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CONTROL-TRANSVERSE DYNAMICS AND BIFURCATION BEHAVIOUR

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Abstract: To every submanifold transverse to the control distribution of a given control system is associated in a canonical way a dynamical behavior. By varying this transverse manifold, we can effect bifurcations in these dynamics. In this paper we explain how this is done and draw the analogies with the more familiar aspects of bifurcation theory. The second crucial point is that the manifold can be made invariant for a control flow and the dynamics in the directions transverse to it can be assigned arbitrarily – in particular, the manifold can be made asymptotically stable.

1. Introduction

The setting for bifurcation theory is to consider systems depending on some parameters and to examine the changes in dynamical behavior as the parameters change *"very slowly"*. There is, in fact, an almost exact analog for control systems and it is the purpose of this paper to outline it and to describe the ways it can be used in control design. What is even better is that the rather ambiguous role of the slow changes in the parameters is replaced by a clear, unambiguous geometric variation in the submanifold transverse to the control

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directions.

This point of view brings out another aspect of bifurcation behavior which is not usually emphasized: the fact that further out from the local, or semi-local bifurcation, the dynamics is fairly robust. Consequently, one can bring into play ideas from the theory of the topological Conley index (itself a far-reaching generalization of the Morse stability index). As an example, we can arrange it so that the dynamics "far away" look like an attractor, yet, in the vicinity of an equilibrium point, we can change the stability of an equilibrium and thus create a stable limit cycle (a Poincaré-Andronov-Hopf "bifurcation"!)

The present work is related to but remains quite distinct from some recent areas of research in Control Dynamics. The Dynamical Systems approach came to control theory in the general framework of Colonius, Kliemann and others [1, 2] and in the explicit consideration of bifurcations and normal forms in control by Kang and Krener [3, 4].[1] It is now an established area within Control Theory (see Murray's CalTech Course CDS140b and [5].) In fact, the author's own interest in the relation between Control and Dynamics came earlier (around 1986), through the problem of large deviations in dissipative dynamics, [6], where it was found that a weaker form of controllability was sufficient to ensure that the small-noise flow exits a region of attraction. It was developed in dealing with the problem of *feedback linearization* [7, 8] and, during a visit to Santa Barbara in 1996-97, was more fully developed [9, 10]. That visit revealed the connections to *Back-stepping*, which was being explored at that time by Kokotovic and co-workers. In a sense, our approach is more general and deals with a more general dynamical framework ([11]).

We shall begin by setting out the geometric viewpoint for control systems which is needed. A special role will be played by the *singular set*, roughly the set where the control set contains the origin of the tangent space. We then set out the approach to control design using control-transverse dynamics. The idea is quite geometric and easy to grasp, yet fundamental in understanding control behavior. It leads to the consideration of an especially simple "normal form" which translates the problem to one of examining dynamics on suitable graphs. We use some simple examples of control systems to illustrate the technique.

[1] A rather distinct branch dealt with the use of control in chaotic dynamics (*chaos control*.)

2. Geometric Definition of Control Systems

The traditional way of writing a control system as a control-dependent system of differential equations

$$\dot{x} = f(x,u), \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m$$

is *local* and leaves rather vague the role of the control. It is also difficult to capture the possible dependence of the control set on the state (in other words $u \in \mathcal{U}(x)$.)

It is clear that, geometrically, a control system is an assignment of a subset $\mathcal{U}(x)$ of the tangent space at each point (all possible velocity vectors at the point.) The variation of the **control set** $\mathcal{U}(x)$ with x must be smooth, in some sense. Thus, the most general form of a control system is a *fibration*, which is a subset of the tangent bundle of the state space manifold. This geometric approach is expounded in detail in the author's work [7, 11, 12, 13], for example.

$$\begin{array}{ccc} \mathcal{U} & \subset & TM \\ & \searrow & \downarrow \\ & & M \end{array}$$

Now a control-affine system $\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)$ is in a sense a linear approximation of the general control system. The span of the control vector fields $g_i(x)$ is the **control distribution**. We shall assume that, locally, it has constant rank, which can be taken to be equal to m.

The **singular set** Σ of a control system is the projection onto the state space of the intersection set of the zero submanifold of the tangent bundle, Z, with the control fibration. In simple terms, it is the set of all states such that the control set contains the zero direction.

It is an elementary result from Algebraic Topology that a vector bundle over a contractible base is trivializable, that is a product. As a consequence, we shall assume that in the domain of interest, the control distribution is actually constant. This implies, of course, that a suitable change of coordinates has been made and we shall decompose the state space (for now a contractible open subset of Euclidean space) so that we can write the state as

$$\left[\begin{array}{c} x \\ y \end{array} \right],$$

where y is the component in the span of the control distribution and x is in its complement. The control system then takes the following simple *"normal* *form"*

$$\begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= u \end{cases}$$

(For simplicity, in this work we assume unbounded control action $u \in \mathbf{R}^m$. This will not affect the applicability of our results, provided we remain close to the invariant manifold to be discussed below.) An attractive feature of this form is that the singular set is now the zero set of the function f:

$$\Sigma = \{(x,y)\,;\; f(x,y) = 0\}.$$

3. Control-Transverse Manifolds and Dynamics

We are now ready to make the connection between the problem of control *design* and the *geometry* of the control system.

The methodology will be as follows: we choose a submanifold of the state space of dimension equal to $n-m$ (the dimension of the state component x) which is transverse to the control distribution. On it is defined a dynamical systems in a canonical way. The dynamics can then be extended away from it —in the transverse directions, which are in fact the control directions— in an arbitrary way.

Assuming the control system is in the normal form we have given, the trasnversality of the submanifold is satisfied in a very easy way: we consider an arbitrary function

$$h : \mathbf{R}^{n-m} \to \mathbf{R}^m;\; x \mapsto y = h(x).$$

The **graph** of this function is automatically transverse to the control distribution! In the simple case of designing a control system with a desirable equilibrium point, we arrange the graph to pass through a suitable point of the singular set.

We claim that on the graph of h is defined a dynamical systems in a canonical way which is described by the system

$$\dot{x} = f(x, h(x)).$$

Clearly, in order for the design process to be effective, we need to make some assumption on the function f (akin to controllability, but in a geometric setting): we need the positioning of the singular set in relation to the control distribution to be generic. This means that we assume that there is no point (in the region of interest) where the tangent space of the singular set is contained in the control distribution.

Invariance of Transverse Manifold

The manifold N on which the above dynamics is defined, the graph of h, can be made *invariant* for a flow defined at least in a neighborhood of N. The reason is that we can use the control action exactly in the directions transverse to N. Starting with $y = h(x)$ and differentiating,

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} \Longrightarrow u = \frac{\partial h}{\partial x} f(x, y)$$

is a feedback control achieving the invariance. Finally, a term to stabilize the invariant manifold can be added, proportional to the distance to the graph. The resulting control takes the form:

$$u = \frac{\partial h}{\partial x} f(x, y) - k(y - h(x)).$$

4. Variations in Transverse Manifolds and Bifurcations

It is clear that, by varying the transverse manifold N, it is possible to change the control-transverse dynamics (see the first part of the example below.) We therefore have a setting where bifurcations occur in the controlled dynamics. The difference is that the manifold on which the dynamics is defined *changes* and also that there are no bifurcation parameters: their role is taken up by the variation of the manifold. Since topologically the manifolds we consider are all the same (graphs of suitable functions), we are justified in talking about bifurcations of dynamics. This is made even more explicit by the fact that, in the context of the normal form, there is just one dynamical system that we consider, namely $\dot{x} = f(x, h(x))$ (some caution is necessary: the system here is a projection of the dynamics on the invariant graph. If the dynamical behaviour is complex, it may happen that the two systems —the original and the projection— are not equivalent. However, for the purposes of our discussion, and while we are dealing with ”rough” features of the dynamics such as asymptotic stability, the equivalence is easier to establish and thus we are justified in studying only the projected dynamics.)

Shaping the Graph

Suppose we have succeeded in finding a graph with dynamics having a compact attractor. In changing the graph it is desirable to *preserve the stable behavior in* *the large*, while trying to simplify or alter the attractor structure —for example, converting a stable limit cycle into an equilibrium point, or vice versa. This means that we should keep the ends of the graph the same, while changing its shape locally. The effect is the same as in bifurcation theory (local –and global!) There is a topological theory of index, the *Conley index*, which can help in this process (described in [13, 15, 14] for example.) It works as follows: we isolate a subset I of the state space containing some (perhaps unknown) isolated invariant set. We can draw conclusions as to the dynamical type of this invariant set by examining the boundary of the subset. More specifically, a quotient space is formed $\partial I/\partial I^{-}$, where the set ∂I^{-} is the exit set on the boundary. It turns out that the topology of the quotient only depends on the invariant set and **not** on the isolating set!

Using the Conley index, and by leaving the graph unchanged outside some set, we can concentrate on a specific finite part where we can try to achieve the dynamics we want, in the certainty that, globally, the dynamic type does not change. As an example, we shall see below how to turn an equilibrium into three equilibria, an analog of the pitchfork bifurcation.

5. An Example

Let us look at the steps of the methodology we just described through an example (since the length of this paper precludes a more detailed presentation.)

Consider the simplest possible system, the two dimensional *linear controllable* system in canonical form:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= u \end{cases}$$

The singular set is the horizontal axis $y = 0$.

Our interest is in showing how even in this linear system, we can use *nonlinear* control to achieve more complicated dynamics than just an asymptotically stable equilibrium.

First, to get us started with this geometric approach to control design, we remark that to every choice of a *linear* control $u = -k_1x - k_2y$ corresponds a choice of a linear function $y = ax$. This is seen by comparing the control arising from our approach for $h(x) = ax$ with the general linear feedback:

$$u = ay - k(y - ax) = -k_1x - k_2y.$$

Geometrically, we see that we get a *stable* system, provided we choose a line with a **negative** slope, so that to the left of zero the invariant dynamics move

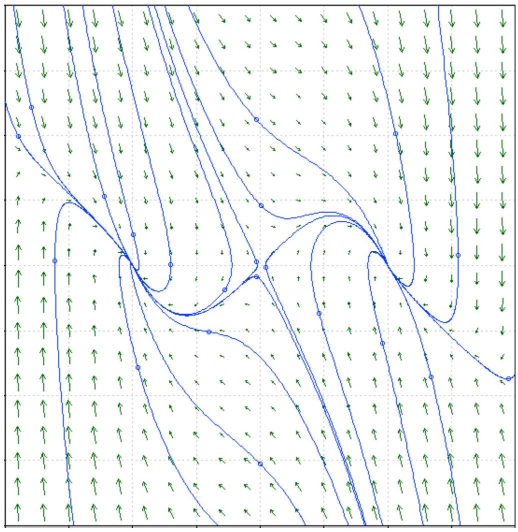

Figure 1: Nonlinear control achieving three equilibria

to the right towards zero ($\dot{x} = y > 0$ there) and to the right of zero they move to the left. Note, however, that this way the linear control dynamics are by necessity a node or sink, never a focus, since there is an invariant stable line. If we wish to achive the dynamics of a stable focus, we need to relax the requirement that the control-transverse line be invariant!

We move now to the use of nonlinear control for this linear system. Clearly, this is doen for illustrative purposes, since the geometry of the situation is especially transparent.

The nonlinear control which achieves dynamics with, for example, three equilibrium points and uses the function $y = -x(x^2 - 1)$ is

$$u = \frac{dh}{dx}y - k(y - h(x) = (-3x^2 + 1)y - k(y + x^3 - x).$$

As before, the first term make the graph invariant and the second makes it asymptotically stable (the value $k = 2$ is shown in the Figure 1.) It can be seen in the figure that the cubic graph is invariant.

Now if we consider a deformation of a linear graph to a cubic one, we get a bifurcation of the control-transverse dynamics which, starting with a single stable equilibrium, finishes with three. The two outside ones are stable, while the middle one is a saddle. But this is just the pitchfork bifurcation, viewed from our control point of view! Depending on the precise deformation we use, we shall encounter in our path a point where the equilibrium point becomes degenerate. By the genericity of the saddle-node bifurcation, we can argue that this will happen with a two-branch bifurcation diagram, not a single one, but singular (as in the symmetric pitchfork bifurcation) —see for example [16] or [17].

A more rigorous treatment requires the concept of continuation of dynamical systems (explained in, for example [14]) but would take us too far afield here.

6. Creation of a Limit Cycle

In this final section, let us briely address the problem of designing control dynamics with a stable "oscillator" (limit cycle.) This is the control system equivalent of the Poincaré-Andronov-Hopf bifurcations (PAH bifurcation.)

The process will be the following:[2] we will assume to begin with that there is a transverse dynamics with an asymptotically stable equilibrium. This guarantees that far away, the system has stable bahavior; this we will preserve by keeping the shape of the graph (say outside a disk of some radius.) We then de-stabilize the equilibrium by twisting the graph. By basic bifurcation results –or simple Poincaré-Bendixson theory in the plane, we are guaranteed the existence of a limit cycle. Further analysis, including *normal forms* for the dynamics yiels more precise reults and conditions for a unique stable limit cycle (to be presented elsewhere; basic references for this bifurcation are [18, 19]. For a control viewpoint related to our approach, see [20].)

Here, we shall present only the part about changing the stability of the equilibrium point. We therefore assume that we have a function $y = h(x)$ on whose graph the control-transverse dynamics are stable (the Conley index is that of an attractor.) We shall preserve this "far away."

Now let us look at the linear part of the system close to the equilibrium. We write the state as (x, y), with $x \in \mathbf{R}^2$ and $y \in \mathbf{R}$ (see footnote.) Partitioning the matrix of the linear part suitably, we have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix},$$

with f, g of higher order. The matrix A_1 is two-by-two while A_2 is a two-by-one column vector and A_4 is a scalar.

[2]We suppose the dimensions are such that the discussion that follows makes sense: either the transverse manifold is two-dimensional, or the remaining directions can be taken care of –i.e. the system is stable in them.

Doing the same for the new graph, we write

$$y = h(x) = \left[\begin{array}{cc} c & d \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] + h_2(x).$$

The key point is now that the dynamics

$$\dot{x} = A_1 x + A_2 h(x) + f(x, h(x))$$

has linear part

$$(A_1 + A_2 \left[\begin{array}{cc} c & d \end{array}\right])$$

and we can arrange for a movement across the imaginary axis of the eigenvalues using the following properties of the *trace* of a matrix:

$$\mathrm{tr}(A+B) = \mathrm{tr}A + \mathrm{tr}B, \quad \mathrm{tr}(AB) = \mathrm{tr}(BA).$$

This is because of the control we have in choosing the coefficients c, d and thus the trace of $A_2 \left[\begin{array}{cc} c & d \end{array}\right]$:

$$\mathrm{tr}(A_2 \left[\begin{array}{cc} c & d \end{array}\right]) = \left[\begin{array}{cc} c & d \end{array}\right] A_2 = ca_{21} + da_{22}.$$

Provided we are near the imaginary axis (or on it), the sign of the trace determines whether the two eigenvalues are stable or unstable —by continuity. If they are not, we will have to make a ”controllabilitty” assumption on the pair (A_1, A_2). Let us note here that it is possible to give geometric *transversality* conditions which yield the traditional stabilizability and controllability conditions of Control Theory (a familiar result, see also [21] in this context.)

In either case, the important thing is that we can obtain a *family* of systems on a parametrized family of graphs, on which the dynamics undergo a PAH bifurcation.

7. Conclusion

In this paper we gave but a brief outline of what we consider to be a powerful geometric method for global control design. An additional advantage is that it is in principle easy to apply. Clearly, what is obvious geometrically in small dimension needs substantially more work for systems in many dimensions, but the main point is that the design process is achieved through the use of geometric objects (analysis of singular set, submanifolds or graphs of functions) which are perhaps of greater ease of use than purely analytic design tools.

Much of what has been presented here can be generalized in a number of ways: the control distribution need not be constant, nor of constant rank (though this complicates matters somewhat.) Compactness of control set can also be taken into account. This is clearly important in applications. A practical example, fully worked out, where the problem of *global stabilization* is examined from the geometric viewpoint of this paper is found in [22]. It deals with the simple nonlinear model of the compressor dynamics of a jet engine, a problem of physical interest which was studied by many workers in Nonlinear Control at the time.

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