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SOLVING NONLINEAR EQUATIONS USING A NEW TENTH-AND SEVENTH-ORDER METHODS FREE FROM SECOND DERIVATIVE

M.A. Hafiz[1 §], Salwa M.H. Al-Goria[2]
[1,2]Department of Mathematics
Faculty of Science and Arts
Najran University, Najran, 1988, SAUDI ARABIA

Abstract: In this paper, we suggest and analyze some new higher-order iterative methods free from second derivative and used for solving of nonlinear equations. These methods based on a Householder iterative method and using predictor–corrector technique. The convergence analysis of our methods are discussed. It is established that the new methods have convergence order ten and seven. Numerical tests show that the new methods are comparable with the well-known existing methods and gives better results.

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Key Words: nonlinear equations, convergence analysis, higher order, iterative methods, householder iterative method

1. Introduction

Finding iterative methods for solving nonlinear equations is an important area of research in numerical analysis at it has interesting applications in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $f(x) = 0$. Due to their importance, several numerical methods have been suggested and analyzed under certain condition. These numerical methods have been constructed using different techniques such as

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§Correspondence author

Taylor series, homotopy perturbation method and its variant forms, quadrature formula, variational iteration method, and decomposition method. For more details, see [1-10]. In this paper, based on a Householder and using predictor-corrector technique, we construct modification of Newton's method with higher-order convergence for solving nonlinear equations. The error equations are given theoretically to show that the proposed techniques have tenth - and seventh - order convergence. Commonly in the literature the efficiency of an iterative method is measured by the *efficiency index* defined as $I \approx p^{1/d}$ [11], where $p$is the order of convergence and $d$is the total number of functional evaluations per step. Therefore these methods have efficiency index $10^{1/5} \approx 1.585$ and $7^{1/4} \approx 1.626$ which are higher than $2^{1/2} \approx 1.4142$ of the Steffensen's method (SM) [12], $3^{1/4} \approx 1.3161$of the DHM method [13]. Several examples are given to illustrate the efficiency and performance of these methods.

2. Iterative Methods

Consider the nonlinear equation of the type

$$f(x) = 0 \tag{1}$$

For simplicity, we assume that r is a simple root of Eq. (1) and x_0 is an initial guess sufficiently close to r. Using the Taylor's series expansion of the function $f(x)$, we have

$$f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) = 0 \tag{2}$$

First two terms of the equation (2) gives the first approximation, as

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{3}$$

This allows us to suggest the following one-step iterative method for solving the nonlinear equation (1).

Algorithm 2.1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which is the Newton method. It is well known that algorithm 2.1 has a quadratic convergence.

Again from (2) we have

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f''(x_0)\,(x - x_0)^2}{2f'(x_0)} \tag{4}$$

Substitution again from (3) into the right hand side of (4) gives the second approximation

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{[f(x_0)]^2 f''(x_0)}{2[f'(x_0)]^3}$$

This formula allows us to suggest the following iterative methods for solving the nonlinear Eq. (1).

Algorithm 2.2. For a given x_0, compute approximates solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3}$$

this Algorithm has cubic convergence, which is known as the Householder [14] iterative method for solving the nonlinear equations, Abbasbandy [15] derived this method by using the Adomian decomposition technique. See also Noor [16] for more details and analysis of these methods.

Now using Algorithm 2.1 as a predictor and Algorithm 2.2 as a corrector, Noor et. al. [17] suggest and analyze a new two-step iterative method for solving the nonlinear equation, which is the main motivation of this paper.

Algorithm 2.3. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{[f(y_n)]^2 f''(y_n)}{2[f'(y_n)]^3}$$

Algorithm 2.3 is a two-step predictor-corrector Householder method and has sixth-order convergence. [17].

Now using the technique of updating the solution, therefore, using Algorithm 2.3 as a predictor and Algorithm 2.1 as a corrector, we suggest and analyze a new three-step iterative methods for solving the nonlinear equation (1), which are the main motivation of this paper.

Algorithm 2.4. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{[f(y_n)]^2 f''(y_n)}{2[f'(y_n)]^3}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

Algorithm 2.4 is called the predictor-corrector Householder's method (PCS) and has twelfth-order convergence. Per iteration of the iterative method 2.3 requires three evaluations of the function, three evaluations of first derivative, and one evaluations of second derivative. We take into account the definition of efficiency index [11], if we suppose that all the evaluations have the same cost as function one, we have that the efficiency index of the method 2.3 is $12^{1/7} \approx 1.426$.

In order to implement Algorithm 2.4, one has to find the second derivative of this function, which may create some problems. To overcome this drawback, several authors have developed involving only the first derivative. This idea plays a significant part in developing some iterative methods free from second derivatives. The second derivative with respect to z, which may create some problems. To overcome this drawback, several authors have developed involving only the first derivatives. This idea plays a significant part in developing our new iterative methods free from second derivatives with respect to$z$. To be more precise, we now approximate $f''(y_n)$, to reduce the number of evaluations per iteration by a combination of already known data in the past steps. Toward this end, an estimation of the function $P_1(t)$ is taken into consideration as follows

$$P_1(t) = a + b(t - y_n) + c(t - y_n)^2 + d(t - y_n)^3$$

and also consider that this approximation polynomial satisfies the interpolation conditions $f(x_n) = P_1(x_n)$, $f(y_n) = P_1(y_n)$, $f'(x_n) = P_1'(x_n)$and $f'(y_n) = P_1'(y_n)$. By substituting the known values in $P_1(t)$ we have a system of three linear equations with three unknowns. By solving this system and simplifying we have

$$f''(y_n) = \frac{2}{y_n - x_n}\left(2f'(y_n) + f'(x_n) - 3\frac{f(y_n) - f(x_n)}{y_n - x_n}\right) = P_1(x_n, y_n). \quad (5)$$

then algorithm 2.4 can be written in the form of the following algorithm.

Algorithm 2.5. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{[f(y_n)]^2 P_1(y_n)}{2[f'(y_n)]^3}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

Algorithm 2.5 is called the predictor-corrector Modified Householder's method (MS1) and has twelfth-order convergence. Per iteration of the iterative method 2.5 requires three evaluations of the function and three evaluations of first derivative. We have that the efficiency index of the method 2.5 is $12^{1/6} \approx 1.513$ which is better than $12^{1/7} \approx 1.426$ of the method 2.4.

To improve the efficiency index, we approximate the first-appeared derivative in the last step $f'(z_n)$ by a combination of already evaluated function values using divided differences. This procedure was used by A. Cordero et al. [18].

To explain the idea, consider the Taylor polynomial of degree 2 for the function $f(z_n)$

$$f(z_n) = f(y_n) + (z_n - y_n)\, f'(y_n) + \frac{(z_n - y_n)^2}{2} f''(y_n) \tag{6}$$

This implies that

$$\begin{aligned} f'(y_n) &= \frac{f(z_n) - f(y_n)}{(z_n - y_n)} - \frac{(z_n - y_n)}{2} f''(y_n) \qquad (7)\\ &= f[z_n, y_n] - \frac{(z_n - y_n)}{2} f''(y_n), \end{aligned}$$

where

$$f[z_n, y_n] = \frac{f(z_n) - f(y_n)}{(z_n - y_n)}$$

then from (7)

$$f''(y_n) = \frac{2\{f[z_n, y_n] - f'(y_n)\}}{(z_n - y_n)} \tag{8}$$

again from (6)

$$f'(z_n) = f'(y_n) + (z_n - y_n) f''(y_n) \tag{9}$$

Substitute the estimation of $f'(y_n)$ and $f''(y_n)$ into the last expression, to get

$$f'(z_n) = f[z_n, y_n] + (z_n - y_n) f[z_n, y_n, y_n] \tag{10}$$

where

$$f[z_n, y_n, y_n] = \frac{f[z_n, y_n] - f'(y_n)}{(z_n - y_n)}$$

from (5) and (7) in (9) we can have another approximation formula to the function $f'(z_n)$as

$$f'(z_n) = f[z_n, y_n] + \frac{1}{2}(z_n - y_n)P_1(y_n) \tag{11}$$

Now by substituting (10)into (5), we obtain the following new proposed three-step iterative method for solving equation (1):

Algorithm 2.6. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{[f(y_n)]^2 P_1(y_n)}{2[f'(y_n)]^3}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + (z_n - y_n)f[z_n, y_n, y_n]}$$

Algorithm 2.6 is called the predictor-corrector Modified Householder's method (MS2) and has tenth-order convergence. Per iteration of the iterative method 2.6 requires three evaluations of the function and two evaluations of first derivative. We have that the efficiency index of the method 2.6 is $10^{1/5} \approx 1.585$ which is better than $12^{1/6} \approx 1.513$ of the method 2.5 and is better than $12^{1/7} \approx 1.426$ of the method 2.4.

Again by substituting (11) into (5), we obtain the following new proposed three-step iterative method for solving equation (1):

Algorithm 2.7. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{[f(y_n)]^2 P_1(y_n)}{2[f'(y_n)]^3}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + 0.5(z_n - y_n)P_1(y_n)}$$

Algorithm 2.7 is called the predictor-corrector Modified Householder's method (MS3) and has tenth-order convergence. Per iteration of the iterative method 2.7 requires three evaluations of the function and two evaluations of first derivative. We have that the efficiency index of the method 2.7 is $10^{1/5} \approx 1.585$ which is better than $12^{1/6} \approx 1.513$ of the method 2.5 and is better than $12^{1/7} \approx 1.426$ of the method 2.4.

To be more precise, we now approximate $f'(y_n)$, to reduce the number of evaluations per iteration by a combination of already known data in the past steps. Toward this end, an estimation of the function $P_2(t)$ is taken into consideration as follows

$$P_2(t) = a + b(t - x_n) + c(t - x_n)^2$$
$$P_2'(t) = b + 2c(t - x_n)$$

By substituting in the known values

$$P_2(y_n) = f(y_n) = a + b(y_n - x_n) + c(y_n - x_n)^2$$
$$P_2'(y_n) = f'(y_n) = b + 2c(y_n - x_n)$$
$$P_2(x_n) = f(x_n) = a$$
$$P_2'(x_n) = f'(x_n) = b$$

we could easily obtain the unknown parameters. Thus we have

$$f'(y_n) = 2\left(\frac{f(y_n) - f(x_n)}{y_n - x_n}\right) - f'(x_n) = P_2(x_n, y_n) \qquad (12)$$

then algorithm 2.7 can be written in the form of the following algorithm.

Algorithm 2.8. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$z_n = y_n - \frac{f(y_n)}{P_2(y_n)} - \frac{[f(y_n)]^2 P_1(y_n)}{2[P_2(y_n)]^3}$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + 0.5(z_n - y_n)P_1(y_n)}$$

Algorithm 2.8 is called the predictor-corrector Modified Householder's method (MS4) and has seventh-order convergence. Per iteration of the iterative method 2.8 requires three evaluations of the function and one evaluations of first derivative. We have that the efficiency index of the method 2.8 is $7^{1/4} \approx 1.626$ which is better than $10^{1/5} \approx 1.585$ of the method 2.7 and this is the main motivation of our paper.

3. Convergence Analysis

Let us now discuss the convergence analysis of the above mentioned methods Algorithm 2.6 and Algorithm 2.8.

Theorem 3.1 *Let r be a semple zero of sufficient differentiable function $f :\subseteq R \to R$ for an open interval I. If $x_0$is sufficiently close to r, then the two step method defined by our algorithm 2.6 has convergence is at least of order ten.*

Proof. Consider to

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{[f(y_n)]^2 P_1(y_n)}{2[f'(y_n)]^3} \tag{13}$$

Let $r$be a simple zero of f. Since $f$is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about r, we get

$$\begin{aligned} f(x_n) &= f(r) + (x_n - r)f'(r) + \frac{(x_n - r)^2}{2!} f^{(2)}(r) \\ &\quad + \frac{(x_n - r)^3}{3!} f^{(3)}(r) + \frac{(x_n - r)^4}{4!} f^{(4)}(r) + \cdots , \end{aligned}$$

Then

$$f(x_n) = f'(r)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \cdots], \tag{14}$$

and

$$f'(x_n) = f'(r)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + \cdots], \tag{15}$$

where $c_k = \frac{1}{k!} \frac{f^{(k)}(r)}{f'(r)}$, $k = 1, 2, 3, \ldots$ and $e_n = x_n - r$.

Now from (14) and (15), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4)e_n^4 + \cdots ,$$

From (13), we get

$$y_n = r + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2 c_3 + 4c_2^3 + 3c_4)e_n^4 + \cdots , \tag{16}$$

From (16), we get,

$$f(y_n) = f'(r)[(y_n - r) + c_2(y_n - r)^2 + c_3(y_n - r)^3 + c_4(y_n - r)^4 + \cdots],$$

$$f'(y_n) = f'(r)[1 + 2c_2^2 e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + (-11c_2^2c_3 + 8c_2^4 + 6c_2c_4)e_n^4 + \cdots].$$

then

$$\begin{aligned} \frac{f(y_n)}{f'(y_n)} &= c_2e_n^2 - 2(c_2^2 - c_3) - (7c_2c_3 - 3c_2^3 - 3c_4)e_n^4 \\ &\quad +2(8c_2^2c_3 - 2c_2^4 - 3c_3^2 - 5c_2c_4 + 2c_5)e_n^5 - (13c_2c_5 - \\ &\quad 22c_4c_2^2 - 5c_5 - 6c_2^5 - 32c_3c_2^3 + 17c_4c_3 - 29c_2c_3^2)e_n^6 + \cdots \end{aligned}$$

$$P_1(y_n) = \frac{2}{y_n - x_n}\left(2f'(y_n) + f'(x_n) - 3\frac{f(y_n) - f(x_n)}{y_n - x_n}\right)$$

$$\begin{aligned} P_1(y_n) &= f'(r)[2c_2 + (6\ c_2\ c_3 - 2c_4)\ e_n^2 - 4(3\ c_3(c_2^2 - \ c_3) \\ &\quad -c_2c_4\ +\ c_5)\ e_n^3 + 2\ (12\ c_2^3\ c_3 -\ 21\ c_2\ c_3^2 \\ &\quad +\ c_2^2c_4 +\ 13\ c_3\ c_4 + (c_2 - 3)\ c_5)\ e_n^4 + \cdots] \end{aligned} \tag{17}$$

Substituting into (13), to get

$$\begin{aligned} z_n &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{[f(y_n)]^2 P_1(y_n)}{2[f'(y_n)]^3} \\ &= r + c_2^2(2\ c_2^3\ -\ c_2\ c_3\ +\ c_4)\ e_n^6 + O(e_n^7) \end{aligned} \tag{18}$$

Now, expand $f(z_n)$ about r to get

$$\begin{aligned} f(z_n) &= f'(r)[c_2^2\ (2\ c_2^3 - c_2\ c_3 + c_4)\ e_n^6 + 2\ c_2\ (-6\ c_2^5 + 9\ c_2^3\ c_3 \\ &\quad -3\ c_2^2\ c_4 +\ 2\ c_3\ c_4 +\ c_2\ (-3\ c_3^2\ +\ c_5))\ e_n^7 + \cdots] \end{aligned}$$

$$\begin{aligned} f[z_n, y_n] + (z_n - y_n)f[z_n, y_n, y_n] &= f'(r)[1 - c_2^2\ c_3\ e_n^4 \\ &\quad +4\ c_2c_3(c_2^2 - c_3)\ e_n^5 + \cdots] \end{aligned}$$

Substituting into (13), to get

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + (z_n - y_n)f[z_n, y_n, y_n]}$$

$$x_{n+1} = r - c_2^4c_3(2\ c_2^3\ -\ c_2\ c_3\ +\ c_4)\ e_n^{10} + O(e_n^{11}) \tag{19}$$

From (19), $e_{n+1} = x_{n+1} - r$ then we will have

$$e_{n+1} = -c_2^4c_3(2\ c_2^3\ -\ c_2\ c_3\ +\ c_4)\ e_n^{10} + O(e_n^{11}) \tag{20}$$

which shows that Algorithm 2.6 is at least a tenth order convergent method, the required result.

Theorem 3.2 *Let r be a semple zero of sufficient differentiable function $f :\subseteq R \to R$ for an open interval I. If $x_0$is sufficiently close to r, then the two step method defined by our algorithm 2.8 has convergence is at least of order seven.*

Proof. Consider to

$$z_n = y_n - \frac{f(y_n)}{P_2(y_n)} - \frac{[f(y_n)]^2 P_1(y_n)}{2[P_2(y_n)]^3}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + 0.5(z_n - y_n)P_1(y_n)} \tag{21}$$

Again by using Taylor's expansion we can get

$$P_2(y_n) = 2\left(\frac{f(y_n) - f(x_n)}{y_n - x_n}\right) - f'(x_n) =$$

$$\begin{aligned} P_2(y_n) = f'(r)[1 + (2\,c_2^2 - c_3)\,e_n^2 - 2\,(2\,c_2^3 - 3\,c_2\,c_3 + c_4)\,e_n^3 \\ +\ (8\,c_2^4 -\ 16\,c_2^2c_3\ +\ 4\,c_3^2 + 8\,c_2\,c_4\ -\ 3\,c_5)\,e_n^4 + \cdots] \end{aligned} \tag{22}$$

from (16), (17) and (22) in (21) we get

$$\begin{aligned} z_n &= y_n - \frac{f(y_n)}{P_2(y_n)} - \frac{[f(y_n)]^2 P_1(y_n)}{2[P_2(y_n)]^3} \\ &= r - c_2c_3e_n^4 +\ (2c_3(c_2^2 -\ c_3) - 2c_2c_4)e_n^5 + \cdots \end{aligned}$$

$$f(z_n) = f'(r)[-c_2\,c_3\,e_n^4 +\ (2c_3(c_2^2 -\ c_3) - 2c_2c_4)e_n^5 + \cdots], \tag{23}$$

$$\begin{aligned} f[z_n, y_n] + \frac{1}{2}(z_n - y_n)P_1(y_n) &= f'(r)[1 - 2c_2c_3\,e_n^3 \\ &\quad + (2c_3(c_2^2 -\ 2c_3) - 3c_2c_4)\,e_n^4 \cdots] \end{aligned} \tag{24}$$

Substituting from (23) and (24) into (21), to get

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + 0.5(z_n - y_n)P_1(y_n)} = r + 2c_2^2c_3^2e_n^7 + O(e_n^8)$$

or, in the final form

$$e_{n+1} = 2c_2^2c_3^2e_n^7 + O(e_n^8) \tag{25}$$

which shows that Algorithm 2.8 has seventh- order of convergence.

4. Numerical Examples

For comparisons, we have used the ninth-order Al-Subaihi method [19] (AS) and Noor et al. [20] (NRM) defined respectively by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{2f(y_n)f'(y_n)}{2[f'(y_n)]^2 - f(y_n)P_1(y_n)}.$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + (z_n - y_n)f[z_n, x_n, x_n]}$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{2f(y_n)f'(y_n)}{2[f'(y_n)]^2 - f(y_n)P_1(y_n)}.$$

$$x_{n+1} = z_n - \frac{f'(x_n) + f'(y_n)}{3f'(y_n) - f'(x_n)} \times \frac{f(z_n)}{f'(x_n)}$$

In this study, we present some numerical examples to illustrate the efficiency and the accuracy of the new developed iterative methods (Tables 1, 2). We compare our new methods namely (MS1) to (MS4), with Al-Subaihi method [19] (AS) and Noor et al. [20] (NRM). Our examples are tested with precision $\varepsilon = 10^{-200}$and the following stopping criteria is used for computer programs:

$$|x_{n+1} - x_n| + |f(x_{n+1})| < \varepsilon.$$

Displayed in Table 1 are the number of iterations (IT), such that the stopping criteria satisfied, the absolute values of the function $f(x_n)$ after the required iterations. Moreover, displayed is the distance of two consecutive approximations $\delta = |x_n - x_{n-1}|$, the time pier second and the computational order of convergence (COC). Where the computational order of convergence (COC) can be approximated using the formula,

$$COC \approx \frac{\ln|(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln|(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}$$

Method	AS	NRM	MS1	MS2	MC3	MS4
f_1 , x_0=1						
IT	4	4	4	4	4	4
$\|f(x_n)\|$	2.21E-4553	7.51E-4458	0	9.86E-7387	2.73E-6980	2.76E-2130
δ	1.97E-506	7.91E-496	3.11E-1116	3.91E-739	1.61E-698	9.92E-305
Time/s	0.219	0.125	0.140	0.171	0.172	0.219
COC	9	9	12	10	10	7
f_2, x_0=1.3						
IT	4	4	4	4	4	4
$\|f(x_n)\|$	3.08E-7159	6.72E-6932	1E-9999	1E-9999	1E-9999	7.76E-3076
δ	4.99E-796	8.30E-771	6.69E-1764	1.40E-1133	4.43E-1065	6.15E-439
Time/s	4.836	4.227	4.820	4.306	4.243	4.290
COC	9	9	12	10	10	7
f_3, x_0=2						
IT	4	4	4	4	4	5
$\|f(x_n)\|$	6.01E-3005	1.03E-2992	0	5.49E-5212	3.86E-5177	1E-3437
δ	5.96E-334	1.38E-332	1.77E-1083	3.36E-521	8.80E-518	3.34E-491
Time/s	1.357	1.077	1.186	1.138	1.154	1.435
COC	9	9	12	10	10	7

Table 1: Comparison of different methods

All the computations are performed using Maple 15 with 10000 significant digits. The following examples are used for numerical testing:

$$\begin{aligned}
f_1(x) &= x^3+4x^2-10, & x_0 &= 1\,.\\
f_2(x) &= \sin^2 x - x^2 + 1, & x_0 &= 1.3\,.\\
f_3(x) &= x^2 - e^x - 3x + 2, & x_0 &= 2\,.\\
f_4(x) &= \cos x - x, & x_0 &= 1.7\,.\\
f_5(x) &= (x-1)^3 - 1, & x_0 &= 2.5\,.\\
f_6(x) &= x^3 - 10, & x_0 &= 2\,.\\
f_7(x) &= e^{x^2+7x-30} - 1, & x_0 &= 3.1\,.
\end{aligned}$$

Results are summarized in Tables 1,2 as it shows, new algorithms are comparable with all of the methods and in most cases gives better or equal results.

Method	AS	NRM	MS1	MS2	MC3	MS4
f_4, x_0=1.7						
IT	4	4	4	4	4	4
$\mid f(x_n) \mid$	1.52E-6046	1.56E-5892	0	0	1.154	5.13E-1777
δ	5.33E-672	7.20E-655	1.52E-1784	2.93E-1008	7.09E-1015	4.92E-254
Time/s	4.758	4.321	4.836	4.274	4.352	4.275
COC	9	9	12	10	10	7
f_5, x_0=2.5						
IT	4	4	4	4	4	5
$\mid f(x_n) \mid$	1.15E-2807	3.28E-2868	1.40E-8376	1.01E-4946	1.03E-4421	9.69E-8633
δ	1.15E-312	2.20E-318	8.62E-699	2.39E-495	7.06E-443	7.59E-1234
Time/s	0.202	0.125	0.141	0.172	0.141	0.187
COC	9	9	12	10	10	7
f_6, x_0=2						
IT	4	4	4	4	4	4
$\mid f(x_n) \mid$	7.72E-7338	1.71E-7415	5.00E-9999	5.00E-9999	1.00E-9998	6.62E-2989
δ	8.50E-816	2.05E-824	2.94E-1873	1.29E-1153	1.39E-1119	2.15E-427
Time/s	0.203	0.125	0.141	0.156	0.140	0.188
COC	9	9	12	10	10	7
f_7, x_0=3.1						
IT	5	4	4	4	4	4
$\mid f(x_n) \mid$	0	1.11E-2165	7.70E-5451	2.76E-2963	2.05E-2594	6.81E-6708
δ	1.25E-1369	4.35E-242	8.79E-456	7.79E-298	5.77E-261	9.53E-960
Time/s	0.967	0.718	0.702	0.749	0.765	1.420
COC	9	9	12	10	10	7

Table 2: Comparison of different methods

5. Conclusions

In this paper, we have suggested new higher-order iterative methods free from second derivative for solving nonlinear equation. We also discussed the efficiency index and computational order of convergence of these new methods. The new methods attain efficiency indices of 1.626 and 1.585, which makes them competitive. In addition, the proposed methods have been tested on a series of examples published in the literature and show good results when compared it with the previous literature.

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