

Proof. Part (a). The linear extension of $H(s,f) \mapsto \chi(s)W^\chi(f)$ gives a well–defined *–homomorphism θ_χ from $\Delta(\mathrm{HG}(E,\sigma))$ onto $\Delta(E,\chi\sigma)$ (these *–algebras are defined by (2.2)). Since $\theta_\chi(H(s,f)) = \theta_\chi(\chi(s)H(0,f))$ but $H(s,f) \neq \chi(s)H(0,f)$ the non–injectivity follows. If Π_χ is a faithful representation of $\mathcal{W}(E,\chi\sigma)$, then $\Pi_\chi \circ \theta_\chi$ is a representation of $\Delta(\mathrm{HG}(E,\sigma))$. Proposition 2.1 implies that $\|A\| \geq \|\Pi_\chi(\theta_\chi(A))\| = \|\theta_\chi(A)\|$ for all $A \in \Delta(\mathrm{HG}(E,\sigma))$, so θ_χ extends continuously.

Part (b). Since by Part (a) we have a *–homomorphism of $C^*(\mathrm{HG}(E,\sigma))$ onto $\mathcal{W}(E,\chi\sigma)$, and the latter is not separable for any $\chi \in \widehat{\mathbb{R}}$ (the inseparability proof in [13] does not depend on the choice of χ), $C^*(\mathrm{HG}(E,\sigma))$ cannot be separable. ☐

Let us denote a representation of $\mathcal{W}(E,\chi\sigma)$ by the symbol Π_χ. If now Π_χ is any non–degenerate representation of $\mathcal{W}(E,\chi\sigma)$, then

$$\Pi \equiv \Pi^\chi := \Pi_\chi \circ \theta_\chi$$

defines a non–degenerate representation of the discrete Heisenberg group C*–algebra $C^*(\mathrm{HG}(E,\sigma))$. For factor representations (with trivial center), and thus especially for irreducible representations, there exists a converse statement.

Theorem 4.3. *The following assertions are valid:*

(a) Let $(\Pi,\mathcal{H})$ be a non–degenerate representation of $C^*(\mathrm{HG}(E,\sigma))$, which is factorial (irreducible). Then there exists a unique $\chi \in \widehat{\mathbb{R}}$ and a unique factorial (irreducible) non–degenerate representation $(\Pi_\chi,\mathcal{H})$ of $\mathcal{W}(E,\chi\sigma)$ such that $\Pi = \Pi_\chi \circ \theta_\chi$.

(b) Let $(\Pi,\mathcal{H})$ be a partially regular, factorial (irreducible), non–degenerate representation of $C^*(\mathrm{HG}(E,\sigma))$. Then there exists a unique $\hbar \in \mathbb{R}$ and a unique factorial (irreducible) non–degenerate representation $(\Pi_\hbar,\mathcal{H})$ of $\mathcal{W}(E,\hbar\sigma)$ such that $\Pi \equiv \Pi^\hbar = \Pi_\hbar \circ \theta_\hbar$.

(c) $\mathrm{HG}(E,\sigma)$, thus $C^*(\mathrm{HG}(E,\sigma))$, and also $C^*(\mathrm{HG}(E,\sigma)_{\mathrm{pr}})$ do not possess any faithful (unitary) factor representation.

(d) Let $(\Pi,\mathcal{H})$ be an irreducible representation of $C^*(\mathrm{HG}(E,\sigma))$ in the infinite dimensional Hilbert space $\mathcal{H}$ and assume σ non–degenerate. Then $\Pi(C^*(\mathrm{HG}(E,\sigma)))$ does not contain any compact operator beside 0.

(e) If σ is non–degenerate and E infinite dimensional, then $C^*(\mathrm{HG}(E,\sigma))$ has overcountably many inequivalent, irreducible representations $(\Pi^\chi,\mathcal{H})$, even for fixed $\chi \in \hat{\mathbb{R}}$.

Proof. Part (a). We put $H_\Pi(\kappa) := \Pi(H(\kappa))$ for all $\kappa \in \mathrm{HG}(E,\sigma)$. Since $\mathbb{R} \equiv \mathbb{R}\times\{0\}$ is a commutative subgroup of $\mathrm{HG}(E,\sigma)$ by the group relations (4.1), it follows from (4.2) that $H_\Pi(s,0)$ commutes with the represented algebra $\Pi(C^*(\mathrm{HG}(E,\sigma)))$ for each $s\in\mathbb{R}$. But the factor property of Π then implies that $H_\Pi(s,0) \in \mathbb{C}1 = \mathbb{C}H_\Pi(0,0)$, and consequently $H_\Pi(s,0) = \chi(s)1$ with a character $\chi \in \widehat{\mathbb{R}}$. Put $W_\Pi(f) := H_\Pi(0,f)$. From $(s,f) = (s,0)\circ(0,f)$ we conclude $H_\Pi(s,f) = \chi(s)W_\Pi(f)$ for all $s\in\mathbb{R}$ and all $f\in E$. The $W_\Pi(f)$ satisfy the Weyl relations (3.2), and thus $f \mapsto W_\Pi(f)$ constitutes a Σ_χ–projective unitary representation of the additive group E. By Theorem 2.1 there exists a unique representation Π_χ of the twisted group C*–algebra $\mathcal{W}(E,\chi\sigma) \equiv C^*(E,\Sigma_\chi)$ so that $W_\Pi(f) = \Pi_\chi(W^\chi(f))$ for all $f \in E$. With Proposition 4.2 we get $\Pi = \Pi_\chi \circ \theta_\chi$. The factor property (irreducibility) and non–degeneracy of Π_χ follows from the resp. properties of Π.

Part (b). If $\mathbb{R} \ni s \mapsto \Pi(H(s,0))$ is strongly continuous, then the character χ is continuous, and thus $\chi(s) \equiv \chi_\hbar(s) = \exp\{-i\hbar s\}$ $\forall s \in \mathbb{R}$ with a unique $\hbar \in \mathbb{R}$ by, e.g., [19, 23.27(e)].

Part (c). By Proposition 4.2 θ_χ is not injective, and so $\Pi = \Pi_\chi \circ \theta_\chi$ cannot be faithful. Nevertheless Π_χ may be faithful.

Part (d). As in Part (a) there is for irreducible $(\Pi,\mathcal{H})$ a non–trivial, irreducible representation $(\Pi\chi,\mathcal{H})$ of $\mathcal{W}(E,\chi\sigma)$, the image of which does not contain any non–trivial compact operator, the Weyl algebra with non–degenerate $\chi\sigma$ being antiliminary. The image of $(\Pi,\mathcal{H})$ is the same as that of $(\Pi\chi,\mathcal{H})$.

Part (e). For fixed $\chi \in \hat{\mathbb{R}}$ there is a one–one correspondence between the irreducible representations $\Pi^\chi = \Pi_\chi \circ \theta_\chi$ of $C^*(\mathrm{HG}(E,\sigma))$ and the irreducible representations Π_χ of $\mathcal{W}(E,\chi\sigma)$, and there are overcountably many Π_χ's (for fixed χ) since $\mathcal{W}(E,\chi\sigma)$ is antiliminary (for the notion of aniliminarity cf. [16], [36], the quasilocality of the Weyl algebra [13] leads to this property). ☐

4.2. Heisenberg Group Versus Continuous Field of Weyl Algebras

In the preceding subsection we saw that a single (represented) Weyl algebra $\mathcal{W}(E,\chi\sigma)$, with character $\chi \in \hat{\mathbb{R}}$, is not rich enough to carry a faithful image of the Heisenberg group algebra. This hint leads us to an algebraic, representation–independent, *–isomorphism between $C^*(\mathrm{HG}(E,\sigma))$ and a field of Weyl algebras, and constitutes a main result of the present work.

Theorem 4.4. *Given an arbitrary pre–symplectic space (E,σ) there exists a unique *–isomorphism*

$$\theta : C^*(\mathrm{HG}(E,\sigma)) \xrightarrow{onto} C^*_{WF}(E,\sigma)$$

from the discrete Heisenberg group C–algebra $C^*(\mathrm{HG}(E,\sigma))$ onto $C^*_{WF}(E,\sigma)$, that is the C*–algebra of the continuous Weyl algebra field $(\{\mathcal{W}(E,\hbar\sigma)\}_{\hbar\in\mathbb{R}},\mathcal{K})$, satisfying*

$$\theta(H(s,f)) = [\hbar \mapsto \exp\{-i\hbar s\}W^\hbar(f)], \quad \forall (s,f) \in \mathbb{R}\times E.$$

It holds $\theta_\hbar = \alpha_\hbar \circ \theta$ for every $\hbar \in \mathbb{R}$, or equivalently, $\theta(A) = [\hbar \mapsto \theta_\hbar(A)]$ for all $A \in C^*(\mathrm{HG}(E,\sigma))$ (where $\theta_\hbar$ is given in Proposition 4.2 and $\alpha_\hbar$ in Lemma 3.2).

Proof. Since the $H(\kappa)$, $\kappa \in \mathrm{HG}(E,\sigma)$, are linearly independent, the linear extension of $H(s,f) \mapsto [\hbar \mapsto \exp\{-i\hbar s\} W^\hbar(f)]$ is a *–homomorphism θ from the *–algebra $\Delta(\mathrm{HG}(E,\sigma))$ onto the dense sub–*–algebra $\Delta_{\mathrm{WF}}(E,\sigma)$ of $C^*_{\mathrm{WF}}(E,\sigma)$ from Section 3. Lemma 3.2(b) yields θ to be injective. We now show that $\|A\| = \|\theta(A)\|$ for all $A \in \Delta(\mathrm{HG}(E,\sigma))$, and arrive at the result.

For every $\hbar \in \mathbb{R}$ let $\omega^\hbar_{\mathrm{tr}}$ be the tracial state on $\mathcal{W}(E,\hbar\sigma)$ given in subsection 2.1, especially in equation (2.3). Then $\omega_\hbar := \omega^\hbar_{\mathrm{tr}} \circ \theta_\hbar$ is a state on $C^*(\mathrm{HG}(E,\sigma))$. By ω_{tr} we denote the tracial state on $C^*(\mathrm{HG}(E,\sigma))$.

For every $\lambda \geq 0$ it holds $[s \mapsto \exp\{-\lambda s^2\}] \in \mathcal{C}(\mathbb{R})$, which we extend trivially to $\mathcal{C}(\mathrm{HG}(E,\sigma))$ by Lemma 2.2 ($\mathbb{R} \equiv \mathbb{R} \times \{0\}$ is a subgroup). The associated state on $C^*(\mathrm{HG}(E,\sigma))$ is denoted by φ_λ, i.e., $\langle \varphi_\lambda; H(s,0)\rangle = \exp\{-\lambda s^2\}$ for all $s \in \mathbb{R}$, but $\langle \varphi_\lambda; H(s,f)\rangle = 0$ for $f \neq 0$. There is a probability measure μ_λ on $\mathbb{R}$ so that $\exp\{-\lambda s^2\} = \int_{\mathbb{R}} \exp\{-i\hbar s\} d\mu_\lambda(\hbar)$, which yields $\langle \varphi_\lambda; H(\kappa)\rangle = \int_{\mathbb{R}} \langle \omega_\hbar; H(\kappa)\rangle d\mu_\lambda(\hbar)$ for all $\kappa \in \mathrm{HG}(E,\sigma)$. Thus $\varphi_\lambda = \int_{\mathbb{R}} \omega_\hbar d\mu_\lambda(\hbar)$ with respect to the weak*–topology by the second part of Theorem 2.1, which leads to $\langle \varphi_\lambda; A^*A\rangle \leq \sup_\hbar \langle \omega_\hbar; A^*A\rangle$ for all $\lambda \geq 0$. In the limit $\lambda \to \infty$ we have $\langle \varphi_\lambda; H(\kappa)\rangle \to \langle \omega_{\mathrm{tr}}; H(\kappa)\rangle$, and hence $\varphi_\lambda \to \omega_{\mathrm{tr}}$ with respect to the weak *–topology, which also follows from Theorem 2.1. Consequently, with equation (2.3) we get for each $A \in \Delta(\mathrm{HG}(E,\sigma))$ that

$$
\begin{aligned}
\|A\|_2^2 = \langle \omega_{\mathrm{tr}}; A^*A\rangle &\leq \sup_{\lambda \geq 0} \langle \varphi_\lambda; A^*A\rangle \leq \sup_{\hbar \in \mathbb{R}} \langle \omega_\hbar; A^*A\rangle \\
&= \sup_{\hbar \in \mathbb{R}} \langle \omega^\hbar_{\mathrm{tr}}; \theta_\hbar(A^*A)\rangle \leq \sup_{\hbar \in \mathbb{R}} \|\theta_\hbar(A)\|^2 \,.
\end{aligned}
$$

Hence $\|A\|' := \sup_\hbar \|\theta_\hbar(A)\|$ defines a further C*–norm on $\Delta(\mathrm{HG}(E,\sigma))$. Since $\|A\|_2 \leq \|A\|'$ the last assertion in Proposition 2.1 implies that $\|A\| = \|A\|' = \sup_\hbar \|\theta_\hbar(A)\| = \|\theta(A)\|_{\mathrm{sup}}$. ☐

Now it is immediate to incorporate also the other two norms $\|.\|_1$ and $\|.\|_2$ from subsection 2.1 on $\Delta_{\mathrm{WF}}(E,\sigma)$ to obtain the estimations

$$
\sqrt{\sum_{k=1}^n |z_k|^2} = \|K\|_2 \leq \|K\|_{\mathrm{sup}} \leq \|K\|_1 = \sum_{k=1}^n |z_k| \tag{4.5}
$$

for arbitrary $K = \sum_{k=1}^n z_k [\hbar \mapsto \exp\{-i\hbar s_k\} W^\hbar(f_k)] \in \Delta_{\mathrm{WF}}(E,\sigma)$ with different tuples $(s_k, f_k) \in \mathbb{R} \times E$. And for the corresponding completions the analoguous inclusions as in equation (2.7) follow,

$$
\begin{aligned}
&\overline{\Delta_{\mathrm{WF}}(E,\sigma)}^1 = \mathrm{l}^1(\mathbb{R} \times E) \;\subseteq\; \overline{\Delta_{\mathrm{WF}}(E,\sigma)} \\
&= C^*_{\mathrm{WF}}(E,\sigma) \;\subseteq\; \overline{\Delta_{\mathrm{WF}}(E,\sigma)}^2 = \mathrm{l}^2(\mathbb{R} \times E)\,.
\end{aligned} \tag{4.6}
$$

Let us finally turn to the trivial symplectic form $\sigma \equiv 0$, leading to the trivial multiplier in equation (3.1). Hence the Weyl relations (3.2) imply $W^0(f)W^0(g) = W^0(f+g) = W^0(g)W^0(f)\ \forall f,g \in E$, and thus the *–algebra $\Delta(E,0)$ from equation (2.2), as well as the group Banach–*–algebra $\overline{\Delta(E,0)}^1 = \mathrm{l}^1(E)$, and the C*–Weyl algebra $\mathcal{W}(E,0)$ all are commutative. Moreover, the Heisenberg group equals now the discrete additive group $\mathbb{R} \oplus E$ and its group C*–algebra

$$C^*(\mathrm{HG}(E,0)) = \mathcal{W}(\mathbb{R} \oplus E, 0) = \mathcal{W}(\mathbb{R},0) \otimes \mathcal{W}(E,0) \tag{4.7}$$

is commutative, too.

For $\sigma \equiv 0$ the continuous field of C*–Weyl algebras $(\{\mathcal{W}(E,0)\}_{\hbar\in\mathbb{R}}, \mathcal{K})$ has, of course, an image algebra, which is constant in $\hbar$. Then $\mathcal{K}_{\mathrm{b}}$ consists of all bounded continuous functions $K : \mathbb{R} \mapsto \mathcal{W}(E,0)$, whereas the C*–algebra $C^*_{\mathrm{WF}}(E,0)$, which is generated by the functions $[\hbar \mapsto \exp\{-i\hbar s\}W^0(f)]$ represents some kind of almost periodic functions on $\mathbb{R}$ with values in $\mathcal{W}(E,0)$. In this connection let us recall that, according to [7], any commutative Weyl algebra $\mathcal{W}(D,0)$, is *–isomorphic to the C*–algebra of the almost periodic, $\sigma(D',D)$–continuous functions $\mathrm{AP}(D')$ on the topological dual D' of D with respect to an arbitrary locally convex Hausdorff topology in D. And in (4.7) D is one of the cases $\mathbb{R} \oplus E$, $\mathbb{R}$, resp. E.

4.3. Partially Decomposable Representations

In spite of being non–separable the C*–algebra $C^*(\mathrm{HG}(E,\sigma))$ may well have faithful representations in separable Hilbert spaces.

From Theorem 4.4 it follows that for each $A = \sum_k c_k H(s_k, 0)$ its norm is equal to $\|A\| = \sup_{\hbar\in\mathbb{R}} |\sum_k c_k \exp\{-i\hbar s_k\}|$ (where the discontinuous characters do not come into play). Thus we have

$$C^*(\mathrm{HG}(E,\sigma)) \supset \overline{\mathrm{LH}\{H(s,0) \,|\, s \in \mathbb{R}\}}^{\|\cdot\|} =: \mathsf{Z}(\mathbb{R}) \cong \mathrm{AP}(\mathbb{R}), \tag{4.8}$$

realizing the abstract subalgebra $\mathsf{Z}(\mathbb{R})$ by the commutative C*–algebra of all continuous, almost periodic functions $\mathrm{AP}(\mathbb{R})$. As illustrates, e.g., the factor representation $\Pi^\chi = \Pi_\chi \circ \theta_\chi$ from Theorem 4.3, with χ a discontinuous character, $\mathsf{Z}(\mathbb{R})$ may nevertheless be represented in terms of discontinuous characters (if $\aleph_1$ is the cardinality of the continuum then we have $\aleph_1$–many continuous and $2^{\aleph_1}$–many discontinuous characters $\chi \in \hat{\mathbb{R}}$ [19]).

Let $(\Pi, \mathcal{H})$, $\mathcal{H}$ separable, a representation of $C^*(\mathrm{HG}(E,\sigma))$ and form the canonically associated von Neumann algebra $\mathcal{M}_\Pi = \Pi(C^*(\mathrm{HG}(E,\sigma)))'' \subset \mathcal{B}(\mathcal{H})$, with center $\mathcal{Z}_\Pi = \mathcal{M}_\Pi \cap \mathcal{M}'_\Pi$. Then $\Pi(\mathsf{Z}(\mathbb{R}))'' =: \mathcal{Z}_{\mathsf{\Pi}}(\mathbb{R})$ is a von Neumann subalgebra of $\mathcal{Z}_\Pi \subset \mathcal{M}'_\Pi$.

Since $\mathcal{H}$ is separable there exists a desintegration $\mathcal{M}_\Pi = \int_\Gamma^\oplus \mathcal{M}^\gamma \, d\mu_\Pi(\gamma)$ such that $\mathcal{Z}_\Pi(\mathbb{R})$ are the diagonal operators in the desintegrated Hilbert space $\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}_\gamma \, d\mu_\Pi(\gamma)$, where (Γ, μ_Π) is a standard measure space (e.g. [16], [12]). That means especially that

$$H_\Pi(s,0) = \int_\Gamma^\oplus \chi_\gamma(s)\, 1_\gamma \, d\mu_\Pi(\gamma), \quad \text{with } \chi_\gamma(s) \in \mathbb{C}_1. \tag{4.9}$$

Since $C^*(\mathrm{HG}(E,\sigma))$ is not separable, it is hard to formulate criteria for a corresponding decomposition of Π (cf. the first and second edition of [12]). Thus we simply make the following definition, concerning the decomposability of Π with respect to the special Abelian von Neumann algebra $\mathcal{Z}_\Pi(\mathbb{R})$ in the commutant $\mathcal{M}'_\Pi$.

Definition 4.5. A representation $(\Pi, \mathcal{H})$ of $C^*(\mathrm{HG}(E,\sigma))$ is called *partially decomposable*, if $\mathcal{H}$ is separable and if

$$\Pi = \int_\Gamma^\oplus \Pi^\gamma \, d\mu_\Pi(\gamma), \quad \mathcal{H} = \int_\Gamma^\oplus \mathcal{H}_\gamma \, d\mu_\Pi(\gamma), \tag{4.10}$$

where the Hilbert space decomposition is that of the decomposition of the representation von Neumann algebra, induced by $\Pi(\mathsf{Z}(\mathbb{R}))'' =: \mathcal{Z}_\Pi(\mathbb{R})$. Here, $\Gamma \ni \gamma \mapsto (\Pi^\gamma, \mathcal{H}_\gamma)$ denotes a μ_Π–measurable family of representations of $C^*(\mathrm{HG}(E,\sigma))$ (meaning $\gamma \mapsto \Pi^\gamma(A)$ is measurable $\forall A \in C^*(\mathrm{HG}(E,\sigma))$).

Note that a factor representation is always partially decomposable in terms of a point measure. Only in a partially decomposable representation one can say more about the functions $s \mapsto \chi_\gamma(s)$ in (4.9).

Proposition 4.6. *Let $(\Pi, \mathcal{H})$ denote a partially decomposable representation of $C^*(\mathrm{HG}(E,\sigma))$. Then there is a μ_Π–null set N such that for all $\gamma \in \Gamma \backslash N$ the scalar functions $\mathbb{R} \ni s \mapsto \chi_\gamma(s)$ in (4.9) are characters.*

If, moreover, $(\Pi, \mathcal{H})$ is partially regular, then there is a μ_Π–null set N' such that for all $\gamma \in \Gamma \backslash N'$ the scalar functions $s \mapsto \chi_\gamma(s)$ in (4.9) are continuous characters. In this case the standard measure space (Γ, μ_Π) may be replaced, in terms of a bi–measurable bijection $\mathbb{R} \ni \hbar \mapsto \gamma(\hbar) \in \Gamma$, by an isomorphic real measure space $(\mathbb{R}, \nu_\Pi)$.

In a partially regular and partially decomposable representation Π ('prpd' for short) one has

$$\Pi^{\gamma(\hbar)} =: \Pi^\hbar = \Pi_\hbar \circ \theta_\hbar, \quad \text{for } \nu_\Pi - a.a.\ \hbar \in \mathbb{R}. \tag{4.11}$$

Here $(\Pi_\hbar, \mathcal{H}_\hbar)$ is a representation of $\mathcal{W}(E, \hbar\sigma)$.

A prpd representation Π is faithful, if $d\nu_\Pi(\hbar) \cong d\hbar$ and all $(\Pi_\hbar, \mathcal{H}_\hbar)$ are faithful representations of $\mathcal{W}(E, \hbar\sigma)$, respectively. For non–degenerate σ (im-

plying $\mathcal{W}(E,\hbar\sigma)$ to be simple), the latter condition is fulfilled automatically, since it may be arranged that ν_Π supports the non–trivial $(\Pi_\hbar,\mathcal{H}_\hbar)$ only.

Proof. In a partially decomposable representation, the multiplication law for the $H_\Pi(s,0)$ induces the multiplication law for the $\chi_\gamma(s)$, for μ_Π–a.a. γ.

The partially decomposable representation is partially regular only, if there is almost no discontinuous $s \mapsto \chi_\gamma(s)$. Since the continuous ones are indexed by $\hbar \in \mathbb{R}$ there is a function $\hbar \mapsto \gamma(\hbar)$, which is μ_Π–a.e. invertible and bi–measurable with respect to μ_Π and the transferred measure ν_Π. The formula (4.11) is proved as in Theorem 4.3.

If in a prpd representation ν_Π is quasi–equivalent to the Lebesgue measure, then it is supported by $\mathbb{R}$. For the decomposed operator

$$H_\Pi(s,f) = \int_{\mathbb{R}}^{\oplus} \exp\{-i\hbar s\}\, \Pi_\hbar(W^\hbar(f))\, d\nu_\Pi(\hbar), \tag{4.12}$$

one has generally $\|H_\Pi(s,f)\| = \sup_{\hbar\in\operatorname{supp}\nu_\Pi} \|\exp\{-i\hbar s\}\,\Pi_\hbar(W^\hbar(f))\|$ which then is equal to $\|H(s,f)\|$, according to Theorem 4.4. For the last conclusion observe that the norm of the represented element $\Pi_\hbar(W^\hbar(f))$ is equal to the abstract norm $\|W^\hbar(f)\|$, since each $\Pi_\hbar$ is assumed faithful. □

Notice that in a prpd representation Π we have an explicit expression for the selfadjoint generator

$$\frac{d\,H_\Pi(s,0)}{i\,ds}|_{s=0} =: Z_\Pi = \int_{\mathbb{R}}^{\oplus} -\hbar\, 1_\hbar\, d\nu_\Pi(\hbar) \tag{4.13}$$

with the domain

$$\operatorname{dom} Z_\Pi = \{\int_{\mathbb{R}}^{\oplus} \Omega_\hbar\, d\nu_\Pi(\hbar) \in \int_{\mathbb{R}}^{\oplus} \mathcal{H}_\hbar\, d\nu_\Pi(\hbar) \mid \int_{\mathbb{R}} \hbar^2\, \|\Omega_\hbar\|^2\, d\nu_\Pi(\hbar) < \infty\}. \tag{4.14}$$

The selfadjoint Z_Π is unbounded, if $\operatorname{supp}\nu_\Pi$ is unbounded, but is always affiliated with the von Neumann algebra $\mathcal{M}_\Pi$, since it is approximated by elements from $\mathcal{M}_\Pi$, resp. commutes with elements from $\mathcal{M}_\Pi'$.

If σ is non–degenerate there is, in some sense, a minimal type of a faithful prpd representation of $C^*(\mathrm{HG}(E,\sigma))$, namely

$$(\Pi,\mathcal{H}) = \int_{\mathbb{R}}^{\oplus} (\Pi_\hbar \circ \theta_\hbar,\, \mathcal{H}_\hbar)\, d\hbar, \tag{4.15}$$

with $\mathbb{R} \ni \hbar \mapsto (\Pi_\hbar,\mathcal{H}_\hbar)$ a measurable family of irreducible, non–degenerate representations of the $\mathcal{W}(E,\hbar\sigma)$, respectively. In this case $\mathcal{Z}_\Pi(\mathbb{R})$ equals the center $\mathcal{Z}_\Pi$. That means, that the decomposition is the central one and is extremal for each vector state on $(\Pi,\mathcal{H})$. Given any irreducible representation $(\Pi_1,\mathcal{H}_1)$ of

$\mathcal{W}(E,\sigma)$ one obtains a continuous (hence measurable) family of irreducible representations $\mathbb{R} \ni \hbar \mapsto (\Pi_\hbar, \mathcal{H}_\hbar)$ of $\mathcal{W}(E,\hbar\sigma)$ by setting $\Pi_\hbar := \Pi_1 \circ \eta_\hbar$. Here $\{\eta_\hbar : \mathcal{W}(E,\hbar\sigma) \overset{onto}{\to} \mathcal{W}(E,\sigma)\}_{\hbar\in\mathbb{R}}$ denotes an easy–to–construct, continuous family of *–isomorphism, and $\mathcal{H}_\hbar := \mathcal{H}_1$, for all $\hbar \in \mathbb{R}$. The $(\Pi_\hbar \circ \theta_\hbar, \mathcal{H}_\hbar)$, $\hbar \in \mathbb{R}$, are pair–wise disjoint, irreducible representations of $C^*(\mathrm{HG}(E,\sigma))$.

5. Heisenberg Group and Weyl Quantization

5.1. Continuous and Strict Weyl Deformation Quantization

Let us recapitulate here some recent results on the strict and continuous Weyl quantization from [8] in order to discuss in the following subsection the relationship to the Heisenberg group approach.

As formalized in subsection 2.2, the starting point for a quantization is a Poisson algebra $(\mathcal{P}, \{.,.\})$, which owns, beside the anticommutative Poisson product, also a commutative product. For the Weyl quantization, associated with a pre–symplectic space (E,σ), we choose first the abstract commutative *–algebra $\Delta(E,0)$, consisting of (complex) linear combinations of classical Weyl elements $W^0(f) \in \mathcal{W}(E,0)$.

The Poisson bracket originates from realizing the abstract $W^0(f)$ by functions $W^0(f)[F] := \exp\{iF(f)\}$ for all $F \in E'_\tau$. E'_τ is the dual of E in some locally convex vector space topology τ for E. That means, that E'_τ is considered as the classical phase space. As usual, linear combinations and the commutative product for functions on E'_τ are considered pointwisely, and the *–operation is defined by the complex conjugation. The topology in E'_τ is that of point–wise convergence. If one assumes a differentiable structure on E'_τ, making E'_τ to an (in general infinite dimensional) differentiable manifold, and requires σ to be jointly τ–continuous then one may define

$$\{A,B\}[F] := -\sigma(d_F A, d_F B) \quad \text{for } A,B \in C^\infty_{\mathbb{R}}(E'_\tau)\,, \tag{5.1}$$

provided the differentials are elements of $E \subset E'_{\tau'}$. The latter condition plays an essential role already for the Banach Poisson manifolds of [33].

Since the function realization of the Poisson algebra $\mathcal{P} = \Delta(E,0)$ requires complex functions, to have a nontrivial *–operation, we have to extend the Poisson tensor $-\sigma$ *complex*–bilinearly ($-\sigma$ is a '2–tensor' with respect to the differentiable manifold E'_τ only in the weak sense of being defined on the subspace $E \times E \subset E''_\tau \times E''_\tau$). One obtains for the Weyl phase space functions

$$d_F W^0(f)[G] = iW^0(f)[F]\ \Phi^0(f)[G],\ \Phi^0(f)[G] := G(f),\qquad \forall f \in E,\ \forall F, G \in E'_\tau\,. \quad (5.2)$$

The classical field $\Phi^0(f)$ is by definition originally an element in $E''_\tau \subset C^\infty_{\mathbb{R}}(E'_\tau)$. Considered as an embedded element from $E \subset E''_\tau$ we denote it simply by f and obtain $d_F W^0(f) = iW^0(f)[F]\, f$. This leads to the basic Poisson brackets

$$\{W^0(f), W^0(g)\}[F] = \sigma(f,g) W^0(f+g)[F]\,, \quad \forall f,g \in E,\ \forall F \in E'_\tau\,. \quad (5.3)$$

We observe that $d_F A \in E$, after E having been complexified, is satisfied for all $A \in \Delta(E,0) \subset C^\infty(E'_\tau)$.

For the further developemments of quantization theory the function realization is not helpful and we simply *define* the Poisson bracket $\{.,.\}$ by the bilinear extension of the basic Poisson brackets (5.3) (dropping F).

An enlarged Poisson algebra is obtained with the help of a semi–norm ς on E satisfying

$$|\sigma(f,g)| \leq c\, \varsigma(f)\, \varsigma(g)\,, \quad \forall f,g \in E\,, \quad (5.4)$$

for some constant $c > 0$. The set of all finite or countable sums defined by

$$\mathcal{P}_\varsigma := \{ \sum_k z_k W^0(f_k)\ \mid \sum_k |z_k|\, \varsigma(f_k)^m < \infty \text{ for all } m \in \mathbb{N}_0\} \ \subset\ \overline{\Delta(E,0)}^1$$

(always with different f_k's) constitutes a commutative Fréchet–*–algebra. This means that the *–operation and the commutative product inherited from $\overline{\Delta(E,0)}^1$ are continuous resp. jointly continuous with respect to the Fréchet topology given by the increasing system of norms

$$\| \sum_k z_k W^0(f_k) \|^n_\varsigma := \sum_{m=0}^n \left(\sum_k |z_k| \varsigma(f_k)^m \right)\,, \quad n \in \mathbb{N}\,.$$

The Poisson bracket $\{.,.\}$ from equation (5.3) extends Fréchet–continuously from $\Delta(E,0)$ to the Fréchet–*–algebra $\mathcal{P}_\varsigma$.

Theorem 5.1. *Let $(\mathcal{P}, \{.,.\})$ denote one of the above introduced Poisson algebras $\Delta(E,0)$ or $\mathcal{P}_\varsigma$. Define for the continuous field of C^*–Weyl algebras $(\{\mathcal{W}(E,\hbar\sigma)\}_{\hbar\in\mathbb{R}}, \mathcal{K})$ from Theorem 3.1 the global quantization map $Q : \overline{\Delta(E,0)}^1 \to \overline{\Delta_{WF}(E,\sigma)}^1 \subseteq C^*_{WF}(E,\sigma)$ as the linear and $\|.\|_1$–preserving extension of*

$$Q(W^0(f)) := [\hbar \mapsto W^\hbar(f)], \quad \forall f \in E.$$

Then one obtains a continuous quantization of $(\mathcal{P}, \{.,.\})$ in the sense of subsection 2.2.

Furthermore, with the quantization maps

$$Q_\hbar := \alpha_\hbar \circ Q, \quad \forall \hbar \in \mathbb{R},$$

it follows that the structure $(\mathcal{W}(E,\hbar\sigma),Q_\hbar)_{\hbar\in\mathbb{R}}$ *is a strict deformation quantization of each Poisson algebra* $(\mathcal{P},\{.,.\})$. *It holds true that* $Q_\hbar : \overline{\Delta(E,0)}^1 \to \overline{\Delta(E,\hbar\sigma)}^1 \subseteq \mathcal{W}(E,\hbar\sigma)$ *is a* $\|.\|_1$*–preserving, linear, *–preserving bijection satisfying* $Q_\hbar(W^0(f)) = W^\hbar(f)$ *for all* $f \in E$.

If σ is jointly τ–continuous with respect to some locally convex Hausdorff topology τ on E, then it is known that there always exists a τ–continuous semi–norm ς on E such that the estimate (5.4) is fulfilled.

As shortly mentioned in Introduction, the qualification 'deformation' for a 'strict quantization' (= C*–algebraic formalism!) appeals to the fact, that the quantization map is invertible. So one could, also in our infinite dimensional case, introduce a non–commutative, so–called '*–product' for functions on E'_τ by transforming the operator product of the quantized observables back to the pre–images. One should stress in this connection, that the phase space functions obtain in this manner not only a non–commutative product but also a non–classical statistics.

5.2. Heisenberg Group Algebra and Quantization

Theorem 5.1 expresses a quantization strategy which is based on the Weyl elements and results into the mapping

$$\sum_k z_k W^0(f_k) \quad \overset{\text{Weyl Quant. } Q_\hbar}{\longrightarrow} \quad \sum_k z_k W^\hbar(f_k)\,, \tag{5.5}$$

with $z_k \in \mathbb{C}$, $f_k \in E$. This Weyl quantization is performed, however, in two steps: In the first step one has not yet specified a value for the Planck constant but deals globally with functions of the variable Planck parameter. In terms of continuous fields of C*–Weyl algebras one formulates strong continuity properties for the behaviour of the quantized observables in the classical limit $\hbar \to 0$ (cf., Proposition 2.6). It has turned out, that also for the investigation of the more modest continuity of the Rieffel condition in the frame of strict quantization (cf., Definition 2.3), it is formally advantageous to deal first with the continuous quantization.

One principal insight of the present investigation is that the continuous Weyl quantization is closely related to a Heisenberg group approach, since the suitably defined C*-group algebra of the Heisenberg group $C^*(\mathrm{HG}(E,\sigma))$ is *–isomorphic to the C*–algebra of Weyl fields $C^*_{\mathrm{WF}}(E,\sigma)$, according to Theorem 4.4. This *–isomorphism is interesting for itself, since it confirms, especially in the infinite dimensional case, that the introduced Heisenberg group algebra has a proper mathematical structure. As a result of this *–isomorphism, the continuous Weyl quantization may alternatively be performed by mapping the classical Poisson algebra, generated by classical Weyl elements, into the Heisenberg group algebra. In terms of the previously introduced mappings this mapping Q_H may be expressed as

$$(\mathcal{P}, \{.,.\}) \quad \overset{Q_H = \theta^{-1} \circ Q}{\longrightarrow} \quad C^*(\mathrm{HG}(E,\sigma)) = \theta^{-1}\left(C^*_{\mathrm{WF}}(E,\sigma)\right) \tag{5.6}$$

and is written for single elements

$$\sum_k z_k W^0(f_k) \quad \overset{Q_H}{\longrightarrow} \quad \sum_k z_k H(0,f_k) = \theta^{-1}\left(\sum_k z_k[\hbar \mapsto W^\hbar(f_k)]\right). \tag{5.7}$$

The mentioned quantization mappings perform on the algebraic stage and have to be supplemented by a representation map on the quantized side. This is especially essential for infinite dimensional E. From this point of view, the quantization map Q_H gives already valuable instructions for operator orderings in the quantized theory. It enables the transfer of the vast experience, concerning the field quantization in Weyl form, to the concepts of the Heisenberg group algebra.

Beside the choice of a representation of $C^*(\mathrm{HG}(E,\sigma))$ there should be mentioned another ambiguity: One obtains a *–isomorphic Heisenberg group algebra, if one replaces the 2–cocycle $\frac{1}{2}\sigma \in Z^2(E,\mathbb{R})$ by an equivalent one $\frac{1}{2}\sigma(f,g) + v(f) + v(g) - v(f+g)$, where $v : E \to \mathbb{R}$ is a certain function. In the interpretation of a Weyl quantization, one may then arrive at an equivalent strict (deformation) quantization in the sense of Definition 2.7. This is treated (for multiplicative cocycles) in [20], where also the relation to the operator ordering of field products is discussed.

For the evaluation of physical quantities it is, of course, only the (present) experimental value of the Planck parameter $\hbar > 0$, which is of relevance (not mentioning classical approximations). Thus, a continuous quantization, ending up with the Heisenberg group algebra, must be followed by the application of the evaluation map $\theta_\hbar \cong \alpha_\hbar$. In [10] and [9] just the second step is elaborated in the context of systems with information transmission capacities. The choice of special representations enables there optical interpretations, and the classical limit leads to geometrical optics.

The intended quantization strategy becomes clearer by considering the (representation dependent) Heisenberg Lie algebras. In this context, the missing s–dependence in (5.7) can also be addended, if another choice of a function realization of the classical Poisson algebra is made. We can, however only sketch these ideas in the next subsection and defer the elaboration of the details to another occasion.

5.3. Heisenberg Lie Algebra and Quantization

Whereas quantization via the Heisenberg group algebra employs bounded "operators" only and may be performed on the algebraic stage, quantization in terms of the Heisenberg Lie algebra meets the problems of unbounded operators and requires a Hilbert space representation.

Definition 5.2. Let us introduce the following notions, related to representations of the Heisenberg group $HG(E,\sigma)$ and its group algebra $C^*(\mathrm{HG}(E,\sigma))$.

(a) First we have now to discriminate notationally between the two meanings of (s,f) :

$$E_{ex} := \mathbb{R}\times E \ni (s,f) \equiv \kappa \text{ and } HG(E,\sigma) \ni \gamma(s,f) \equiv \gamma(\kappa) \neq (s,f).$$

For the (s,f) we have now only the linear structure and for the $\gamma(s,f)$ we have only the group multiplication (4.1).

(b) A representation $\Pi : C^*(\mathrm{HG}(E,\sigma)) \longrightarrow \mathcal{B}(\mathcal{H})$ is called *regular*, if

$$\mathbb{R} \ni \lambda \mapsto H_\Pi(\lambda\, s, \lambda\, f)$$

is strongly operator continuous for all $(s,f) \in E_{ex}$. Here, as before, $H_\Pi(s,f)$ is short for $\Pi(H(s,f))$, $H(s,f) \in C^*(\mathrm{HG}(E,\sigma))$.

(c) In a non–degenerate representation $(\Pi,\mathcal{H})$ of the Heisenberg group algebra $C^*(\mathrm{HG}(E,\sigma))$ we denote by $C^*_\Pi(\mathrm{HG}(E,\sigma))$ its image and write for the set of unitary operators $\{H_\Pi(s,f)\,|\; (s,f) \in E_{ex}\}$ the symbol $\mathrm{HG}_\Pi(E,\sigma)$ (note that $H_\Pi(s,f)$ may be also viewed as a unitary representative of the group element $\gamma(s,f)$).

(d) In a regular representation, we define for each $(s,f) \in E_{ex}$ the selfadjoint operator

$$\frac{1}{i}\,\frac{dH_\Pi(\lambda\, s, \lambda\, f)}{d\lambda}|_0 =: L_\Pi(s,f) \tag{5.8}$$

and call it *Lie element* of $\mathrm{HG}_\Pi(E,\sigma)$.

(e) We divide the Lie elements into two classes as follows:

$$L_\Pi(s,0) =: Z_\Pi(s),\; s\in\mathbb{R}, \quad L_\Pi(0,f) =: \Phi_\Pi(f),\; f\in E. \tag{5.9}$$

Proposition 5.3. *Let* $(\Pi,\mathcal{H})$ *a regular representation of* $C^*(\mathrm{HG}(E,\,\sigma))$.

(a) *For each finite dimensional subspace*

$$M \;\subset\; E_{ex} \;=\; \mathbb{R}\times E$$

there exists a common, dense subspace $\mathcal{D}(M)\subset\mathcal{H}$ *of analytic vectors for all* $L_\Pi(\kappa),\kappa\in M$. *That implies, that there exist on* $\mathcal{D}(M)$ *all (finite) polynomials of the* $L_\Pi(\kappa),\kappa\in M$, *where formulas, concerning their sum and (non–commutative) operator product, are, of course, uniquely given by the algebraic relations in* $C^*_\Pi(\mathrm{HG}(E,\sigma))$. *Especially, one finds by differentiation the subsequent relations.*

(b) *Let the* $\kappa_i=(s_i,f_i)=(s_i,0)+(0,f_i)$ *always be in the finite dimensional subspace* $M\subset E_{ex}$ *and the* $\lambda_i\in\mathbb{R}$. *Then*

$$L_\Pi(\sum_{i=1}^{n}\lambda_i\,\kappa_i)=\sum_{i=1}^{n}\lambda_i\;L_\Pi(\kappa_i),\quad \textit{on}\quad \mathcal{D}(M). \tag{5.10}$$

Especially one has, with $Z_\Pi:=Z_\Pi(1)$,

$$L_\Pi(\kappa_i)=s_i\,Z_\Pi+\Phi_\Pi(f_i)\quad \textit{on}\quad \mathcal{D}(M). \tag{5.11}$$

(c) *The commutator for two selfadjoint Lie elements calculates to*

$$[L_\Pi(\kappa_i),L_\Pi(\kappa_j)]_-=[\Phi_\Pi(f_i),\Phi_\Pi(f_j)]_-=i\sigma(f_i,f_j)\,Z_\Pi,\quad \textit{on}\quad \mathcal{D}(M). \tag{5.12}$$

(d) *If the regular* Π *is also partially decomposable in the sense of Subsection 4.3, we obtain*

$$(\Pi,\mathcal{H})=\int_{\mathbb{R}}^{\oplus}(\Pi^\hbar,\mathcal{H}_\hbar)\;d\nu_\Pi(\hbar),\quad (\Pi^\hbar,\mathcal{H}_\hbar)\ \textit{regular}\ \nu_\Pi-\text{a.e.} \tag{5.13}$$

Thus, not bothering about ν_Π*–null sets anymore, we may differentiate*

$$H_\Pi(s,f)=\int_{\mathbb{R}}^{\oplus}H^\hbar_\Pi(s,f)\;d\nu(\hbar),\quad H^\hbar_\Pi(s,f):=\Pi^\hbar(H(s,f)), \tag{5.14}$$

component–wise and obtain

$$L_\Pi(s,f)=\int_{\mathbb{R}}^{\oplus}L^\hbar_\Pi(s,f)\;d\nu(\hbar),\quad L^\hbar_\Pi(s,f)=-s\,\hbar\,1^\hbar_\Pi\,+\,\Phi^\hbar_\Pi(f), \tag{5.15}$$

on the resp. domains of definition, which we do not specify here. The decomposability of Π *reduces algebraic operations to that of the components giving also*

$$[L^\hbar_\Pi(\kappa_i),L^\hbar_\Pi(\kappa_j)]_-=[\Phi^\hbar_\Pi(f_i),\Phi^\hbar_\Pi(f_j)]_-=-i\,\hbar\,\sigma(f_i,f_j)1^\hbar_\Pi. \tag{5.16}$$

Thus, we find in the original representation

$$[L_\Pi(\kappa_i),L_\Pi(\kappa_j)]_-=[\Phi_\Pi(f_i),\Phi_\Pi(f_j)]_- =-i\int_{\mathbb{R}}^{\oplus}\hbar\,\sigma(f_i,f_j)1^\hbar_\Pi\;d\nu(\hbar). \tag{5.17}$$

Recall that, by operator algebraic reduction theory of partially decomposable representations Π of the Heisenberg group algebra (not containing an $\hbar$–parameter), we obtain an $\hbar$–spectrum, namely $\operatorname{supp}\nu_\Pi$, which possibly comprises the value $\hbar = 0$. By Π–dependent differentiation (in a regular Π) we obtain commutation relations for Lie elements, which – after polarization of E – lead to the usual P–Q commutation relations with variable $\hbar$. The algebraic form, however, of the commutator in (5.12) is representation independent. To interpret it as a product for selfadjoint elements, the commutation must be multiplied by $\pm i$. We multiply the commutator by $-i$ and take it then as the paradigm for the abstract Lie–product (in the $\hbar$–scaled commutator of (2.8) multiplication by $+i$ is incorporated. The different sign is compensated for by the $-\hbar$ in (5.16)).

Definition 5.4. The abstract Heisenberg Lie algebra $h(E,\sigma)$ for an arbitrary pre–symplectic space (E,σ) is the set $\{L(\kappa)\,|\,\kappa \in E_{ex} = \mathbb{R}\times E\}$, endowed with the real linear structure of E_{ex} and with the Lie product (without the comma)

$$[L(s,f)\,L(t,g)] := \sigma(f,g)\,Z, \quad Z := L(1,0), \quad \forall (s,f),(t,g) \in E_{ex}. \tag{5.18}$$

It follows that $h(E,\sigma)$ is distributive, nilpotent, and not associative if σ is non–trivial.

Let us here only shortly indicate, how $h(E,\sigma)$ may be realized by phase space functions, generalizing the finite dimensional theory. In a function realization we decorate the Lie elements by a superscript '0'.

There are mainly two types of phase space realizations. Assuming a locally convex Hausdorff topology τ on the test function space E, the usual phase space is the topological dual space E'_τ. As is explained at the beginning of subsection 5.1, $W^0(f)[F] = \exp\{iF(f)\}$ and thus $\Phi^0(f)[F] = F(f)$, $F \in E'_\tau$, so that $\Phi^0(f) \in E \subset E''_\tau$. Assuming further a differentiable structure on E'_τ we obtain by (5.1)

$$\{\Phi^0(f),\Phi^0(g)\} = -\,\sigma(f,g)\,1, \quad 1[F] = 1, \forall F \in E'_\tau.$$

We set $L^0(0,f) := \Phi^0(f), \forall f \in E$, and $L^0(s,0) \equiv Z^0(s) := -s\,1, \forall s \in \mathbb{R}$ obtaining the function realization for the abstract Lie element $L(s,f)$ by $L^0(s,f) = -s\,1 + \Phi^0(f)$. Since $d_F Z^0(s) = 0, \forall F \in E'_\tau$ we get for the Poisson bracket

$$\{L^0(s,f),L^0(t,g)\} = \{\Phi^0(f),\Phi^0(g)\} = -\,\sigma(f,g)\,1. \tag{5.19}$$

Thus, in the function realization, the Poisson bracket constitutes a realization of the abstract Lie product $[L(s,f)\,L(t,g)]$.

In the second type of a function realization the phase space is $E'_{ex} = \mathbb{R}\times E'_\tau$.

Let us denote this duality relation by

$$\langle . ; . \rangle : E'_{ex} \times E_{ex} \longrightarrow \mathbb{C}. \tag{5.20}$$

If we assume σ to be τ–bi–continuous, then not only the linear combinations in $\mathfrak{h}(E,\sigma)$ but also the Lie product is (bi–) continuous, and we arrive at a topological algebra $\mathfrak{h}(E,\sigma)_\tau$. The extended phase space may now considered as the topological dual of $\mathfrak{h}(E,\sigma)_\tau$, that is (denoting the dual of an algebra by a $*$, the dash meaning 'commutant')

$$E'_{ex} \cong \mathfrak{h}(E,\sigma)^*_\tau.$$

We realize now the Lie algebra elements, without changing their notation, by continuous, linear functions $L^0(s,f)$ on $\mathfrak{h}(E,\sigma)^*_\tau$, defining

$$L^0(s,f)[r,F] := \langle (r,F); L(s,f) \rangle = -r\,s + F(f), \quad (r,F) \in E'_{ex}. \tag{5.21}$$

For differentiable functions $A, B \in C^\infty(E'_{ex})$ the canonical Poisson bracket may now be formulated quite analogously to (5.1). Since the differentials are assumed to be in the extended predual E_{ex}, they may also be considered as elements in $\mathfrak{h}(E,\sigma)_\tau$ and the Poisson bracket may alternatively be expressed in terms of the Lie product of the differentials (what we use in (5.24) below).

We find for the adjoint action of $HG(E,\sigma) \ni \gamma(s,f)$ in $\mathfrak{h}(E,\sigma)_\tau$

$$Ad_{\gamma(t,g)}\, L(s,f) = \gamma(t,g)L(s,f)\gamma(-t,-g) = L(s+\sigma(g,f),f) \tag{5.22}$$

and for the coadjoint action on $(r,F) \in E'_{ex} \cong \mathfrak{h}(E,\sigma)^*_\tau$

$$Ad^*_{\gamma(t,g)}\,(r,F) = (r, \sigma^\flat(g) + F), \quad \text{with } \sigma^\flat(g)(f) := \sigma(g,f). \tag{5.23}$$

Thus, the subsets

$$E'_r := \{r\} \times E' \subset E'_{ex},\ r \in \mathbb{R},$$

are invariant under the coadjoint action, and the coadjoint orbits

$$\mathcal{O}(r,F) : \{(r, \sigma^\flat(g) + F) \,|\, g \in E\}, \quad (r,F) \in E'_{ex},$$

are subsets of E'_r. If E is finite dimensional and σ non–degenerate $\mathcal{O}(r,F) = E'_r$, $\forall F \in E'$. The canonical Poisson bracket for Lie functions $L^0(s,f)$, evaluated at points $(r,F) \in E'_r$, gives

$$\{L^0(s,f), L^0(t,g)\}[r,F] := \langle (r,F); [L(s,f)L(t,g)] \rangle$$
$$= -r\sigma(f,g)1[r,F]. \tag{5.24}$$

Using the correspondence $i\{.,.\} \leftrightarrow [.,.]_-$ we observe the striking analogy to (5.16). It almost appears, that a Planck parameter already arises from the Poisson reduction theory for function realizations of the Heisenberg Lie algebra. A more detailed comparison, emphasizing the infinite dimensional case, will be deferred to a future investigation.

Here, let us emphasize only that the correspondence (5.24) $\leftrightarrow$ (5.16) covers merely linear functions of the canonical variables. Thus it would be misleading to consider any formalism, in which the Heisenberg Lie algebra appears, to be 'quantized', if there is no prescription for quantizing higher powers of the canonical variables. To quantize a certain class of nonlinear functions of the canonical variables, generated by linear combinations of Weyl elements, the present work advocates the use of the Heisenberg group algebra.

In regular operator representations Π of the Heisenberg group algebra, we have from (5.8) $\exp\{iL_\Pi(s,f)\} = H_\Pi(s,f)$. Imitating this for the function realization on E'_{ex} we find $\exp\left\{iL^0(s,f)\right\}[r,F] = \exp\{-irs\}\, W^0(f)[F]$ (commutative exponentiation!). We may use the linear combinations of these phase space functions to generate extended classical Poisson algebras $\mathcal{P}_{ex}$ in the same manner as in subsection 5.1. Since the phase space function $\exp\left\{iL^0(s,f)\right\}$ is equivalent to the family of its restrictions to the leaves E'_r, $r \in \mathbb{R}$, we may express this symbolically by

$$\exp\left\{iL^0(s,f)\right\} \equiv [r \mapsto \exp\{-irs\}\, W^0(f)].$$

The scheme (5.7) of a continuous Weyl quantization may now be extended to

$$\begin{aligned}\sum_k z_k[r \mapsto \exp\{-irs_k\}\, W^0(f_k)] \xrightarrow{Q_H} \quad & \sum_k z_k H(s_k, f_k) \\ &= \theta^{-1}\left(\sum_k z_k[\hbar \mapsto \exp\{-i\hbar s_k\}\, W^\hbar(f_k)]\right). \end{aligned} \tag{5.25}$$

By a rescaling one could arrange that the dequantization of the quantized theory leads to a prescribed r–sector of the classical theory. A general statistical state on $\mathcal{P}_{ex}$ leads to a marginal measure $d\nu(r)$ for the distribution of the r–sectors and is the classical counterpart of the partial decomposition measure $d\nu_\Pi(\hbar)$ of the quantized theory (cf. subsection 4.3).

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