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NONLINEAR WAVE EQUATIONS WITH ACOUSTIC BOUNDARY CONDITIONS

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Abstract: In this paper we investigate the existence and uniqueness of solution of a initial boundary value problem for a nonlinear wave operator with weak internal damping of the type

$$L(u) = \frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho + \beta \frac{\partial u}{\partial t},\ \rho > 1,\ \beta > 0,$$

with acoustic boundary conditions.

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1. Introduction and Main Result

Motivated by a nonlinear theory of measons field, cf. Schiff [12], Jörgens [4]

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initiated the investigation, from a mathematical point of view, of a nonlinear model for partial differential equation of the type

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + F'\left(|u|^2\right) u = 0, \tag{1.1}$$

for a real function $u = u(x,t)$, $x \in \mathbb{R}^n$ and $t \geq 0$. Imposing restrictions on the function
$F : \mathbb{R} \longrightarrow \mathbb{R}$ and on the initial conditions $u(x,0)$, $\frac{\partial u}{\partial t}(x,0)$, he proved existence and uniqueness for the initial boundary value problem for (1.1).

Motivated by Jörgens [4] and [6], J. L. Lions - W. A. Strauss [10] developed a large program of research on nonlinear evolutions equations applying techniques of nonlinear functional analysis, cf. [8]. In [8] and [10] the authors considered nonlinearities of the type $F(s) = |s|^\rho s$, $\rho > 0$. Strauss [14] considered nonlinearity of the type $F : \mathbb{R} \longrightarrow \mathbb{R}$, continuous and $F(s)\, s \geq 0$, for all $s \in \mathbb{R}$.

In the present paper we investigate the nonlinear wave operator

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho + \beta \frac{\partial u}{\partial t} = 0, \quad \rho > 1,$$

with acoustic boundary conditions as in [3] and [7]. Observe that this nonlinearity is not included in [8], [10], [14] and [11].

Acoustic boundary conditions were introduced for wave propagation by Beale and Rosencrans [1]. The acoustic boundary condition says that each point on the boundary reacts to the excess pressure of the wave like a resistive harmonic oscillator, this is

$$\alpha u' + f\delta'' + g\delta' + h\delta = 0, \tag{1.2}$$

where $\delta(x,t)$ is the normal displacement to the boundary at time t with the boundary point x, α is the fluid density and f, g, h are nonnegative functions on the boundary. Condition (1.2) must be coupled with a impenetrability boundary condition expressed by

$$\frac{\partial u}{\partial \nu} = \delta',$$

where by ν we represent the unit outward normal.

We call attention to the fact that the nonlinearity $|u|^\rho$ brings troubles in the process of calculus of a priori estimate, by energy method, because we get in certain point of our proof a term of the type

$$\int_\Omega |\nabla u|^2\, dx + \frac{2}{\rho+1} \int_\Omega |u|^\rho\, u dx,$$

which one cannot control the sign. In this point of the proof we employ an argument contained in Tartar [15] plus contradiction process.

Let us consider Ω an open, bounded and connect set of $\mathbb{R}^n$ with smooth boundary denoted by Γ. Suppose $\Gamma = \Gamma_0 \cup \Gamma_1$ where Γ_0 is a measurable subset of Γ such that $meas\,(\Gamma_0) > 0$ and $\Gamma_1 = \Gamma - \Gamma_0$. By $Q = \Omega \times (0,T)$, for $T > 0$ a real number, we denote a cylinder of $\mathbb{R}^{n+1}$ with lateral boundary $\Sigma = \Gamma \times (0,T) = \Sigma_0 \cup \Sigma_1$, with $\Sigma_0 = \Gamma_0 \times (0,T)$ and $\Sigma_1 = \Gamma_1 \times (0,T)$.

We shall investigate the existence and uniqueness of solutions to the initial boundary value problem

$$\left| \begin{array}{ll} u'' - \Delta u + |u|^{\rho} + \beta u' = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma_0 \\ \alpha u' + f\delta'' + g\delta' + h\delta = 0 & \text{on } \Sigma_1 \\ \dfrac{\partial u}{\partial \nu} - \delta' = 0 & \text{on } \Sigma_1 \\ u\,(x,0) = u_0\,(x)\,, \quad u'\,(x,0) = u_1\,(x) & \text{in } \Omega \\ \delta\,(x,0) = \delta_0\,(x)\,, \quad \delta'\,(x,0) = \delta_1\,(x) & \text{on } \Gamma, \end{array} \right. \tag{1.3}$$

where the derivatives are in the sense of the theory of distributions, Δ represents the usual Laplace operator in $\mathbb{R}^n$, α and β are positive real constants, $1 < \rho \le \dfrac{n}{n-2}$, for $n \ge 3$ and $\rho > 1$, for $n = 2$. For $\rho = 2$ look [7], see also [3] for acoustic boundary conditions.

In the study of the problem (1.1) the symbols $(\cdot,\cdot)\,,\ (\cdot,\cdot)_{\Gamma}\,,\ |\cdot|^2$ and $|\cdot|_{\Gamma}^2$ denote the inner products and norms of the Hilbert spaces $L^2\,(\Omega)$ and $L^2\,(\Gamma)$, respectively.

We consider the Hilbert space $H\,(\Delta,\Omega) = \left\{u \in H^1\,(\Omega)\,;\ \Delta u \in L^2\,(\Omega)\right\}$ with the norm

$$\|u\|_{H(\Delta,\Omega)} = \left(\|u\|_{H^1(\Omega)}^2 + |\Delta u|^2\right)^{1/2},$$

where $H^1\,(\Omega)$ is the usual real Sobolev space of first order.

By V we denote the functional space defined by

$$V = \left\{v \in H^1\,(\Omega)\,;\ \gamma_0\,(v) = 0 \text{ a.e. on } \Gamma_0\right\},$$

where $\gamma_0 : H^1\,(\Omega) \longrightarrow H^{1/2}\,(\Gamma)$ is the trace map of order zero of v. Observe that in V the norm

$$\|u\|_V = \left(\sum_{i=1}^{n} \int_{\Omega} \left(\frac{\partial u}{\partial x_i}\right)^2 dx\right)^{1/2}$$

and the norm of the real Sobolev Space $H^1(\Omega)$ are equivalents. Thus we consider V with the above gradient norm.

By Sobolev embedding theorem, we have $H^1(\Omega) \hookrightarrow L^q(\Omega)$, with $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$, that is, $q = \frac{2n}{n-2}$ for $n \geq 3$. In the case $n = 2$, $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$. In the proof of our result, we need the embedding of the space $L^{2\rho}(\Omega)$ into $L^{\rho+1}(\Omega)$. Thus, if we fixe $\rho > 0$ such that $1 < \rho \leq \frac{n}{n-2}$ than $2\rho \leq \frac{2n}{n-2} = q$ and, therefore, $L^q(\Omega) \hookrightarrow L^{2\rho}(\Omega)$. Summarizing, for $1 < \rho \leq \frac{n}{n-2}$, we obtain

$$V \hookrightarrow H^1(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^{2\rho}(\Omega) \hookrightarrow L^{\rho+1}(\Omega),$$

continously.

About the real functions f, g and h we consider the following hypotheses

$$\left| \begin{array}{l} f, g, h \text{ are real valued functions of class } C^0 \text{ in } x \in \overline{\Gamma}_1; \\ 0 < f_1 \leq f(x), \quad 0 < g_1 \leq g(x), \quad 0 < h_1 \leq h(x) \text{ for all } x \in \Omega, \end{array} \right. \tag{1.4}$$

where f_1, g_1 and h_1 are constants.

The concept of solution for the mixed problem (1.3) is established in the following definition:

Definition 1.1. A regular global solution for the nonlinear initial boundary value problem (1.3) is a pair of real valued functions $\{u, \delta\}$ defined on $\{\Omega \times [0,T)\} \times \{\Gamma \times [0,T)\}$, with $T > 0$ arbitrary, such that

$$\begin{array}{l} u \in L^\infty(0,T;V), \ u(t) \in H(\Delta, \Omega) \text{ a.e. in } [0,T], \\ u' \in L^\infty(0,T;V), \ u'' \in L^\infty\left(0,T;L^2(\Omega)\right), \\ \delta, \ \delta', \ \delta'' \in L^\infty\left(0,T;L^2(\Gamma)\right), \end{array} \tag{1.5}$$

and

$$u'' - \Delta u + |u|^\rho + \beta u' = 0 \text{ a.e. in } Q. \tag{1.6}$$

Moreover, $\{u, \delta\}$ satisfying the conditions in $(1.3)_{2,3,4}$ and the initial conditions $(1.3)_{5,6}$.

The main result is contained in the following Theorem:

Theorem 1.1. *Given $u_0 \in V \cap H^2(\Omega)$, $u_1 \in V$, δ_0, $\delta_1 \in L^2(\Gamma)$. Set*

$$\gamma = \alpha |u_1|^2 + \alpha \|u_0\|^2 + \frac{2\alpha}{\rho+1} \int_\Omega |u_0|^\rho u_0 dx + \left|\sqrt{f}\delta_1\right|_\Gamma^2 + \left|\sqrt{h}\delta_0\right|_\Gamma^2,$$

and suppose

$$\|u_0\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)} \tag{1.7}$$

and

$$\gamma < \alpha \left(\frac{1}{2C_0^2}\right)^{\frac{\rho+1}{\rho-1}}, \tag{1.8}$$

where C_0 is the constant of the embedding of V into $L^{\rho+1}(\Omega)$, $1 < \rho \leq \dfrac{n}{n-2}$, $n \geq 3$. Then, there exists a unique global solution in the sense of definition 1.1.

In the next two sections we will proof Theorem 1.1.

2. Existence of Solutions

The proof of existence of solutions will be done by the Faedo-Galerkin method. In fact, let $\{w_i\}_{i\in\mathbb{N}}$ and $\{z_j\}_{j\in\mathbb{N}}$ be orthonormal bases of $V \cap H^2(\Omega)$ and $L^2(\Gamma)$ respectively. For each $m \in \mathbb{N}$ we consider

$$u_m(x,t) = \sum_{i=1}^{m} \xi_{i,m}(t)\, w_i(x),\ x \in \Omega \quad \text{and} \quad t \in [0, T_m],$$

$$\delta_m(x,t) = \sum_{j=1}^{m} \eta_{j,m}(t)\, z_j(x),\ x \in \Gamma \quad \text{and} \quad t \in [0, T_m],$$

which are solutions of the approximate problem:

$$\begin{aligned} &(u''_m(t), w) + (\nabla u_m(t), \nabla w) - (\delta'_m(t), \gamma_0(w))_\Gamma + \\ &(|u_m(t)|^\rho, w) + (\beta u'_m(t), w) = 0, \\ &(\alpha\gamma_0(u'_m(t)) + f\delta''_m(t) + g\delta'_m(t) + h\delta_m(t), z)_\Gamma = 0, \end{aligned} \tag{2.1}$$

with initial conditions

$$\begin{aligned} &u_m(x,0) = u_{0m}(x) \longrightarrow u_0 \text{ in } V \cap H^2(\Omega), \\ &u'_m(x,0) = u_{1m}(x) \longrightarrow u_1 \text{ in } V, \\ &\delta'_m(x,0) = \gamma_1(u_{0m}(x)) \longrightarrow \gamma_1(u_0(x)) \text{ in } L^2(\Gamma), \\ &\delta_m(x,0) = \gamma_0(\delta_{0m}(x)) \longrightarrow \gamma_0(\delta_0(x)) \text{ in } L^2(\Gamma), \\ &\delta'_m(x,0) = \gamma_0(\delta_{1m}(x)) \longrightarrow \gamma_0(\delta_1(x)) \text{ in } L^2(\Gamma), \end{aligned} \tag{2.2}$$

for all

$$w \in [w_1, w_2, ..., w_m] = Span\{w_1, w_2, ..., w_m\}$$

and

$$z \in [z_1, z_2, ..., z_m] = Span\{z_1, z_2, ..., z_m\}.$$

The local existence, for some $T_m > 0$, is consequence of results about systems of nonlinear ordinary deifferential equations.

We need estimates permiting to pass the limit in the approximate solutions $u_m(t)$ and $\delta_m(t)$.

Estimate 1. Taking $w = 2u'_m(t)$ in $(2.1)_1$ and $z = 2\delta'_m(t)$ in $(2.1)_2$ we have

$$\begin{aligned}&\frac{d}{dt}\left(\left|u'_m(t)\right|^2 + \|u_m(t)\|^2 + \frac{2}{\rho+1}\int_\Omega |u_m(t)|^\rho u_m(t)\,dx + \right) - \\ &-2\left(\delta'_m(t), \gamma_0(u'_m(t))\right)_\Gamma + 2\beta\left|u'_m(t)\right|^2 = 0\end{aligned} \tag{2.3}$$

$$\frac{d}{dt}\left(\left|\sqrt{f}\delta'_m(t)\right|_\Gamma^2 + \left|\sqrt{h}\delta_m(t)\right|_\Gamma^2\right) + 2\alpha\left(\delta'_m(t), \gamma_0\left(u'_m(t)\right)\right)_\Gamma + 2\left|\sqrt{g}\delta'_m(t)\right|_\Gamma^2 = 0 \tag{2.4}$$

Multiplying (2.3) by α and adding the resulting expression to (2.4), we get

$$\begin{aligned}&\frac{d}{dt}\left(\alpha\left|u'_m(t)\right|^2 + \alpha\|u_m(t)\|^2 + \frac{2\alpha}{\rho+1}\int_\Omega |u_m(t)|^\rho u_m(t)\,dx + \right. \\ &\left. + \left|\sqrt{f}\delta'_m(t)\right|_\Gamma^2 + \left|\sqrt{h}\delta_m(t)\right|_\Gamma^2\right) + 2\alpha\beta\left|u'_m(t)\right|^2 + 2\left|\sqrt{g}\delta'_m(t)\right|_\Gamma^2 = 0\end{aligned} \tag{2.5}$$

From the last equality and the hypothesis (1.4) on g, it follows that

$$\begin{aligned}&\frac{d}{dt}\left(\alpha\left|u'_m(t)\right|^2 + \alpha\|u_m(t)\|^2 + \frac{2\alpha}{\rho+1}\int_\Omega |u_m(t)|^\rho u_m(t)\,dx \right. \\ &\left. + \left|\sqrt{f}\delta'_m(t)\right|_\Gamma^2 + \left|\sqrt{h}\delta_m(t)\right|_\Gamma^2\right) \le 2\alpha\beta\left|u'_m(t)\right|^2 + 2C_1\left|\delta'_m(t)\right|_\Gamma^2.\end{aligned} \tag{2.6}$$

Integrating (2.6) from 0 to $t < T_m$ we find

$$\begin{aligned}
&\alpha \left|u_m'(t)\right|^2 + \alpha \left\|u_m(t)\right\|^2 + \frac{2\alpha}{\rho+1}\int_\Omega \left|u_m(t)\right|^\rho u_m(t)\,dx + \\
&+ \left|\sqrt{f}\delta_m'(t)\right|_\Gamma^2 + \left|\sqrt{h}\delta_m(t)\right|_\Gamma^2 \leq \alpha\left|u_1\right|^2 + \alpha\left\|u_0\right\|^2 + \\
&+ \frac{2\alpha}{\rho+1}\int_\Omega \left|u_0\right|^\rho u_0 dx + \left|\sqrt{f}\delta_1\right|_\Gamma^2 + \left|\sqrt{h}\delta_0\right|_\Gamma^2 + \\
&+ 2\alpha\beta\int_0^t \left|u_m'(s)\right|^2 ds + 2C_1\int_0^t \left|\delta_m'(s)\right|_\Gamma^2 ds.
\end{aligned} \tag{2.7}$$

The main question in this point of the proof is that we don't know the sign of

$$J(u) = \frac{1}{2}\left\|u\right\|^2 + \frac{2}{\rho+1}\int_\Omega |u|^\rho u\ dx,$$

for $u = u_m(t)$ and $u = u_0$ in the inequality (2.7).

This is the main point of the proof. To improve on this difficulty we will do some computation. In fact, we first have:

$$\begin{aligned}
&\left|\int_\Omega \left|u_m(x,t)\right|^\rho u_m(x,t)\,dx\right| \leq \int_\Omega \left|u_m(x,t)\right|^{\rho+1} dx = \left|u_m(t)\right|_{L^{\rho+1}(\Omega)}^{\rho+1} \leq \\
&\leq C_0^{\rho+1}\left\|u_m(t)\right\|^{\rho+1},
\end{aligned} \tag{2.8}$$

by hypothesis on ρ, Sobolev Theorem and $V \hookrightarrow L^{\rho+1}(\Omega)$.

Thus, by (2.8) we have

$$\int_\Omega \left|u_m(x,t)\right|^\rho u_m(x,t)\,dx \geq -C_0^{\rho+1}\left\|u_m(t)\right\|^{\rho+1}.$$

We go back to $J(u)$ and get

$$\frac{1}{2}\left\|u\right\|^2 + \frac{2}{\rho+1}\int_\Omega |u|^\rho u\ dx \geq \frac{1}{2}\left\|u\right\|^2 - \frac{2C_0^{\rho+1}}{\rho+1}\left\|u\right\|^{\rho+1} \tag{2.9}$$

which we employ for $u = u_m(t)$ and $u = u_0$.

Thus, the sign of both sides of (2.7) depends on the sign of the function

$$P(\lambda) = \frac{\lambda^2}{2} - \frac{2C_0^{\rho+1}}{\rho+1}\lambda^{\rho+1},$$

for $\lambda \geq 0$ and $1 < \rho \leq \dfrac{n}{n-2}$, $n \geq 3$.

Remark 2.1. From the definition of $P(\lambda)$ we have that it is increasing in $\left(0, \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}\right)$ and has a maximum value at $\left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}$.

Now let us go back to (2.7). By the hypothesis (1.7) of Theorem 1.1, we have that $P(\|u_0\|) > 0$. Since, by (2.9), $J(u) \geq P(\|u\|)$, we obtain

$$\frac{1}{2}\|u_0\|^2 + \frac{2}{\rho+1}\int_\Omega |u_0|^\rho u_0 \, dx > 0.$$

Thus, the right hand side of (2.7) is positive.

To analyse the left hand side of (2.7) we need of the following Lemma:

Lemma 2.1. *If we have the conditions* (1.8) *and* (1.7) *it implies*

$$\|u_m(t)\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)},$$

for all $t \in [0, T_m)$ *and* $m \in \mathbb{N}$, *for the approximate solution* $u_m(t)$.

Proof. We will employ a contradiction argument. In fact, suppose there exists $m_0 \in \mathbb{N}$ such that

$$\|u_{m_0}(t)\| \geq \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}$$

for some $0 < t < T_m$. By $(2.2)_1$ and the hypothesis (1.7) we have that

$$0 < \|u_{m_0}(0)\| \leq \|u_0\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}. \tag{2.10}$$

From (2.10) and the continuity of $\|u_{m_0}(t)\|$, we conclude that there exists $t_0 > 0$ such that

$$0 < \|u_{m_0}(t)\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)} \quad \text{for all } t \in (0, t_0).$$

Hence the set

$$\left\{ t > 0;\ \|u_{m_0}(t)\| \geq \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)} \right\}$$

is non-empty, closed and bounded below. Thus, there exists a minimum, say t^*. By continuity of $\|u_{m_0}(t)\|$ we have

$$\left|\begin{array}{l} \|u_{m_0}(t)\| < \left(\dfrac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}, \quad 0 \le t < t^* \\[2ex] \|u_{m_0}(t^*)\| = \left(\dfrac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}. \end{array}\right. \tag{2.11}$$

On the other hand, from (2.5) we have

$$\begin{gathered} \frac{d}{dt}\left(\alpha\left|u'_{m_0}(t)\right|^2 + \alpha\|u_{m_0}(t)\|^2 + \frac{2\alpha}{\rho+1}\int_\Omega |u_{m_0}(t)|^\rho u_{m_0}(t)\,dx + \right. \\ \left. + \left|\sqrt{f}\delta'_{m_0}(t)\right|_\Gamma^2 + \left|\sqrt{h}\delta_{m_0}(t)\right|_\Gamma^2\right) \le 0. \end{gathered} \tag{2.12}$$

Integrating (2.12) from 0 to t^* we get

$$\begin{gathered} \alpha\left|u'_{m_0}(t^*)\right|^2 + \alpha\|u_{m_0}(t^*)\|^2 + \frac{2\alpha}{\rho+1}\int_\Omega |u_{m_0}(t^*)|^\rho u_{m_0}(t^*)\,dx + \\ + \left|\sqrt{f}\delta'_{m_0}(t^*)\right|_\Gamma^2 + \left|\sqrt{h}\delta_{m_0}(t^*)\right|_\Gamma^2 \le \alpha|u_1|^2 + \alpha\|u_0\|^2 + \\ + \frac{2\alpha}{\rho+1}\int_\Omega |u_0|^\rho u_0 dx + \left|\sqrt{f}\delta_1\right|_\Gamma^2 + \left|\sqrt{h}\delta_0\right|_\Gamma^2. \end{gathered} \tag{2.13}$$

By (2.11) we conclude

$$\frac{\alpha}{2}\|u_{m_0}(t^*)\|^2 + \frac{2\alpha}{\rho+1}\int_\Omega |u_{m_0}(t^*)|^\rho u_{m_0}(t^*)\,dx > 0$$

and, therefore, from (2.13) follows

$$\frac{\alpha}{2}\|u_{m_0}(t^*)\|^2 \le \gamma < \alpha\left(\frac{1}{2C_0^2}\right)^{\frac{\rho+1}{\rho-1}}, \tag{2.14}$$

in this last inequality we have used the hypothesis (1.8).

From (2.14) we get

$$\|u_{m_0}(t^*)\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)},$$

that is a contradiction with $(2.11)_2$. This proofs the Lemma 2.1. □

Now, by Lemma 2.1, we know that

$$\frac{\alpha}{2}\left\|u_m(t)\right\|^2+\frac{2\alpha}{\rho+1}\int_\Omega\left|u_m(t)\right|^\rho u_m(t)\,dx>0$$

and therefore returning to (2.7) we get

$$\begin{aligned}&\alpha\left|u_m'(t)\right|^2+\frac{\alpha}{2}\left\|u_m(t)\right\|^2+\left|\sqrt{f}\delta_m'(t)\right|_\Gamma^2+\left|\sqrt{h}\delta_m(t)\right|_\Gamma^2\leq\\&\leq\gamma+C_2\int_0^t\left(\left|u_m'(s)\right|^2+\left|\delta_m'(s)\right|_\Gamma^2\right)ds,\end{aligned}\tag{2.15}$$

where $C_2=\max\{2\alpha\beta,2C_1\}$.

From (2.15), Gronwall's inequality gives

$$\left|u_m'(s)\right|^2+\left|\delta_m'(s)\right|_\Gamma^2\leq C_3.$$

This and (2.15) imply that there exists a constant C_4 independent of m and $t\in[0,T_m]$ such that

$$\alpha\left|u_m'(t)\right|^2+\frac{\alpha}{2}\left\|u_m(t)\right\|^2+\left|\sqrt{f}\delta_m'(t)\right|_\Gamma^2+\left|\sqrt{h}\delta_m(t)\right|_\Gamma^2\leq C_4,\tag{2.16}$$

which complete the first estimate.

Estimate 2. Taking $t=0$ in $(2.1)_{1,2}$ we get

$$\begin{aligned}&(u_m''(0),w)+(\nabla u_m(0),\nabla w)-(\delta_m'(0),\gamma_0(w))_\Gamma+\\&(|u_m(0)|^\rho,w)+(\beta u_m'(0),w)=0,\end{aligned}$$

$$(\alpha\gamma_0(u_m'(0))+f\delta_m''(0)+g\delta_m'(0)+h\delta_m(0),z)_\Gamma=0.$$

Putting $w=u_m''(0)$ and $z=\delta_m''(0)$ we obtain

$$\begin{aligned}&|u_m''(0)|^2\leq(|\Delta u_m(0)|+||u_m(0)|^\rho|+\beta|u_m'(0)|)|u_m''(0)|\\&f_1|\delta_m''(0)|_\Gamma^2\leq(\alpha|\gamma_0(u_m'(0))|_\Gamma+|g\gamma_1(u_m(0))|_\Gamma+|h\delta_m(0)|)|\delta_m''(0)|_\Gamma.\end{aligned}\tag{2.17}$$

Now we note that from the continuity of the traces mapping γ_0 and γ_1, the hypoyhesis about g, h and the continous embedding of V in $L^{2\rho}(\Omega)$ we obtain

$$
\begin{aligned}
&|\gamma_0(u'_m(0))|_\Gamma \leq C_5 \|u'_m(0)\| \\
&|g\gamma_1(u_m(0))|_\Gamma \leq \max_{x\in\overline{\Gamma}} |g(x)|_\mathbb{R} |\gamma_1(u_m(0))|_\Gamma \leq C_6 \|u_m(0)\|_{H^2(\Omega)} \\
&|h\delta_m(0)|_\Gamma \leq \max_{x\in\overline{\Gamma}} |h(x)|_\mathbb{R} |\delta_m(0)|_\Gamma = C_7 |\delta_m(0)|_\Gamma \\
&\|u_m(0)\|^\rho_{L^{2\rho}(\Omega)} \leq C_8^\rho \|u_m(0)\|^\rho,
\end{aligned}
\tag{2.18}
$$

returning with (2.18) in (2.17) we get

$$
\begin{aligned}
&|u''_m(0)| \leq |\Delta u_m(0)| + C_8^\rho \|u_m(0)\|^\rho + \beta |u'_m(0)| |u''_m(0)| \leq C_9 \\
&f_1 |\delta''_m(0)|^2_\Gamma \leq \alpha C_5 \|u'_m(0)\| + C_6 \|u_m(0)\|_{H^2(\Omega)} + C_7 |\delta_m(0)|_\Gamma \leq C_{10}.
\end{aligned}
\tag{2.19}
$$

Differentiating $(2.1)_{1,2}$ with respect to t and taking $w = 2u''_m(t)$ and $z = 2\delta''_m(t)$ we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left|u''_m(t)\right|^2 + \frac{d}{dt}\|u_m(t)\|^2 - 2\left(\delta''_m(t), \gamma_0\left(u''_m(t)\right)\right) + 2\beta\left|u''_m(t)\right|^2 \leq \\
&\leq 2\rho \int_\Omega |u_m(x,t)|^{\rho-1} |u'_m(x,t)| |u''_m(x,t)| \, dx
\end{aligned}
\tag{2.20}
$$

and

$$
\begin{aligned}
&-2\alpha\left(\delta''_m(t), \gamma_0\left(u''_m(t)\right)\right) \\
&\qquad = \frac{d}{dt}\left|\sqrt{f}\delta''_m(t)\right|^2_\Gamma + 2\left|\sqrt{g}\delta''_m(t)\right|^2_\Gamma + \frac{d}{dt}\left|\sqrt{h}\delta'_m(t)\right|^2_\Gamma.
\end{aligned}
\tag{2.21}
$$

Multiplying (2.20) by α and using the equality in (2.21) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\alpha\left|u''_m(t)\right|^2 + \alpha\left\|u'_m(t)\right\|^2 + \left|\sqrt{f}\delta''_m(t)\right|^2_\Gamma + \left|\sqrt{h}\delta'_m(t)\right|^2_\Gamma\right) + 2\alpha\beta\left|u''_m(t)\right|^2 + \\
&+2\left|\sqrt{g}\delta''_m(t)\right|^2_\Gamma \leq 2\alpha\rho \int_\Omega |u_m(x,t)|^{\rho-1} |u'_m(x,t)| |u''_m(x,t)| \, dx.
\end{aligned}
\tag{2.22}
$$

Now we recall that $q = \dfrac{2n}{n-2}$, thus $\dfrac{1}{n}+\dfrac{1}{q}+\dfrac{1}{2} = 1$ and by Hölder's inequality we get

$$\int_\Omega |u_m(x,t)|^{\rho-1} \left|u_m'(x,t)\right| \left|u_m''(x,t)\right| dx$$
$$\leq \left(\int_\Omega |u_m(x,t)|^{n(\rho-1)} dx\right)^{1/n} \left(\int_\Omega \left|u_m'(x,t)\right|^q dx\right)^{1/q},$$

$$\left(\int_\Omega \left|u_m''(x,t)\right|^2 dx\right)^{1/2} = \left\||u_m(t)|^{\rho-1}\right\|_{L^n(\Omega)} \left\|u_m'(t)\right\|_{L^q(\Omega)} \left|u_m''(t)\right|. \quad (2.23)$$

By the hypothesis about ρ we have that $n(\rho-1) \leq q$ thus

$$\left\||u_m(t)|^{\rho-1}\right\|_{L^n(\Omega)} = \|u_m(t)\|^{\rho-1}_{L^{n(\rho-1)}} \leq C_{11}^{\rho-1} \|u_m(t)\|^{\rho-1}_{L^q(\Omega)}$$

therefore

$$\left\||u_m(t)|^{\rho-1}\right\|_{L^n(\Omega)} \leq C_{12}^{\rho-1} \|u_m(t)\|^{\rho-1}, \quad (2.24)$$

because $V \hookrightarrow L^q(\Omega)$.

Returning with (2.24) in (2.23) we conclude that

$$\int_\Omega |u_m(x,t)|^{\rho-1} \left|u_m'(x,t)\right| \left|u_m''(x,t)\right| dx$$
$$\leq C_{12}^{\rho-1} \|u_m(t)\|^{\rho-1} \left\|u_m'(t)\right\| \left|u_m''(t)\right|. \quad (2.25)$$

From (2.22), (2.25) and the first estimate we get

$$\frac{d}{dt}\left(\alpha \left|u_m''(t)\right|^2 + \alpha \left\|u_m'(t)\right\|^2 + \left|\sqrt{f}\delta_m''(t)\right|_\Gamma^2 + \left|\sqrt{h}\delta_m'(t)\right|_\Gamma^2\right) \leq$$

$$\leq C_{13}\left(\left\|u_m'(t)\right\|^2 + \left|u_m''(t)\right|^2\right).$$

Integrating over $(0,t)$, using (2.19) and applying Gronwall's inequality we have

$$\alpha \left|u_m''(t)\right|^2 + \alpha \left\|u_m'(t)\right\|^2 + \left|\sqrt{f}\delta_m''(t)\right|_\Gamma^2 + \left|\sqrt{h}\delta_m'(t)\right|_\Gamma^2 \leq C_{14} \quad (2.26)$$

which is the second estimate. □

Limits of approximate solution. From the limitations on (2.16) and (2.26) we obtain that there exists a subsequence, still denoted by $(u_m)_{m\in\mathbb{N}}$, such that

$$\begin{aligned}
&u_m \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;V),\\
&u'_m \overset{*}{\rightharpoonup} u' \text{ in } L^\infty(0,T;V),\\
&u''_m \overset{*}{\rightharpoonup} u'' \text{ in } L^\infty\left(0,T;L^2(\Omega)\right),\\
&\delta_m \overset{*}{\rightharpoonup} \delta \text{ in } L^\infty\left(0,T;L^2(\Gamma)\right),\\
&\delta'_m \overset{*}{\rightharpoonup} \delta' \text{ in } L^\infty\left(0,T;L^2(\Gamma)\right),\\
&\delta''_m \overset{*}{\rightharpoonup} \delta'' \text{ in } L^\infty\left(0,T;L^2(\Gamma)\right).
\end{aligned} \tag{2.27}$$

Now we observe that if we define

$$W(0,T) = \left\{u \in L^2(0,T;V); \quad u' \in L^2\left(0,T;L^2(\Omega)\right)\right\},$$

since $V \overset{c}{\hookrightarrow} L^2(\Omega)$ from the compactness theorem of Lions-Aubin [9], we get

$$W(0,T) \overset{c}{\hookrightarrow} L^2\left(0,T;L^2(\Omega)\right).$$

From the first estimate we have that

$$(u_m)_{m\in\mathbb{N}} \text{ is bounded in } W(0,T)$$

and thus we can extract a subsequence, still denoted by $(u_m)_{m\in\mathbb{N}}$, such that

$$u_m \longrightarrow u \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right),$$

this is

$$|u_m|^\rho \longrightarrow |u|^\rho \text{ a.e. in } Q. \tag{2.28}$$

On the other hand, since $V \hookrightarrow L^{2\rho}(\Omega)$, we have that

$$\begin{aligned}
&||u_m(t)|^\rho|^2_{L^2(\Omega)} = \int_\Omega |u_m(x,t)|^{2\rho}\,dx =\\
&= |u_m(t)|^{2\rho}_{L^{2\rho}(\Omega)} \le C^{2\rho}\,\|u_m(t)\|^{2\rho} \le C_{15}.
\end{aligned} \tag{2.29}$$

From (2.28) and (2.29), thanks to Lions [8] Lemma 1.3, we conclude that

$$|u_m|^\rho \rightharpoonup |u|^\rho \text{ in } L^2\left(0,T;L^2(\Omega)\right). \tag{2.30}$$

Taking into account the convergences in (2.27) and (2.30) we can passing to the limit in the approximate equations and obtain

$$\begin{aligned} &(u''(t), w) + (\nabla u(t), \nabla w) - (\delta'(t), \gamma_0(w))_\Gamma + \\ &+ (|u(t)|^\rho, w) + (\beta u'(t), w) = 0, \\ &(\alpha\gamma_0(u'(t)) + f\delta''(t) + g\delta'(t) + h\delta(t), z)_\Gamma = 0, \end{aligned} \tag{2.31}$$

for all $w \in V$, $z \in L^2(\Gamma)$ a.e. in $[0, T]$.

From $(2.31)_1$ we obtain

$$\begin{aligned} &\int_\Omega u''(x,t)\varphi(x)\,dx - \langle \Delta u(t), \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} + \\ &+ \int_\Omega |u(x,t)|^\rho \varphi(x)\,dx + \beta\int_\Omega u'(x,t)\varphi(x)\,dx = 0, \end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$, a.e. in $[0, T]$.

Therefore $\Delta u(t) \in L^2(\Omega)$ a.e. in $[0, T]$ and

$$u'' - \Delta u + |u|^\rho + \beta u' = 0 \quad \text{a.e. in } Q. \tag{2.32}$$

From $(2.31)_2$ we can see that $\{u, \delta\}$ satisfy the boundary condition $(1.3)_3$. In order to get the sense of equality in $(1.3)_4$, we proceeded as follows. Multiplying (2.32) by $w \in V$ and integrating over Ω we find

$$\big(u''(t), w\big) - (\Delta u(t), w) + (|u(t)|^\rho, w) + \beta\big(u'(t), w\big) = 0.$$

Since $u(t) \in H(\Delta, \Omega)$ a.e. in $[0, T]$, using the generalized Green's formula we have

$$\begin{aligned} &(u''(t), w) + (\nabla u(t), \nabla w) - \langle \gamma_1(u(t)), \gamma_0(w)\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} + \\ &+ (|u(t)|^\rho, w) + \beta(u'(t), w) = 0, \end{aligned} \tag{2.33}$$

where by $\gamma_1 : H(\Delta, \Omega) \longrightarrow H^{-1/2}(\Gamma)$ we denote the trace map of order one, this is,

$$\gamma_1(u) = \left(\frac{\partial u}{\partial \nu}\right)_{|\Gamma}.$$

Comparing $(2.31)_1$ with (2.33) we conclude that

$$\langle \gamma_1(u(t)), \gamma_0(w)\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = \big(\delta'(t), \gamma_0(w)\big)_\Gamma,$$

for all $w \in V$ and a.e. in $[0, T]$, which proves $(1.3)_4$.

The initial conditions can be proved in a standard way and this complete the proof of the existence of solution. □

3. Uniqueness of Solutions

The uniqueness of solution to the problem (1.3) is obtained by energy method as follows.

Let $\{u_1, \delta_1\}$ and $\{u_2, \delta_2\}$ be two solutions of (1.3) in the sense of the definition 1.1. We have that $\psi = u_1 - u_2$ and $\theta = \delta_1 - \delta_2$ satisfy

$$\begin{aligned}
&(\psi''(t), w) + (\nabla\psi(t), \nabla w) - (\theta'(t), \gamma_0(w))_\Gamma + \\
&+ (|u_1(t)|^\rho - |u_2(t)|^\rho, w) + (\beta\psi'(t), w) = 0, \\
&(\alpha\gamma_0(\psi'(t)) + f\theta''(t) + g\theta'(t) + h\theta(t), z)_\Gamma = 0, \\
&\psi(x, 0) = 0; \quad \psi'(x, 0) = 0 \text{ in } \Omega, \\
&\theta(x, 0) = 0; \quad \theta'(x, 0) = 0 \text{ on } \Gamma,
\end{aligned} \tag{3.1}$$

for all $w \in V$, $z \in L^2(\Gamma)$ a.e. in $[0, T]$.

Taking in (3.1) $w = 2\psi'(t)$ and $z = 2\theta'(t)$ we find

$$\begin{aligned}
&\frac{d}{dt}|\psi'(t)|^2 + \frac{d}{dt}\|\psi(t)\|^2 - 2\left(\theta'(t), \gamma_0\left(\psi'(t)\right)\right)_\Gamma + \\
&+2\left(|u_1(t)|^\rho - |u_2(t)|^\rho, \psi'(t)\right) + 2\beta|\psi'(t)|^2 = 0, \\
&2\alpha\left(\gamma_0(\psi'(t)), \theta'(t)\right)_\Gamma + \frac{d}{dt}\left(\left|\sqrt{f}\theta'(t)\right|_\Gamma^2 + \left|\sqrt{h}\theta(t)\right|_\Gamma^2\right) + 2\left|\sqrt{g}\theta'(t)\right|_\Gamma^2 = 0.
\end{aligned} \tag{3.2}$$

Multiplying the equation in $(3.2)_1$ by α and adding the resulting expression to $(3.2)_2$, we get

$$\begin{aligned}
&\alpha\frac{d}{dt}|\psi'(t)|^2 + \alpha\frac{d}{dt}\|\psi(t)\|^2 + \frac{d}{dt}\left(\left|\sqrt{f}\theta'(t)\right|_\Gamma^2 + \left|\sqrt{h}\theta(t)\right|_\Gamma^2\right) + \\
&+2\alpha\left(|u_1(t)|^\rho - |u_2(t)|^\rho, \psi'(t)\right) + 2\alpha\beta|\psi'(t)|^2 + 2\left|\sqrt{g}\theta'(t)\right|_\Gamma^2 = 0.
\end{aligned} \tag{3.3}$$

From (3.3) we obtain, after some calculations, that

$$\begin{aligned}
&\frac{d}{dt}\left(\alpha|\psi'(t)|^2 + \alpha\|\psi(t)\|^2 + \left|\sqrt{f}\theta'(t)\right|_\Gamma^2 + \left|\sqrt{h}\theta(t)\right|_\Gamma^2\right) \leq \\
&\qquad \leq C_{15}|\theta'(t)|_\Gamma^2 + 2\alpha\left(\|u_1(t)\|_{L^{2\rho}(\Omega)}^\rho + \|u_2(t)\|_{L^{2\rho}(\Omega)}^\rho\right)|\psi'(t)|^2
\end{aligned}$$

$$+ 2\alpha\beta \left|\psi'(t)\right|^2, \quad (3.4)$$

where $C_{15} = 2\max\limits_{x\in\overline{\Gamma}} |g(x)|_{\mathbb{R}}$.

Now we observe that, since u_1 and u_2 are solutions of (1.3) and $V \hookrightarrow L^{2\rho}(\Omega)$ then

$$\|u_1(t)\|^{\rho}_{L^{2\rho}(\Omega)} + \|u_2(t)\|^{\rho}_{L^{2\rho}(\Omega)} \leq C_{16}.$$

Applying this to (3.4) we obtain

$$\frac{d}{dt}\left(\alpha\left|\psi'(t)\right|^2 + \alpha\|\psi(t)\|^2 + \left|\sqrt{f}\theta'(t)\right|^2_{\Gamma} + \left|\sqrt{h}\theta(t)\right|^2_{\Gamma}\right) \leq$$
$$\leq C_{17}\left(\left|\theta'(t)\right|^2_{\Gamma} + \left|\psi'(t)\right|^2\right).$$

Integrating the last inequality from 0 to t and using $(3.1)_{3,4}$ we get

$$\alpha\left|\psi'(t)\right|^2 + \alpha\|\psi(t)\|^2 + \left|\sqrt{f}\theta'(t)\right|^2_{\Gamma} + \left|\sqrt{h}\theta(t)\right|^2_{\Gamma}$$
$$\leq C_{17}\int_0^t \left(\left|\theta'(s)\right|^2_{\Gamma} + \left|\psi'(s)\right|^2\right) ds. \quad (3.5)$$

From (3.5) and Gronwall's inequality we conclude the uniqueness of solutions of the system (1.3) and this complte the proof of the Theorem 1.1. □

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