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# A MODIFIED ITERATIVE DECOMPOSITION METHOD FOR SOLVING BOUNDARY LAYER PROBLEM IN UNBOUNDED DOMAIN

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**Abstract:** In this paper, a modified iterative decomposition method is employed to obtain the approximate solution of the boundary layer problem in unbounded domain. This method is combined along with the diagonal Padé approximants. The proposed scheme finds the solution without any discretization or restrictive assumptions and is free from round off errors and therefore, reduces computational burden to a great extent. Comparison is made between the obtained results and those in the literature. The results has excellent agreement with series solutions of other methods in the literature and further showed the effectiveness and simplicity of the new technique.

**AMS Subject Classification:** 65L05, 65L10, 65L20
**Key Words:** modified iterative decomposition method, boundary layer problem, Padé approximants

## 1. Introduction

The iterative decomposition method (see [5, 6, 7, 8, 14]) is used to solve ef-

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fectively, easily and accurately a large class of linear and nonlinear differential equations with appropriate solutions which converges rapidly to exact solutions. This work deals with numerical approximation for a boundary layer problem in unbounded domain which arises in the mathematical modeling of problems arising in fluid mechanics, see [1, 3, 4, 10, 11]. In this paper, the modified iterative decomposition method is combined with the diagonal Padé approximants. The implementation of the technique has shown reliable results such that fewer terms are needed to obtain either the exact solution or to find an approximate solution of a reasonable degree of accuracy to real physical problems, see [17, 18, 19, 20, 21].

In earlier works of some researchers, the behaviours of the infinite series solution was a great concern, see [9, 12, 13, 15, 16]. Boyd [2], Wazwaz [12], and Baker [22] have shown that power series in isolation may not be useful to handle boundary value problems. This can be attributed to the possibility that the radius of convergent may not be sufficiently large to contain the boundaries of the domain. This justifies the combination of the modified iterative decomposition method or any series solution with diagonal Padé approximants to provide an effective tool to handle boundary value problems in infinite or semi-definite domains.

## 2. Analysis of the Method

In order to elucidate the solution procedure of the proposed modified iterative method. We consider the general form of a second order nonlinear nonhomogeneous differential equation with initial conditions as given below

$$u''(x) + b_1(x)u'(x) + b_2(x)u(x) + N[u(x)] = g(x), \tag{1}$$

$$u(0) = \alpha, \quad u'(x) = \beta. \tag{2}$$

Equation (1) can be put in operator form as

$$Lu + Ru + Nu = g(x), \tag{3}$$

where the operator $L = \frac{d^2}{dx^2}(.)$. Because $L$ is invertible, $L^{-1}$ exist and it is a two-fold definite integral defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) ds ds.$$

Applying $L^{-1}$ to both sides of the equation (3) and using the initial conditions, we get

$$u(x) = G(x) - L^{-1}[N(u)], \tag{4}$$

where

$$G(x) = \alpha + \beta t + L^{-1}[g(x) - Ru(x)]. \tag{5}$$

According to the iterative method proposed by Daftadar-Geji and Jafari [14], the non-linear operator $N(u)$ is decomposed as

$$N\left(\sum_{n=0}^{\infty} u_n\right) = N(u_0) + \sum_{n=0}^{\infty}\left\{N\left(\sum_{j=0}^{\infty} u_j\right) - N\left(\sum_{j=0}^{n-1} u_j\right)\right\}. \tag{6}$$

On further simplification, we obtain the recursive relation

$$\begin{aligned} u_0 =& G(x), \\ u_1 =& L^{-1}[N(u_0)], \\ u_{n+1} =& L^{-1}\left[N\left(\sum_{i=1}^{n} u_i\right)\right] - L^{-1}\left[N\left(\sum_{i=1}^{n-1} u_i\right)\right], \quad n \geq 1, \end{aligned} \tag{7}$$

where $G(x)$ represents the term arising from source term and the prescribed initial conditions. The proposed method or technique is based on the assumption that the zeroth component $G(x)$ can be divided into two equal parts, viz $G_0(x) + G_1(x)$ such that

$$G(x) = G_0(x) + G_1(x). \tag{8}$$

The slight variation suggests that only the part $G_0$ be assigned the zeroth component of $u_0$ while the remaining part $G_1$ be combined only with other terms in $u_1$ of equation (5). Moreover, since the current trend in numerical solution of differential equations is towards efficiency, simple algorithm devoid of complexity and high level of accuracy. A modified recursive algorithm is given as

$$\begin{aligned} u_0 =& G_0(x), \\ u_1 =& G_1(x) + L^{-1}[N(u_0)], \\ u_{n+1} =& L^{-1}\left[N\left(\sum_{i=1}^{n} u_i\right)\right] - L^{-1}\left[N\left(\sum_{i=1}^{n-1} u_i\right)\right], \quad n \geq 1. \end{aligned} \tag{9}$$

The success of equation (9) depends on the proper selection of $G_0$ as the initial solution to avoid noise oscillation during the iteration procedure. The choice of $G_0$ is based mainly on trial criteria. Also equation (9) yields a slight reduction in terms of which reduces computational burden when compared to iterative method. Equation (9) may also give exact solution by two or fewer iterations only without necessarily using special polynomials such as Adomian Polynomials, Bell's Polynomials and He's Polynomials.

## 3. Application

In this section, we apply the modified iterative decomposition method combined with the diagonal Padé approximants to a third order boundary layer problem and the results obtained are presented in a tabular form for easy comparison with some of the different methods mentioned in the literature.

$$f'''(x) + (n-1)f(x)f'(x) - 2n(f'(x))^2 = 0, \quad n > 0,$$

with boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0,$$

where $f''(0) = \alpha < 0$.

By applying the recursive algorithm (8), subject to the initial conditions, yields

$$\begin{aligned}
f_0(x) =& x, \\
f_1(x) =& \frac{x^2}{2}\alpha + \frac{nx^3}{3}, \\
f_2(x) =& \frac{\alpha(3n+1)}{24}x^4 + \left(\frac{n}{30} + \frac{n^2}{30} + \frac{n}{40}\alpha^2 + \frac{\alpha^2}{120}\right)x^5 \\
&+ \left(\frac{\alpha n}{90} + \frac{\alpha n^2}{45}\right)x^6 + \left(\frac{n^2+2n^3}{315}\right)x^7, \\
f_3(x) =& \frac{1}{315}n^2x^7 + \frac{1}{42}\alpha n x^6 + \frac{1}{240}\alpha n^2 x^6 + \frac{1}{6449625}x^{15}n^4 + \frac{16}{19348875}x^{15}n^5 \\
&+ \frac{4}{2764125}x^{15}n^6 + \frac{1}{61425}x^{13}n^4 + \frac{179}{8108100}x^{13}n^5 + \frac{31}{8108100}x^{13n^3} \\
&+ \frac{1}{7700}x^{11}n^3 + \frac{4}{17325}x^{11}n^4 + \frac{1}{712800}x^{11}\alpha^4 + \frac{1}{44550}x^{11}n^2 \\
&+ \frac{1}{64800}x^{10}\alpha^3 + \frac{19}{22680}x^9n^2 + \frac{1}{567}x^9n^3 + \frac{1}{24192}x^9\alpha^2 + \frac{11}{40320}x^8\alpha^3 \\
&+ \frac{11}{5040}x^7\alpha^2 + \frac{1}{315}x^7n + \frac{1}{240}x^6\alpha + \frac{16}{19348875}x^{15}n^7
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{103950}x^{13}n^{6}+\frac{23}{207900}x^{11}n^{5}+\frac{13}{22680}x^{9}n^{4}+\frac{829}{362880}x^{9}\alpha^{2}n^{2}\\
&+\frac{59}{72576}x^{9}n\alpha^{2}+\frac{19}{6720}x^{8}\alpha n+\frac{1}{960}x^{8}\alpha^{3}n+\frac{1}{120}x^{7}n\alpha^{2}\\
&+\frac{557}{362880}x^{9}\alpha^{2}n^{3}+\frac{3}{4480}x^{8}\alpha^{3}n^{2}+\frac{3}{560}x^{7}n^{2}\alpha^{2}+\frac{8}{1289925}x^{14}\alpha n^{6}\\
&+\frac{313}{16216200}x^{13}n^{5}\alpha^{2}+\frac{37}{498960}x^{12}\alpha n^{5}+\frac{7}{237600}x^{12}n^{4}\alpha^{3}+\frac{1}{4950}x^{11}n^{4}\alpha^{2}\\
&+\frac{1}{52800}x^{11}n^{3}\alpha^{4}+\frac{53}{75600}x^{16}n^{4}+\frac{1}{4800}x^{10}n^{3}\alpha^{3}+\frac{17}{4032}x^{8}\alpha n^{3}\\
&+\frac{8}{1289925}x^{14}\alpha n^{4}+\frac{2}{184275}x^{14}\alpha n^{5}+\frac{1}{859950}x^{14}\alpha n^{3}+\frac{1}{56700}x^{13}n^{3}\alpha^{2}\\
&+\frac{1}{30800}x^{13}n^{4}\alpha^{2}+\frac{101}{32432400}x^{12}\alpha n^{4}+\frac{59}{623700}x^{12}\alpha n^{3}+\frac{247}{1663200}x^{12}\alpha n^{4}\\
&+\frac{13}{285120}x^{12}n^{3}\alpha^{3}+\frac{1}{44550}x^{12}n^{2}\alpha^{3}+\frac{1}{285120}x^{12}\alpha^{3}n+\frac{97}{4989600}x^{12}\alpha n^{2}\\
&+\frac{1}{5400}x^{11}\alpha^{2}n^{2}+\frac{1}{32400}x^{11}n\alpha^{2}+\frac{41}{118800}x^{11}n^{3}\alpha^{2}+\frac{1}{39600}x^{11}n^{2}\alpha^{4}\\
&+\frac{1}{95040}x^{11}n\alpha^{4}+\frac{1}{8640}x^{10}\alpha^{3}n+\frac{1}{16200}x^{10}\alpha n+\frac{151}{113400}x^{10}\alpha n^{3}\\
&+\frac{1}{3600}x^{10}n^{2}\alpha^{3}+\frac{47}{75600}x^{10}\alpha n^{2}+\frac{73}{10080}x^{8}\alpha n^{2}.\\
&\vdots
\end{aligned}
$$

The series solution is given as

$$
\begin{aligned}
f(x)=&x+\frac{1}{2}\alpha x^{2}+\frac{1}{3}nx^{3}+\left(\frac{1}{24}\alpha+\frac{1}{8}n\alpha\right)x^{4}+\left(\frac{1}{30}n^{2}+\frac{1}{40}n\alpha^{2}\right.\\
&\left.+\frac{1}{120}\alpha^{2}+\frac{1}{30}n\right)x^{5}+\left(\frac{19}{720}n^{2}\alpha+\frac{1}{240}\alpha+\frac{1}{40}n\alpha\right)x^{6}\\
&+\left(\frac{1}{120}n\alpha^{2}+\frac{1}{315}n+\frac{2}{315}n^{3}+\frac{11}{540}\alpha^{2}+\frac{3}{560}n^{2}\alpha^{2}+\frac{2}{315}n^{2}\right)x^{7}\\
&+\left(\frac{11}{40320}\alpha^{3}+\frac{33}{4480}\alpha^{2}n^{2}+\frac{3}{4480}\alpha^{3}n^{2}+\frac{23}{5760}n\alpha+\frac{1}{2688}\alpha+\frac{167}{40320}n^{3}\alpha\right.\\
&\left.+\frac{1}{960}\alpha^{3}n\right)x^{8}+\left(\frac{1}{3780}n+\frac{527}{362880}n^{3}\alpha^{2}+\frac{19}{11340}n^{3}+\frac{709}{362880}n\alpha^{2}\right.\\
&\left.+\frac{23}{8064}n^{2}\alpha^{2}+\frac{23}{226780}n^{2}+\frac{13}{22680}n^{4}+\frac{43}{120960}\alpha^{2}\right)x^{9}+\cdots
\end{aligned}
$$

| n | Padé approx imants | MADM [23] | MVIM [4] | MLDM [3] | Present Method |
|---|---|---|---|---|---|
| 0.2 | [2,2] | -0.3872983347 | -0.3872983347 | -0.3872983347 | -0.3872983347 |
| | [3,3] | -0.3821533832 | -0.3821533832 | -0.3821533832 | -0.3821533832 |
| | [4,4] | -0.3819153845 | -0.3819153845 | -0.3819153845 | -0.3819153845 |
| | [5,5] | -0.3819148088 | -0.3819148088 | -0.3819148088 | -0.3819148088 |
| | [6,6] | -0.3819121854 | -0.3819121854 | -0.3819121854 | -0.3819121854 |
| 0.3 | [2,2] | -0.5773502692 | -0.5773502692 | -0.5773502692 | -0.5773502692 |
| | [3,3] | -0.5615999244 | -0.5615999244 | -0.5615999244 | -0.5615999244 |
| | [4,4] | -0.5614066588 | -0.5614066588 | -0.5614066588 | -0.5614066588 |
| | [5,5] | -0.5614481405 | -0.5614481405 | -0.5614481405 | -0.5614481405 |
| | [6,6] | -0.5614491934 | -0.5614491934 | -0.5614491934 | -0.5614491934 |
| 0.4 | [2,2] | -0.6451506398 | -0.6451506398 | -0.6451506398 | -0.6451506398 |
| | [3,3] | -0.6391000575 | -0.6391000575 | -0.6391000575 | -0.6391000575 |
| | [4,4] | -0.6389732578 | -0.6389732578 | -0.6389732578 | -0.6389732578 |
| | [5,5] | -0.6389892681 | -0.6389892681 | -0.6389892681 | -0.6389892681 |
| | [6,6] | -0.6389734794 | -0.6389734794 | -0.6389734794 | -0.6389734794 |

| n | Padé approx imants | MADM [23] | MVIM [4] | MLDM [3] | Present Method |
|---|---|---|---|---|---|
| 0.6 | [2,2] | -0.8407961591 | -0.8407961591 | -0.8407961591 | -0.8407961591 |
| | [3,3] | -0.8393603021 | -0.8393603021 | -0.8393603021 | -0.8393603021 |
| | [4,4] | -0.8396060478 | -0.8396060478 | -0.8396060478 | -0.8396060478 |
| | [5,5] | -0.8395875381 | -0.8395875381 | -0.8395875381 | -0.8395875381 |
| | [6,6] | -0.8396056769 | -0.8396056769 | -0.8396056769 | -0.8396056769 |
| 0.8 | [2,2] | -1.0079832070 | -1.0079832070 | -1.0079832070 | -1.0079832070 |
| | [3,3] | -1.0077969810 | -1.0077969810 | -1.0077969810 | -1.0077969810 |
| | [4,4] | -1.0076468280 | -1.0076468280 | -1.0076468280 | -1.0076468280 |
| | [5,5] | -1.0076468280 | -1.0076468280 | -1.0076468280 | -1.0076468280 |
| | [6,6] | -1.0077921000 | -1.0077921000 | -1.0077921000 | -1.0077921000 |

Table 1: Comparison of the numerical values of $\alpha = f''(x)$ for $0 < x < 1$ by using Padé approximation

The series solution accuracy and further understanding of the solution behaviour is enhanced by the diagonal Padé approximations exhibited below using the series solution $f(x)$ obtained earlier.

| n | MLDM [3] | MADM [23] | HPM [2] | MVIM [4] | Present Method |
|---|---|---|---|---|---|
| 4 | -2.483954032 | -2.483954032 | -2.5568 | -2.483954032 | -2.483954032 |
| 10 | -4.026385103 | -4.026385103 | -4.0476 | -4.026385103 | -4.026385103 |
| 100 | -12.84334315 | -12.84334315 | -12.8501 | -12.84334315 | -12.84334315 |
| 1000 | -40.65538218 | -40.65538218 | -40.6556 | -40.65538218 | -40.65538218 |
| 5000 | -104.8420672 | -104.8420672 | -90.9127 | -104.8420672 | -104.8420672 |

Table 2: Comparison of the numerical value of $\alpha = f''(0)$, using diagonal Padé approximants

## 4. Conclusion

The modified iterative decomposition method combined with diagonal Padé approximants for solving boundary layer problem in unbounded domain has been presented. The convergence of the technique is also exhibited in table 1 and 2. An analytic approach was used to obtain numerical values of $f^{''}(x)$ for various values of $n$. The method further attests to the fact that $f^{''}(x)$ decays algebraically for $0 < n < 1$ and decays exponentially for $n > 1$ as x tends to infinity as earlier claimed by Kuiken[15,16]. Comparison of the present method agrees favourably with the existing solution in the literature. The method prove simple in its principles and convenient for computer algorithms.

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