International Journal of Differential Equations and Applications
Volume 12 No. 4 2013, 131-137
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v12i4.1176

acadpubl.eu

BLOW UP FOR COUPLED NONLINEAR WAVE EQUATIONS WITH WEAK DAMPING TERMS

Erhan Pişkin
Department of Mathematics
Dicle University
Diyarbakir, TURKEY

Abstract: In this paper, we consider a coupled nonlinear wave equations with weak damping terms. We prove the blow up of solutions with negative initial energy.

AMS Subject Classification: 35B44, 58J45
Key Words: blow up, weak damping term, wave equations

1. Introduction

In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_{tt}+\left|u_t\right|^{p-1}u_t=\operatorname{div}\left(\rho\left(\left|\nabla u\right|^2\right)\nabla u\right)+f_1\left(u,v\right), & (x,t)\in\Omega\times(0,T), \\ v_{tt}+\left|v_t\right|^{q-1}v_t=\operatorname{div}\left(\rho\left(\left|\nabla v\right|^2\right)\nabla v\right)+f_2\left(u,v\right), & (x,t)\in\Omega\times(0,T), \\ u\left(x,t\right)=v\left(x,t\right)=0, & (x,t)\in\partial\Omega\times(0,T), \\ u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right), & x\in\Omega, \\ v\left(x,0\right)=v_0\left(x\right),\ v_t\left(x,0\right)=v_1\left(x\right), & x\in\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n=1,2,3$;

Received: July 12, 2013

$p, q \geq 1$; $f_i : R^2 \longrightarrow R$ are given functions to be specified later. Problems of this type arise in material science and physics.

Throughout this paper, we define ρ by

$$\rho(s) = b_1 + b_2 s^m, \; q \geq 0, \tag{1.2}$$

where b_1, b_2 are nonnegative constants, and $b_1 + b_2 > 0$. Set $P(s) = \int_0^s \rho(\xi)\, d\xi$, $s \geq 0$.

(**A1**). Let $F(u,v) = a|u+v|^{r+1} + 2b|uv|^{\frac{r+1}{2}}$ with $a, b > 0$, $r \geq 3$ if $n = 1, 2$ and $r = 3$ if $n = 3$; $f_1(u,v) = \frac{\partial F}{\partial u}$, $f_2(u,v) = \frac{\partial F}{\partial v}$; $p, q \geq 1$ if $n = 1, 2$ and $1 \leq p, q \leq 5$ if $n = 3$.

One can easily verify that

$$u\ f_1(u,v) + v f_2(u,v) = (r+1) F(u,v), \; \forall (u,v) \in R^2. \tag{1.3}$$

Lemma 1. (see [2]) *There exist two positive constants c_0 and c_1 such that*

$$c_0 \left(|u|^{r+1} + |v|^{r+1} \right) \leq F(u,v) \leq c_1 \left(|u|^{r+1} + |v|^{r+1} \right)$$

is satisfied.

Wu et al. [6] obtained the global existence and blow up of the solution of problem (1.1) under some suitable conditions. Fei and Hongjun [1] considered problem (1.1) and improved the blow up result obtained in [6], for a large class of initial data in positive initial energy, using the some techniques as in Payne and Sattinger [4] and some estimates used firstly by Vitillaro [5]. Recently, Pişkin and Polat [3] studied the local and global existence, energy decay and blow of the solution of problem (1.1). Also, for more information about (1.1), see references [1, 3].

In this paper, under some restrictions on the initial data, we establish blow up of solutions with negative initial energy, using the same techniques as in [7].

Throughout this paper, $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

This paper is organized as follows. In Section 2, we state the local existence result. In Section 3, we show the blow up properties of solution in the case $p = q = 1$.

2. Local Existence

In this section, we state the local existence and uniqueness of the solution of problem (1.1).

Definition 1. A pair of functions (u,v) is said to be a weak solution of (1.1) on $[0,T]$ if $u,v \in C\left([0,T];W_0^{1,2(m+1)}(\Omega)\cap L^{r+1}(\Omega)\right)$, $u_t \in C\left([0,T];L^2(\Omega)\right)\cap\ L^{p+1}(\Omega\times(0,T))$ and $v_t \in C\left([0,T];L^2(\Omega)\right)\cap L^{q+1}(\Omega\times (0,T))$. In additon, (u,v) satisfies

$$\begin{aligned} &\int_\Omega u'(t)\,\phi dx - \int_\Omega u_1(t)\,\phi dx + \int_\Omega \left(\rho\left(|\nabla u|^2\right)\nabla u\right)\nabla\phi dx \\ &\qquad + \int_0^t\int_\Omega |u'|^{p-1}u'\phi dx d\tau \\ &\qquad = \int_0^t\int_\Omega f_1(u(\tau),v(\tau))\,\phi dx d\tau, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\int_\Omega v'(t)\,\varphi dx - \int_\Omega v_1(t)\,\varphi dx + \int_\Omega \left(\rho\left(|\nabla v|^2\right)\nabla v\right)\nabla\varphi dx \\ &\qquad + \int_0^t\int_\Omega |v'|^{q-1}v'\varphi dx d\tau \\ &\qquad = \int_0^t\int_\Omega f_2(u(\tau),v(\tau))\,\varphi dx d\tau, \end{aligned} \tag{2.2}$$

for all test functions $\phi \in W_0^{1,2(m+1)}(\Omega)\cap L^{p+1}(\Omega)$, $\varphi \in W_0^{1,2(m+1)}(\Omega)\cap L^{q+1}(\Omega)$ and for almost all $t\in[0,T]$.

Now, we state the local existence theorem (see [3] for the proof).

Theorem 2. *(Local existence). Assume (A1) holds. Then, for any initial data $u_0, v_0 \in W_0^{1,2(m+1)}(\Omega)\cap L^{r+1}(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$, there exists a unique local weak solution (u,v) of problem (1.1) (in the sense of Definition 2) defined in $[0,T]$ for some $T>0$, and satisfies the energy identity*

$$E(t) + \int_0^t \left(\|u_\tau(\tau)\|_{p+1}^{p+1} + \|v_\tau(\tau)\|_{q+1}^{q+1}\right)d\tau = E(0), \tag{2.3}$$

where

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\int_\Omega \left(P\left(|\nabla u|^2\right) + P\left(|\nabla v|^2\right)\right) dx - \int_\Omega F(u,v)\, dx. \quad (2.4)$$

3. Blow Up of Solutions

In this section, we are going to consider the blow up of the solution for problem (1.1), when $p = q = 1$. For this purpose, we give the following lemma.

Lemma 2. (see [7]) *Suppose that $\psi(t)$ is a twice continuously differentiable function satisfying*

$$\begin{cases} \psi''(t) + \psi'(t) \geq C_0 \psi^{1+\alpha}(t), & t > 0, \\ \psi(0) > 0, \quad \psi'(0) \geq 0, \end{cases}$$

where $C_0 > 0$, $\alpha > 0$ are constants. Then, $\psi(t)$ blows up in finite time.

Theorem 3. *Let the assumptions of Theorem 3 hold. Assume further that $p = q = 1$. If initial data satisfies*

$$E(0) \leq 0, \quad \int_\Omega (u_0 u_1 + v_0 v_1)\, dx \geq 0,$$

then the corresponding solution blows up in finite time. In other words, there exists a positive constant T^* such that $\lim_{t \longrightarrow T^*} \left(\|u\|^2 + \|v\|^2\right) = \infty$.

Proof. To apply Lemma 4, we define

$$\psi(t) = \frac{1}{2}\int_\Omega \left(|u|^2 + |v|^2\right) dx. \quad (3.1)$$

Therefore

$$\psi'(t) = \int_\Omega (u u_t + v v_t)\, dx, \quad (3.2)$$

and

$$\psi''(t) = \int_\Omega \left(u_t^2 + v_t^2\right) dx + \int_\Omega (u u_{tt} + v v_{tt})\, dx. \quad (3.3)$$

Then, eq (1.1) is used to estimate (3.3) as follows

$$\begin{aligned}\psi''(t) &= \int_\Omega \left(u_t^2+v_t^2\right)dx - \int_\Omega \left(\rho\left(|\nabla u|^2\right)|\nabla u|^2 + \rho\left(|\nabla v|^2\right)|\nabla v|^2\right)dx \\ &\quad - \int_\Omega (uu_t+vv_t)\,dx + (r+1)\int_\Omega F(u,v)\,dx \\ &= \int_\Omega \left(u_t^2+v_t^2\right)dx - b_1\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &\quad - b_2\left(\|\nabla u\|_{2(m+1)}^{2(m+1)}+\|\nabla v\|_{2(m+1)}^{2(m+1)}\right) \\ &\quad - \int_\Omega (uu_t+vv_t)\,dx + (r+1)\int_\Omega F(u,v)\,dx. \qquad (3.4)\end{aligned}$$

Now, we exploit (3.4) to substitute for $\|\nabla u\|_{2(m+1)}^{2(m+1)}+\|\nabla v\|_{2(m+1)}^{2(m+1)}$; thus (3.4) takes the form

$$\begin{aligned}\psi''(t)+\psi'(t) &= (m+2)\left(\|u_t\|^2+\|v_t\|^2\right)+b_1 m\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &\quad -2(m+1)E(t)+(r-2m-1)\int_\Omega F(u,v)\,dx \\ &\geq (m+2)\left(\|u_t\|^2+\|v_t\|^2\right)+b_1 m\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &\quad -2(m+1)E(t)+c_0(r-2m-1)\left(\|u\|_{r+1}^{r+1}+\|v\|_{r+1}^{r+1}\right) \\ &\geq \gamma\left(\|u\|_{r+1}^{r+1}+\|v\|_{r+1}^{r+1}\right), \qquad (3.5)\end{aligned}$$

where $\gamma = c_0(r-2m-1)$.

Now, Hölder inequality are used to estimates $\|u\|_{r+1}^{r+1}$ and $\|v\|_{r+1}^{r+1}$ as follows

$$\int_\Omega |u|^2\,dx \leq \left(\int_\Omega |u|^{r+1}\,dx\right)^{\frac{2}{r+1}}\left(\int_\Omega 1dx\right)^{\frac{r-1}{r+1}}.$$

W_n is called the volume of the domain Ω, then

$$\|u\|_{r+1}^{r+1} \geq \left(\int_\Omega |u|^2\,dx\right)^{\frac{r+1}{2}}(W_n)^{-\left(\frac{r-1}{2}\right)}, \qquad (3.6)$$

and similarly

$$\|v\|_{r+1}^{r+1} \geq \left(\int_\Omega |v|^2\,dx\right)^{\frac{r+1}{2}}(W_n)^{-\left(\frac{r-1}{2}\right)}. \qquad (3.7)$$

Consequently, we have

$$\psi''(t)+\psi'(t)\geq\gamma(W_n)^{-\left(\frac{r-1}{2}\right)}\left[\left(\int_\Omega|u|^2\,dx\right)^{\frac{r+1}{2}}+\left(\int_\Omega|v|^2\,dx\right)^{\frac{r+1}{2}}\right]. \quad (3.8)$$

In order to estimate the right-hand side in (3.8), we make use of the following inequality

$$(X+Y)^\rho\leq 2^{\rho-1}(X^\rho+Y^\rho),$$

$X,Y\geq 0$, $1\leq\rho<\infty$, applying the above inequality we have

$$2^{-\left(\frac{r-1}{2}\right)}\left(\int_\Omega|u|^2\,dx+\int_\Omega|v|^2\,dx\right)^{\frac{r+1}{2}}\leq\left(\int_\Omega|u|^2\,dx\right)^{\frac{r+1}{2}}+\left(\int_\Omega|v|^2\,dx\right)^{\frac{r+1}{2}}.$$

Consequently, (3.8) becomes

$$\begin{aligned}\psi''(t)+\psi'(t)&\geq 2^{-\left(\frac{r-1}{2}\right)}\gamma(W_n)^{-\left(\frac{r-1}{2}\right)}\left(\int_\Omega|u|^2\,dx+\int_\Omega|v|^2\,dx\right)^{\frac{r+1}{2}}\\&=2\gamma(W_n)^{-\left(\frac{r-1}{2}\right)}\psi^{\frac{r+1}{2}}(t).\end{aligned}$$

It is easy to verify that the requirements of Lemma 4 are satisfied by

$$C_0=2\gamma(W_n)^{-\left(\frac{r-1}{2}\right)}>0 \text{ and } \alpha=\frac{r+1}{2}>0.$$

Therofere $\psi(t)$ blows up in finite. □

References

[1] L. Fei, G. Hongjun, Global nonexistence of positive initial-energy solutions for coupled nonlinear wave equations with damping and source terms, *Abstr. Appl. Anal.*, **2011** (2011), 1-14.

[2] S.A. Messaoudi, B.S. Houari, Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms, *J. Math. Anal. Appl.*, **365** (2010), 277-287.

[3] E. Pişkin, N. Polat, Global existence, decay and blow up solutions for coupled nonlinear wave equations with damping and source terms, *Turk. J. Math.*, **37** (2013), 633-651.

[4] L.E. Payne, D.H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Isr. J. Math.*, **22**, No-s: 3-4 (1975), 273-303.

[5] E. Vitillaro, Global existence theorems for a class of evolution equations with dissipation, *Arch. Rational Mech. Anal.*, **149** (1999), 155-182.

[6] J. Wu, S. Li, S. Chai, Existence and nonexistence of a global solution for coupled nonlinear wave equations with damping and source, *Nonlinear Anal.*, **72** (2010), 3969-3975.

[7] Y. Zhou, Global existence and nonexistence for a nonlinear wave equation with damping and source terms, *Math. Nachr.*, **278**, No. 11 (2005), 1341-1358.