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SCALE INVARIANT THEORY ON AXIALLY SYMMETRIC SPACE-TIME WITH PERFECT FLUID

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Abstract: In this paper, we have studied the perfect fluid distribution in the scale invariant theory of gravitation, when the space time is described by axially symmetric metric with a time dependent gauge function. The cosmological equations for this space-time with gauge function are solved. Some physical properties of the model are discussed.

AMS Subject Classification: 83F05
Key Words: axially symmetric, space-time, perfect fluid

1. Introduction

Einstein's general theory of gravitation has been successful in describing gravitational phenomena. His theory has been served as a basis for different models of the universe. The homogeneous isotropic expanding model based on general relativity appears to provide a grand approximation to the observed large scale properties of the universe. However, since Einstein first published his theory of gravitation several modifications have been proposed from time to time which seek to incorporate certain desirable features lacking in the original theory. One of the modifications he himself pointed out that Mach's principle is not substantiated by general relativity. So, there have been considerable attempts made to generalize the general theory of relativity by incorporating Mach's principle

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and other desired features which are lacking in the original theory.

Concerning elementary particles, there are three main scale invariant theories of gravity, proposed by Dirac [6,7], Hoyle and Narlikar [8] and Canuto et al. [4,5]. Wesson [18] formulated a scale invariant theory of gravitation, which also admits a variable G as a viable alternative to scale covariant theory of gravitation (Canuto et al. [5]). In the scale invariant theory of gravitation, Einstein equations have been written in a scale-independent way by performing the conformal or scale transformation as

$$\bar{g_{ij}} = \beta^2(x^k)g_{ij} \tag{1}$$

where the gauge function $\beta(0 < \beta < 1)$,in its most general formulation, is a function of all space-time coordinates. Thus, using the conformal transformation of the type given by equation (1), Wesson [18] transforms the usual Einstein field equations into

$$G_{ij} + 2\frac{\beta_{;ij}}{\beta} - 4\frac{\beta_{,i}\beta_{,j}}{\beta^2} + (g^{ab}\frac{\beta_{,a}\beta_{,b}}{\beta^2} - 2g^{ab}\frac{\beta_{;ij}}{\beta})g_{ij} + \Lambda_0\beta^2 g_{ij} = -\kappa T_{ij} \tag{2}$$

where

$$G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij} \tag{3}$$

Here, G_{ij} is the conventional Einstein tensor involving g_{ij}. Semicolon and comma respectively denote covariant differentiation with respect to g_{ij} and partial differentiation with respect to coordinates. R_{ij} is the Ricci tensor, and R is the Ricci scalar. The cosmological term Λg_{ij} of Einstein theory is now transformed to $\Lambda_0\beta^2 g_{ij}$ in scale invariant theory with dimensionless cosmological constant Λ_0. G and κ are the Newtonian and Wesson's gravitational parameters respectively. T_{ij} is the energy momentum tensor of the matter field and $\kappa = 8\pi G/c^4$. A particular feature of this theory is that no independent equation for β exists.

In this theory Mohanty and Daud [15] have studied the cosmological model governed by vacuum field equations when the space-time described by homogeneous and anisotropic Bianchi type I metric with a gauge function. In that paper, they have shown that the model reduce to Kasner model [9] when cosmological constant is zero, but for non-zero cosmological constant, the model isotropize as in Einstein theory. Moreover, Mohanty and Mishra [16] have studied the feasibility of Bianchi type VIII and IX space-times combinely in this theory and constructed the radiating model of the universe for the feasible Bianchi type VIII space-time.

Beesham [1,2,3], Reddy and Venkateswarlu [17], Mishra [10,11,12], Mishra et. al. [13,14] have investigated several aspects of this theory of gravitation. However, axially symmetric space-time has not been considered, so far, in scale invariant theory of gravitation. Hence, in this paper, we consider the axially symmetric space-time in the scale invariant theory of gravitation. In Section 2, the metric and energy momentum tensor are set up. In Section 3, the field equations are set up and explicit solution to the field equations are obtained. In Section 4, some properties of the model are discussed. Concluding remarks are given in Section 5.

2. Metric and Energy Momentum Tensor

We consider the axially symmetric metric with a gauge function $\beta = \beta(ct)$ of the form

$$ds_W^2 = \beta^2 ds_E^2 \tag{4}$$

with

$$ds_E^2 = c^2dt^2 - A^2(dx^2 + f^2(x)d\varphi^2) - B^2dz^2 \tag{5}$$

Where x is the radial coordinate, z is the vertical coordinate of the cylindrical coordinate system, φ is the azimuthal angle. The metric potentials A and B are functions of t only and f is a function of x only, c is the velocity of light. ds_W^2 and ds_E^2 respectively represent the intervals in Wesson and Einstein theory. Further, $x^i, i = 1, 2, 3, 4$ respectively denote for x, φ,z and t only.

The energy momentum tensor for perfect fluid distribution is given by

$$T_{ij}^m = (p_m + \rho_m c^2)U_iU_j - p_m g_{ij} \tag{6}$$

together with

$$g_{ij}U^iU^j = 1 \tag{7}$$

where U^i is the four velocity vector of the fluid, p_m and ρ_m are respectively the proper isotropic pressure and energy density of the matter.

3. Field Equations and its Solutions

The non-vanishing components of conventional Einstein's tensor (3) for the metric (5) are:

$$G_{11} = f^2G_{22} = \frac{A^2}{c^2}\left(\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB}\right) \tag{8}$$

$$G_{33} = \frac{B^2}{c^2}\left(\frac{A_4^2}{A^2} + 2\frac{A_{44}}{A} - \frac{c^2 f''}{A^2 f}\right) \tag{9}$$

$$G_{44} = -\left(\frac{A_4^2}{A^2} + 2\frac{A_4 B_4}{AB} - \frac{c^2 f''}{A^2 f}\right) \tag{10}$$

The suffix 4 after a field variable denotes exact differentiation with respect to time t only and over head dash denotes differentiation with respect to x.

Using the comoving coordinates (0,0,0,c), the non-vanishing components of the field equations (2) can now be written explicitly as

$$G_{11} = f^2 G_{22} = -\kappa p_m A^2 c^2 - A^2\left[2\left(\frac{A_4}{A} + \frac{B_4}{B}\right)\frac{\beta_4}{\beta} + 2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2\right] \tag{11}$$

$$G_{33} = -\kappa p_m B^2 c^2 - B^2\left[4\left(\frac{A_4}{A}\right)\frac{\beta_4}{\beta} + 2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2\right] \tag{12}$$

$$G_{44} = -\kappa \rho_m c^4 + \left[2\left(2\frac{A_4}{A} + \frac{B_4}{B}\right)\frac{\beta_4}{\beta} + 3\frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2\right] \tag{13}$$

Equation (2) and equations (11) - (13) suggest the definition vacuum pressure p_v and vacuum density ρ_v that involve neither the Einstein tensor of conventional theory nor the properties of conventional matter (Wesson, [18]). These two quantities can be obtained as

$$2\left(\frac{A_4}{A} + \frac{B_4}{B}\right)\frac{\beta_4}{\beta} + 2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2 = \kappa p_v c^2 \tag{14}$$

$$4\left(\frac{A_4}{A}\right)\frac{\beta_4}{\beta} + 2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2 = \kappa p_v c^2 \tag{15}$$

$$2\left(2\frac{A_4}{A} + \frac{B_4}{B}\right)\frac{\beta_4}{\beta} + 3\frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2 = -\kappa \rho_v c^4 \tag{16}$$

In this case, when there is no matter and the gauge function β is a constant, one recovers the relation

$$c^2 \rho_v = c^4 \frac{\lambda_{GR}}{8\pi G} = -p_v \quad i.e. \quad c^2 \rho_v + p_v = 0 \tag{17}$$

which is the equation of state for vacuum. Here $\lambda_{GR} = \lambda_0 \beta^2 = constant$, is the cosmological constant in general relativity. Also, p_v being dependent on the constants λ_{GR}, c and G, is uniform in all directions and hence isotropic in nature.

It is evident from the aforesaid equations that p_v being isotropic, is consistent only when

$$A = c_1 B, \quad as \quad \beta_4 \neq 0 \tag{18}$$

where c_1 is an integrating constant.

Making use of the consistency condition (18), the vacuum pressure p_v and vacuum energy density ρ_v can be obtained as

$$p_v = \frac{1}{\kappa c^2}\left[4\left(\frac{A_4}{A}\right)\frac{\beta_4}{\beta} + 2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2\right] \tag{19}$$

$$\rho_v = -\frac{1}{\kappa c^4}\left[\left(6\frac{A_4}{A}\right)\frac{\beta_4}{\beta} + 3\frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2\right] \tag{20}$$

The total pressure p_t and total energy density ρ_t can be obtained as:

$$p_t \equiv p_m + p_v \tag{21}$$

$$\rho_t \equiv \rho_m + \rho_v \tag{22}$$

By using the components of Einstein Tensor (8) - (10) and the results obtained in equations (18) - (20) with the aforesaid of (21) - (22), the field equations (11) - (13) can be written in the following explicit form:

$$2\frac{A_{44}}{A} + \frac{A_4^2}{A^2} = -\kappa p_t c^2 \tag{23}$$

$$2\frac{A_{44}}{A} + \frac{A_4^2}{A^2} - \frac{c^2 f''}{A^2 f} = -\kappa p_t c^2 \tag{24}$$

$$3\frac{A_4^2}{A^2} - \frac{c^2 f''}{A^2 f} = \kappa \rho_t c^4 \tag{25}$$

The functional dependency of the metric together with eqns. (23) and (24) imply $\frac{f''}{f} = 0$, as $c \neq 0$. Hence, $f(x) = c_2 x + c_3$, where c_2 and c_3 are integrating constants. Without loss of generality by choosing the constants suitable, we obtained

$$f(x) = x \tag{26}$$

Hence, eqns. (23)-(25) reduces to

$$2\frac{A_{44}}{A} + \frac{A_4^2}{A^2} = -\kappa p_t c^2 \tag{27}$$

$$3\frac{A_4^2}{A^2} = \kappa \rho_t c^4 \tag{28}$$

Now, we have two field equations with three unknowns viz., p_t, ρ_t and A. For the complete determinacy one extra condition is needed. We therefore, consider the equation of state $p_t = \frac{\rho_t c^2}{3}$, the radiation model of the universe.

From eqns. (27)- (28), we obtained

$$A = c_1 B = (c_4 t + c_5)^{\frac{1}{2}} \tag{29}$$

where c_4 and c_5 are integrating constants.

Without loss of generality, we take $c_1 = 1$ in eqn. (18). Subsequently, we have

$$A = B = (c_4 t + c_5)^{\frac{1}{2}} \tag{30}$$

Now the total pressure p_t and total energy density ρ_t can be calculated as

$$p_t = \frac{\rho_t c^2}{3} = \frac{1}{4\kappa c^2}\left[\frac{c_4^2}{(c_4 t + c_5)^2}\right] \tag{31}$$

Considering Dirac gauge in the form $\beta = \frac{1}{ct}$, the pressure and energy density corresponding to vacuum case can be calculate as

$$p_v = \frac{1}{\kappa c^2}\left[-\frac{2c_4}{t(c_4 t + c_5)} + \frac{3 - \Lambda_0}{t^2}\right] \tag{32}$$

$$\rho_v = \frac{1}{\kappa c^4}\left[-\frac{3c_4}{t(c_4 t + c_5)} + \frac{3 - \Lambda_0}{t^2}\right] \tag{33}$$

The matter pressure p_m and matter energy density ρ_m can be calculated as

$$p_m = \frac{1}{\kappa c^2}\left[\frac{c_4^2}{4(c_4 t + c_5)^2} + \frac{2c_4}{t(c_4 t + c_5)} + \frac{\Lambda_0 - 3}{t^2}\right] \tag{34}$$

$$\rho_m = \frac{1}{\kappa c^4}\left[\frac{3c_4^2}{4(c_4 t + c_5)^2} + \frac{3c_4}{t(c_4 t + c_5)} + \frac{\Lambda_0 - 3}{t^2}\right] \tag{35}$$

So, the axially symmetric model in scale invariant theory is given by equations (30) and (31). The metric in this case is

$$ds_E^2 = c^2 dt^2 - Q(t)[dx^2 + x d\varphi^2 + dz^2], \quad where \;\; Q(t) = (c_4 t + c_5) \tag{36}$$

4. Some Physical Properties of the Model

We investigate some physical properties of the model obtained in the previous section. The scalar expansion of the model $\theta = U^{i}_{;i}$, which gives the rate of expansion or contraction of the model is found to be $\theta(t) = \frac{3c_4}{2c(c_4t+c_5)}$. Consequently, $\theta \to 0$ when $t \to \infty$. Thus the universe is expanding with increase in time and the rate of expansion is slow with increase in time.

It is observed that $\frac{\rho_m}{\theta^2} \to \infty$ when $t \to 0$ and $\frac{\rho_m}{\theta^2} \to constant$ when $t \to \infty$, which confirms the homogeneity nature of the space-time.

It is also found that, $\rho_m \to 0$ as $t \to \infty$ and $\rho_m \to \infty$ when $t \to 0$, which indicats that there is a big bang like singularity at the initial epoch.

The shear scalar σ for the model vanishes which indicats that the shape of the universe remain unchanged during evolution. Moreover, as $\frac{\sigma^2}{\theta^2} = 0$, the space-time is isotropized during evolution in this theory.

The acceleration $\dot{U}^i = 0$ confirms that the matter particles follow geodesic in this theory. The vorticity W of the radiating fluid of the model also vanishes. Thus U^i is hypersurface orthogonal.

Corresponding to the model (36), the value of the Hubble parameter H is found to be $H = \frac{Q_4(t)}{Q(t)} = \left[\frac{c_4}{c_4t+c_5}\right]$, which determines the present rate of expansion.

However, $H \to$ constant as $t \to 0$ and $H \to 0$ as $t \to \infty$, which indicates the rate of expansion is accelerated or decelerated depending on the signature of the parameter.

Also, the deceleration parameter for the model (36) becomes q=0.

5. Conclusions

The significance of the present work deals with the modification of gravitational and geometrical aspects of Einstein's equations. These are 1) scale invariant theory of gravitation which describes the interaction between matter and gravitation in scale free manner; and 2) the gauge transformation, which represents a change of units of measurements and hence gives a general scaling of physical system. The nature of the cosmological model with modified gravity that would reproduce the kinematical history and evolution of perturbation of the universe is investigated.

Here, axially symmetric radiating model is obtained in the presence of perfect fluid distribution in scale invariant theory of gravitation. As far as matter is concerned the model does not admit either big bang or big crunch during evolution till infinite future.

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