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A NEW THEOREM OF EXISTENCE TO FOURTH-ORDER BOUNDARY VALUE PROBLEM

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Abstract: A new theorem of existence for fourth-order boundary value problem

$$\begin{aligned} \Delta^4 x(t-2) &= a(t)f(x(t)), \quad t \in [2,T], \\ x(0) &= x(T+2) = 0, \\ \Delta^2 x(0) &= \Delta^2 x(T) = 0 \end{aligned}$$

is obtained by using a Fixed Point Theorem.

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1. Introduction

Recently, the existence of solutions for boundary value problems of difference equations has received much attention. For example, see [1, 2, 4, 5, 6] and the references therein.

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In this paper we consider the following fourth-order boundary value problem

$$\begin{aligned} \Delta^4 x(t-2) &= a(t)f(x(t)), \quad t\in[2,T], \\ x(0) &= x(T+2)=0, \\ \Delta^2 x(0) &= \Delta^2 x(T)=0, \end{aligned} \tag{1}$$

where $T>2$ is a fixed positive integer, Δ^m denotes the mth forward difference operator with stepsize 1, and $[a,b]=\{a,a+1,\cdots,b-1,b\}\subset Z$ the set of all integers.

Our purpose here is to give a new criteria of the existence to the boundary value problem (1) and the conditions we need are very easy to prove.

In order to prove our main result, the following Fixed Point Theorem is needed.

Lemma 1. (see [3]) *Let X be a Banach space, and $F: X\to X$ be completely continuous. Assume that $A: X\to X$ is a bounded linear operator such that* 1 *is not a eigenvalue of A and*

$$\lim_{\|x\|\to\infty}\frac{\|F(x)-A(x)\|}{\|x\|}=0.$$

Then F has a fixed point in X.

2. Main Results

For the convenience, we let the Banach space $X=\{x:[0,T+2]\longrightarrow R\}$ be equipped with the norm

$$\|x\|=\max_{t\in[0,T+2]}|x(t)|,$$

$$G_1(t,s)=\frac{1}{T}\begin{cases} (t-1)(T+1-s), 1\le t\le s\le T, \\ (s-1)(T+1-t), 2\le s\le t\le T+1, \end{cases}$$

and

$$G_2(t,s) = \frac{1}{T+2}\begin{cases} t(T+2-s), 0 \le t \le s \le T+1, \\ s(T+2-t), 1 \le s \le t \le T+2. \end{cases}$$

Our main result is the following theorem.

Theorem 1. *Assume that $a : [2,T] \to [0,\infty)$ is not identical zero. Let $f \in C(R,R)$ be such that*

$$\lim_{s\to\infty} \frac{f(s)}{s} = m.$$

If $|m| < d = [\max_{t\in[0,T+2]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)]^{-1}$, then the boundary value problem (1) has a solution x^* and $x^* \neq 0$ when $f(0) \neq 0$.

Proof. Suppose that the operator $F : X \to X$ is defined by

$$(Fx)(t) = \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)f(x(v)), \qquad t \in [0,T+2]\,,$$

then it is noted that F is completely continuous and that solving (1) is equivalent to finding an $x \in X$ such that $x = F(x)$.

We consider the following boundary value problem

$$\begin{aligned} \Delta^4 x(t-2) &= ma(t)x(t), \quad t \in [2,T], \\ x(0) &= x(T+2) = 0, \\ \Delta^2 x(0) &= \Delta^2 x(T) = 0. \end{aligned} \tag{2}$$

Let the operator $A : X \to X$ be defined by

$$(Ax)(t) = m\sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)x(v), \qquad t \in [0,T+2]\,,$$

then it is easy to know that A is completely continuous (so bounded) and linear, and that solving (2) is equivalent to finding an $x \in X$ such that $x = Ax$.

First, we claim that 1 is not a eigenvalue of A.

In fact, if $m = 0$, then it is obvious that the boundary value problem (2) has not a solution which is nontrivial.

Let $m \neq 0$, if the boundary value problem (2) has a nontrivial solution x, then $\|x\| > 0$, and so

$$\begin{aligned}
\|x\| &= \|Ax\| = \max_{t\in[0,T+2]} \left| m \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)x(v) \right| \\
&= |m| \max_{t\in[0,T+2]} \left| \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)x(v) \right| \\
&\leq |m| \max_{t\in[0,T+2]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v) \left| x(v) \right| \\
&\leq |m| \|x\| \max_{t\in[0,T+2]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v) \\
&< d \|x\| \frac{1}{d} = \|x\|,
\end{aligned}$$

which is impossible. Hence, the boundary value problem (2) has not a solution which is nontrivial, i.e., 1 is not a eigenvalue of A.

Next, we will prove that

$$\lim_{\|x\|\to\infty} \frac{\|F(x) - A(x)\|}{\|x\|} = 0.$$

Since $\lim_{s\to\infty} \frac{f(s)}{s} = m$, then for $\forall \varepsilon > 0$, there must exist $M_1 > 0$ such that

$$|f(s) - ms| < \varepsilon |s|, \qquad |s| > M_1. \tag{3}$$

Let $M = \max_{|s|\leq M_1} |f(s)|$. We can choose $L > M_1$ such that

$$\frac{M + |m| M_1}{L} < \varepsilon,$$

then for $\forall x \in X$ and $\|x\| > L$,

(i) If $v \in [2, T]$ and $|x(v)| \leq M_1$, then

$$|f(x(v)) - mx(v)| \leq |f(x(v))| + |m|\,|x(v)| \leq M + |m|\,M_1 < \varepsilon L < \varepsilon \|x\| .$$

(ii) If $v \in [2, T]$ and $|x(v)| > M_1$, then from (3), we have

$$|f(x(v)) - mx(v)| \leq \varepsilon\,|x(v)| \leq \varepsilon \|x\| .$$

Hence,

$$|f(x(v)) - mx(v)| \leq \varepsilon \|x\|, \forall v \in [2, T]. \tag{4}$$

In view of (4), we have

$$\begin{aligned} & \|F(x) - A(x)\| \\ = & \max_{t\in[0,T+2]} \left| \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)[f(x(v)) - mx(v)] \right| \\ \leq & \max_{t\in[0,T+2]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)\,|f(x(v)) - mx(v)| \\ \leq & \varepsilon \|x\| \max_{t\in[0,T+2]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v) \\ = & \frac{\varepsilon}{d} \|x\| , \end{aligned}$$

and so

$$\lim_{\|x\|\to\infty} \frac{\|F(x) - A(x)\|}{\|x\|} = 0.$$

It follows from Lemma 1 that F has a fixed point $x^* \in X$, i.e., x^* is a solution of the boundary value problem (1), and it is obvious that $x^* \neq 0$ when $f(0) \neq 0$. □

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