SYZYGIES OF REDUCIBLE CURVES

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Abstract: Here we use works by Schreyer and Aprodu to check the Green conjecture for certain reducible curves with a nice nodal model inside $\mathbb{P}^1 \times \mathbb{P}^1$.

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For any scheme $X$ and any spanned line bundle $L$ on $X$ and all non-negative integers $a, b$ let $K_{u,v}(X, L)$ denote the Koszul cohomology groups (see [8], [4]).

We first use [11] to get several stable reducible curves $X$ with $\omega_X$ very ample and for which Green's conjecture holds.

**Theorem 1.** Fix integers $q, p, \delta$ such that $q \geq p \geq 3, 0 \leq \delta \leq p - 2$. If $(p, q, \delta) = (3, 3, 1)$, then assume $Y$ irreducible. If $(p, \delta) = (3, 1)$ and $q > 3$, then assume that $Y$ has no component of type $(0, 1)$. Fix a general $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $\sharp(S) = \delta$. Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be any reduced curve of type $(p, q)$ such that $S \subset Y$ and $Y$ has an ordinary node at each point of $S$. Let $u : X \to Y$ be the partial normalization of $Y$ in which we normalize only the points of $S$. Then $X$ is connected and Gorenstein, $p_a(X) = pq - p - q + 1 - \delta$, $\omega_X$ is very ample and $K_{x,1}(X, \omega_X) = 0$ for every $x \geq g - p + 1$.

See Lemma 5 for a list of reducible curves to which Theorem 1 may be applied.

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**Proposition 1.** Fix integers $p, g, c$ such that $p \geq 2$, $g > p(p - 1)$ and $0 \leq c \leq g$. There is an integral nodal curve $X$ such that:

(i) $p_\delta(X) = g$, $\sharp(S_{\text{Sing}}(X)) = c$, there is $M \in \text{Pic}^p(X)$ such that $h^0(X, M) = 2$; while there is no $N \in \text{Pic}^{p-1}(X)$ with $h^0(X, N) \geq 2$;

(ii) there are an integer $y \geq 2g$ and $R \in \text{Pic}^y(X)$ such that $K_{h^0(X,R)-p,1}(X, R) = 0$;

(iii) $K_{x,1}(X, L) = 0$ for every integer $d \geq y + 2g$, every integer $x \geq d + 1 - g - p$, and every $L \in \text{Pic}^x(X)$, i.e. $L$ has property $M_{k-1}$.

In particular (iii) implies that $X$ satisfies the Green-Lazarsfeld conjecture.

For any scheme $A$ and any $P \in A_{\text{reg}}$ let $\{2P, A\}$ denote the closed subscheme of $A$ with $I_P^2$ as its ideal sheaf. For any finite $S \subset A_{\text{reg}}$ set $\{2S, A\} := \cup_{P \in S}\{2P, A\}$.

**Lemma 1.** Fix integers $q, p, \delta$ such that $q \geq p \geq 2$ and $0 \leq \delta \leq p - 2$. Fix a general $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $\sharp(S) = \delta$. Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be any reduced curve of type $(p, q)$ such that $S \subset Y$ and $Y$ has an ordinary node at each point of $S$. Let $u : X \rightarrow Y$ be the partial normalization of $Y$ in which we normalize only the points of $S$. Then $X$ is connected.

**Proof.** Assume that $X$ is not connected and write $X = A \sqcup B$ with $A$ one of its connected components. Set $A' := u(A)$ and $B' := u(B)$. Call $(a, b)$ the bidegree of $A'$. Thus $B'$ has bidegree $(p - a, q - b)$. Without loosing generality we may assume $0 \leq a \leq p/2$. Since $Y$ is nodal, $A' \cap B' \subseteq S$ and $A'$ intersects transversally $B'$, i.e. $\sharp(A' \cap B') = a(q - b) + b(p - a)$. Thus

\[ a(q - b) + b(p - a) \leq p - 2. \quad (1) \]

Since $a \leq p/2$, the second term of the left hand side of (1) gives $b \in \{0, 1\}$. If $b \in \{0, 1\}$, then the first term of the left hand side of (1) gives $a = 0$. Even in the remaining case $(a, b) = (0, 1)$, the inequality (1) fails. \hfill \Box

**Lemma 2.** Fix integers $u \geq 2$ and $v \geq 2$ and a general $S \subset A := \mathbb{P}^1 \times \mathbb{P}^1$ such that $\sharp(S) = uv - u - v + 1$. There is an integral $B \in \mathcal{I}_{\{2S, A\}}(u, v)$ if and only if $(u, v) = (2, 2)$.

**Proof.** The “if” part is obvious. We only check the “only if” part. Fix an integer $t \geq 0$ and let $B \subset A$ be a general subset such that $\sharp(A) = t$. Since $u \geq 2$ and $v \geq 2$, [5], Corollary 4.6, gives that either $h^0(A, \mathcal{I}_{\{2B, A\}}(u, v)) = 0$ (case $3t \geq (u + 1)(v + 1)$) or $h^1(A, \mathcal{I}_{\{2B, A\}}(u, v)) = 0$ (case $3t \leq (u + 1)(v + 1)$), unless $(u, v, t) = (4, 4, 8)$ (in the exceptional case $\mathcal{I}_{\{2B, A\}}(4, 4) = \{2\Gamma\}$, where $\Gamma$ is the only smooth elliptic curve of bidegree $(2, 2)$ containing $B$). If $uv + u + v \geq
3uv - 3u - 3v + 3, i.e. if \(2u + 2v \geq 2uv + 3\), then \(u = v = 2\). No other case may arise.

Proof of Theorem 1. Obviously \(X\) is Gorenstein. Lemma 1 gives the connectedness of \(X\). The adjunction formula on \(\mathbb{P}^1 \times \mathbb{P}^1\) gives the value of \(p_a(Y)\) and hence of \(p_a(X)\).

(a) Here we check that \(X\) is very ample. Assume that \(X\) is not very ample. By [6], Theorem 3.6, \(X\) is either honestly hyperelliptic or \(X\) is not numerically 3-connected.

(a1) Here show that \(X\) is 3-connected. Let \(A\) be a proper subcurve of \(X\). Set \(B := \overline{X \setminus A}, A' := u(A)\) and \(B' := u(B)\). In order to obtain a contradiction we assume length\((A \cap B) \leq 2\). Let \((a, b)\) the bidegree of \(A'\). Hence \(B'\) has bidegree \((p - a, q - b)\). Without losing generality we may assume \(0 \leq a \leq p/2\). Assume length\((A \cap B) \leq 2\). Since each point of \(S\) is an ordinary node of \(Y\), the scheme \(A' \cap B'\) is the disjoint union of a scheme isomorphic to \(A \cap B\) and a subset \(S'\) of \(S\). Set \(z := \sharp(S')\). We get

\[
a(q - b) + b(p - a) \leq z + 2 \leq \delta + 2 \leq p. \tag{2}
\]

Since \(a \leq p - 2\), (2) implies \(b \leq 1\) (the case \((a, b, z, \delta) = (p/2, 2, p - 2, p - 2)\) is excluded, because \(q \geq p \geq 3\)). First assume \(b = 0\). Since \((a, b) \neq (0, 0)\) and \(q \geq p\), we get \(a = 1\) and \(z = \delta = p - 2\). However, the generality of \(S\) implies that each curve of bidegree \((1, 0)\) contains at most one point of \(S\). Hence \(z = \delta = 1\) and \(q = 3\). Hence \(p = 3\). Thus we are in the excluded case \((p, q, \delta) = (3, 3, 1)\). Now assume \(b = 1\). Since \(q \geq b\), (2) gives \(a = 0\) and \(z = \delta = p - 2\). However, the generality of \(S\) implies that every curve of bidegree \((0, 1)\) contains at most one point of \(S\). Thus \(z = \delta = 1\) and \(p = 3\). Hence we are in the excluded case \((p, \delta) = 1\) with \(Y\) having a component of type \((0, 1)\).

(a2) Here we assume that \(X\) is honestly hyperelliptic, i.e. assume the existence of a finite and flat morphism \(f : X \to \mathbb{P}^1\) such that \(\deg(f) = 2\). Set \(R := f^*(\mathcal{O}_{\mathbb{P}^1}(1))\). \(R\) is an ample and spanned line bundle and \(\deg(R) = 2\). The existence of \(f\) shows that either \(X\) is irreducible or it has two irreducible components, each of them isomorphic to \(\mathbb{P}^1\). First assume that \(X\) is reducible and call \(U, V\) the irreducible components of \(X\). Set \(U' := u(U)\) and \(V' := u(V)\). Let \((c, d)\) be the bidegree of \(U'\). Hence \(V'\) has bidegree \((p - c, q - d)\). Without losing generality we may assume \(0 \leq c \leq p/2\). We have \(p_a(U') = cd - c - d + 1\) and \(p_a(V') = (p - c)(q - d) - p + c - q + d + 1\). Since \(U \cong V \cong \mathbb{P}^1\) and each point of \(S\) is an ordinary node of \(Y\), we get \(cd - c - d + 1 + (p - c)(q - d) - p + c - q + d + 1 \leq \delta\), i.e.

\[
pq + 2cd - cq - dp + 2 \leq \delta \leq p - 2. \tag{3}
\]
If $c = 0$, then $d = 1$, because $U$ is irreducible. In this case (3) fails. Now assume $c = 1$. In this case (3) gives $pq + 2d - q - dp \leq p - 4$. To get a contradiction we see that the last inequality fails if either $d \geq q - 2$ (just because $q \geq p \geq 3$) or $d \leq q - 3$ and $2d \geq q$ (because $3p > p - 4$) or $d \leq q - 3$ and $2d < q$ (because $p \geq 3$). Now assume $d = 0$. Since $V$ is irreducible, we get $c = 0$. Hence in this case (3) fails. Now assume $d = 1$. In this case (3) gives $pq + 2c - eq - p + 2 \leq p - 2$. To get a contradiction we see that the last inequality fails if either $c \geq p - 2$ (obvious) or $c \leq p - 3$ (because $3q \geq 3p > p - 4$). If $d = q$, then $c = 1$, because $V$ is connected; we excluded this case. Now assume $d = q - 1$. In this case (3) gives $p + (2q - 2)c \leq 2c + p - 4$, which is obviously false. Hence from now on we may assume $2 \leq c \leq p/2$ and $2 \leq d \leq q - 2$. Since $p_{\delta}(U) = p_{\delta}(V) = 0$ and $Y$ is nodal at each point of $S$, we also get $h^i(U \cap V) = cd - c - d + 1$ and $h^i(U \cap V) = (p - c)(q - d) - p - q + c + d + 1$. The contradiction comes from Lemma 2 applied to the integers $(u, v) = (c, d)$ and $(u, v) = (p - c, p - c)$, even if $p = q = 4$ and $c = d = 2$.

Now assume that $X$ is irreducible. If $\delta = 0$, then $Y = X$ and hence $R$ is a degree 2 spanned line bundle on the irreducible curve $Y$ of bidegree $(p, q)$, contradicting [10] and the assumption $q \geq p \geq 3$. Hence we may assume $\delta > 0$. Since $f$ is finite, $h^i(Y, f_*(R)) = h^i(X, R)$ for all $i$. Since $p_{\delta}(Y) - p_{\delta}(X) = \delta$, Riemann-Roch applied to $R$ on $X$ and to $f_*(R)$ on $Y$ shows that the rank 1 torsion free sheaf $f_*(R)$ has degree $2 + \delta$. By [7] there is a smooth curve $C$ of bidegree $(p, q)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and a degree 2 + $\delta$ line bundle $L$ on $C$ such that $h^0(C, L) \geq h^0(Y, f_*(R))$. Since $h^0(Y, f_*(R)) = h^0(X, R) \geq 2$, the smooth case of [10] gives $\delta = p - 2$. Let $C''$ be any smooth curve of bidegree $(p, q)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Every $g^1_p$ on $C''$ is induced by a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence $C''$ has exactly one (resp. two) $g^1_p$ if and only if $q > p$ (resp. $q = p$). Let $\{C_t\}_{t \in \Delta}$ be a smoothing of $Y$ with $\Delta$ a connected affine curve, $Y = C_o$ for some $o \in \Delta$, $C = C_o$ for some $t_0 \in \Delta \setminus \{o\}$, and $C_t$ smooth for all $t \in \Delta \setminus \{o\}$. A ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ induces a flat family of spanned and locally free $g^1_p$ and $g^1_p$ on the family $\{C_t\}_{t \in \Delta}$. The quoted uniqueness part for $C_t$, $t \neq o$, implies that $f_*(R)$ is one of the limits over $C_o$ of the restriction to $\Delta \setminus \{o\}$ of this relative $g^1_p$. Since the relative compactified Jacobian for integral curves contained in a smooth surface is relatively projective (and hence relatively separated) (see [1], Theorem 9; one can also quote [2] Corollary 6.7 (i), or [2], Theorem 8.5), we get that the sheaf $f_*(R)$ is locally free, i.e. $\delta = 0$, contradiction.

(b) The proof of [11], Proposition 6, shows that the canonical model of our curve $X$ has the same betti numbers as the curve $C$ of [11], Proposition 6 (the case in which $X$ is smooth).
Remark 1. The same proof may be adapted for the linear systems on the Hirzebruch surface $F_n$ used in [4], Corollary 5.

Lemma 3. Fix integers $u, v, z, w$ such that $v \geq u \geq 4$, $0 \leq z \leq u$ and $0 \leq w \leq uv - u - v + 1 - z$. Let $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a general subset such that $\sharp(S) = z$. There is a nodal and integral $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ with bidegree $(u, v)$, $S \subset \text{Sing}(Y)$ and $w$ further ordinary nodes as its only singularities.

Proof. Since $z \leq p \leq q$, we may find a union $T$ of $p$ distinct curves of bidegree $(1, 0)$ and $q$ distinct curves of bidegree $(0, 1)$ such that each point of $S$ is contained in one of these components of bidegree $(1, 0)$ and in a component of bidegree $(0, 1)$, i.e. such that $S \subset \text{Sing}(T)$. Since $\omega_{\mathbb{P}^1 \times \mathbb{P}^1}^* \omega_{\mathbb{P}^1 \times \mathbb{P}^1}^*$ is ample, we may smooth $pq - z - w$ of the points of $\text{Sing}(T) \setminus S$ appyling [12], Lemma 2.2 and Corollary 2.14.

Lemma 4. Fix integers $q \geq p \geq 3$, $0 \leq z \leq p - 2$, $s > 0$ and $a_i \geq 0$, $1 \leq i \leq s$, such that $(a_i, b_i) \neq (0, 0)$ for all $i$, $\sum_{i=1}^s a_i = p$ and $\sum_{i=1}^s b_i = q$. Set $\epsilon_i := 0$ if $(a_i, b_i) \neq (4, 4)$ and $\epsilon_i := 1$ if $(a_i, b_i) = (4, 4)$. Fix integers $\alpha_i \geq 0$, $\eta_i \geq 0$, $1 \leq i \leq s$, such that $\alpha_i = 0$ if either $a_i \leq 1$ or $b_i \leq 1$, $3a_i + \max\{\eta_i, \epsilon_i\} \leq (a_i + 1)(b_i + 1) - 1$ and $2(\sum_{i=1}^s a_i) + \eta_i = 2z$. Fix a general $S \subset A := \mathbb{P}^1 \times \mathbb{P}^1$ such that $\sharp(S) = z$. There is a nodal $Y \subset A$ such that $Y$ has $s$ irreducible components $Y_1, \ldots, Y_s$, each $Y_i$ has bidegree $(a_i, b_i)$, $S \subset \text{Sing}(Y)$ and each $Y_i$ contains exactly $\alpha_i$ singular points of $Y$ lying in a unique irreducible component of $Y$ and $\eta_i$ singular points of $Y$ lying in two irreducible components of $Y$. Moreover, each finite set $S \cap Y_i \cap Y_j$, $1 \leq i < j \leq s$, may be an arbitrary subset $S_{i,j}$ of $S$, with the only restriction that with these choices $\sum_{j=i+1}^s \sharp(S_{i,j}) + \sum_{j=1}^{i-1} \sharp(S_{i,j}) = \eta_i$.

Proof. Apply Lemma 5 $s$ times.

The following lemma says that the bound for the vanishing of $K_{x,1}(X, \omega_X) = 0$, i.e. that $K_{P^1 \times \mathbb{P}^1}(X, \omega_X) \neq 0$. Indeed, it allows to apply the proof of [9]; alternatively, if $q > p$ we may use [11], part (c) of Proposition 5.

Lemma 5. Take $X$ as in in the proof of Theorem 1 and let $R \in \text{Pic}^0(X)$ be the spanned line bundle induced by a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mu_R : H^0(X, R) \otimes H^0(X, \omega_X \otimes R^*)$ is separately injective, i.e. $\mu_R(\alpha \otimes \beta) \neq 0$ for all $\alpha \in H^0(X, R) \setminus \{0\}$ and all $\beta \in H^0(X, \omega_X \otimes R^*) \setminus \{0\}$.

Proof. Set $S := \text{Sing}(Y)$. Thus $0 \leq \delta := \sharp(S) \leq p - 2$ and $S$ is general in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}) = 0$, adjunction theory gives $H^0(X, \omega_X \otimes R^*) \cong H^0(\mathbb{P}^1 \times \mathbb{P}^1, I_S(p-2, q-2))$. Since the restriction map $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0))$ is an isomorphism, it is sufficient to use the separate injectivity of the multipli-
cation map for any two line bundles on \( \mathbb{P}^1 \times \mathbb{P}^1 \), which is true because this is true on an arbitrary integral projective variety.

The proof of [3], Lemma 4.1 (i.e. the case of a smooth curve) gives the following result.

**Lemma 6.** Let \( X \) be an integral projective curve. Fix an integer \( k > 0 \). Fix \( R \in \text{Pic}(X) \) such that \( h^1(X, R) = 0 \) and assume \( K^0_{h^0(X, R) - k, 1}(X, R) = 0 \). Let \( E \subset X_{\text{reg}} \) be a zero-dimensional scheme. Then \( K^0_{h^0(X, R(E)) - k, 1}(X, R(E)) = 0 \) and \( h^1(X, R(E)) = 0 \).

**Lemma 7.** Let \( X \) be an integral projective curve. Set \( g := p_a(X) \). Fix an integer \( k > 0 \). Assume the existence of \( R \in \text{Pic}(X) \) such that \( h^1(X, L) = 0 \) and \( K^0_{h^0(X, L) - k, 1}(X, L) = 0 \). Fix an integer \( d \geq \text{deg}(L) + 2g \). Then \( K^0_{h^0(X, L) - k, 1}(X, L) = 0 \) for every \( L \in \text{Pic}^d(X) \).

**Proof.** Fix \( L \in \text{Pic}^d(X) \). Since \( d - \text{deg}(R) \geq 2g \), the line bundle \( L \otimes R^* \) is spanned. Hence the zero-locus of a general section of \( L \otimes R^* \) is an effective divisor \( E \subset X_{\text{reg}} \). Apply Lemma 6.

**Proof of Proposition 1.** Set \( q := p_a(X) \) and \( z := \). Take \( Y \) as in the statement of Lemma 2 and let \( X \) be the partial normalization of \( Y \) in which we normalize only the points of \( S \). Part (ii) follows from the proof of the case \( e = 0 \) of [4]. Lemma 7 shows that part (ii) implies part (iii) and hence that \( X \) satisfies the Green-Lazarsfeld conjecture.

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**References**


