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CONTINUOUS ATTRACTORS IN HOPFIELD NEURAL NETWORKS

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Abstract: In this paper, we demonstrate periodic motion, 3-chaos and 4-torus in a five-dimensional Hopfield neural networks for some weight matrices. This purpose is achieved by using Matlab code for computing the Lyapunov spectrum.

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Key Words: Hopfield neural networks, periodic motion, 3-chaos, 4-torus

1. Introduction

The Hopfield neural network [5] abstracted from brain dynamics is a significant model in artificial neuro- computing. However, there are criticisms that it is too simple because it is just a gradient descent system converging to an equilibrium point, and the brains seem more dynamical. Many have suggested that chaos plays a central role in memory storage and retrieval. From this point of view, many artificial networks have been proposed in order to realize more dynamical attractors as chaos, Hyperchaos in artificial neural dynamics. [1, 2]. In this paper, we consider a 5D Hopfield neural network and show that periodic motion, 3-chaos and 4-torus dynamics can occur for some weight matrices.

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An n-dimensional continuous-time (autonomous) smooth dynamical system is defined by the differential equation

$$x' = F(x) \tag{1}$$

Where $x' = \frac{dx}{dt}$, $x(t) \in \mathbb{R}^n$ is the state vector at time t and $F : U \to \mathbb{R}^n$ is a C^r function ($r \geq 1$) on an open set $U \subset \mathbb{R}^n$. The space of dependent variables is often referred to as the phase space $M$of the system, which, in our discussion, will be$\mathbb{R}^n$. The function $F$is a called *vector field.* The vector field F generates the *flow*

$$\phi : U \times \mathbb{R} \to \mathbb{R}^n,$$

where $\phi_t(x) = \phi(t, x)$ is a C^r function defined for all $x \in U$ and $t \in \mathbb{R}$, such that $\phi_t'(x) = F(\phi_t(x))$ for all $x \in U$, $t \in \mathbb{R}$.

Given an initial state $x_0 \in U$, the solution of (1) is the function $\phi_t(x_0)$: $\mathbb{R} \to \mathbb{R}^n$ such that $\phi_0(x_0) = x_0$. The set $\{\phi_t(x_0) : t \in R\}$is called the *trajectory* of the system through x_0. A point$p$is an ω-limit point of x if there are points $\phi_{t_1}(x)$, $\phi_{t_2}(x)$,... on the the trajectory of x such that $\phi_{t_i}(x) \to p$as $t_i \to \infty$. The ω-limit set $\Omega\,(x)$is the set of all the ω-limit point of x. Moreover, an ω-limit set Ω is *attracting* if there exists an open neighborhood U of $\Omega(x)$ such that $\Omega\,(x) = \Omega$ for all $x \in U$. The *basin* of *attraction* B_Ω of an attracting set Ω is the union of all such neighborhoods U. In other words, B_Ω is the set off initial conditions x that tend toward Ω as $t \to \infty$.

There are four foundamental types of limit sets, corresponding to as many types of solutions of differential equations.

A: *Fixed points.* A fixed point is a point $x \in M$such that $\phi_t(x) = x$ for all t.

B: *Periodic motions.* A periodic motion is a solution $\phi_t$such that $\phi_t(x) = \phi_{t+T}(x)$ for some fixed constant $T > 0$ (the period) and all t. The limit set corresponding to a periodic solution is the closed curv traced out by $\phi_t(x)$ over one period, which is topologically equivalent to a circle S^1.

C: *Quasiperiodic motions.* A Quasiperiodic solution of a dynamical system is a function $\phi : R \to R^n$that can be represented in the form $f(t) = H(\omega_1 t, ..., \omega_n t)$, where H is periodic of period 2π in each argument, and the real numbers form a finite set of base frequencies. A Quasiperiodic solution with q base frequencies is called q- periodic. The limit set of a q- periodic solution is a diffemorphic copy of a q-dimensional torus $T^q = S^1 \times \cdots \times S^1$, where each S^1represents one of the base frequencies.

D: *Chaotic motions.* Avoiding a formal definition, we say that chaotic dynamics are characterized by three properties: (a) they are bounded random-like state trajectories distinct from the previous kinds of motion; (b) they converge to a set in the phase space, called a strange attractor, which is not a simple manifold like a point, circle, or torus, but has a complex (fractal) geometrical structure with a fractional Hausdorff dimension [2]; (c) they exhibit sensitive dependence to initial conditions, that is chaotic trajectories locally diverge away from each other and small changes in starting conditions build up exponentially fast into large changes in evolution.

2. The Lyapunov Spectrum Defined

We now define [1, 7] the spectrum of Lyapunov exponents in the manner most relevant to spectral calculations. Given a continuous dynamical system in an n-dimensional phase space, we monitor the long-term evolution of an infinitesimal n-sphere of initial conditions; the sphere will become an n-ellipsoid due to the locally deforming nature of the flow. The ith one-dimensional Lyapunov exponent is then defined in terms of the length of the ellipsoidal principal axis

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \log_2 \frac{p_i(t)}{p_i(0)}$$

where the λ_i are ordered from largest to smallest. Notice that the linear extent of the ellipsoid grows as $e^{\lambda_1 t}$, the area defined by the first two principal axes grows as $e^{(\lambda_1+\lambda_2)t}$, the volume defined by the first three principal axes grows as $e^{(\lambda_1+\lambda_2+\lambda_3)t}$, and so on. This property yields another definition of the spectrum of exponents: the sum of the first j exponents is defined by the long term exponential growth rate of a j-volume element. This alternate definition will provide the basis of our spectral technique for experimental data. Any continuous time-dependent dynamical system without a fixed point will have at least one zero exponent [4]. Axes that are on the average expanding (contracting) correspond to positive (negative) exponents. The sum of the Lyapunov exponents is the time-averaged divergence of the phase space velocity; hence any dissipative dynamical system will have at least one negative exponent, the sum of all of the exponents is negative, and the posttransient motion of trajectories will occur on azero volume limit set, an attractor. The exponential expansion indicated by a positive Lyapunov exponent is incompatible with motion on a bounded attractor unless some sort of *folding* process merges widely separated trajectories. Each positive exponent reflects a "direction" in which the system experiences the repeated stretching and folding that decorrelates nearby states on the attractor. Therefore, the long-term behavior of an initial condition that is specified with *any* uncertainty cannot be predicted; this is chaos. An attractor for a dissipative system with one or more positive Lyapunov exponents is said to be "strange" or "chaotic".

As previously mentioned, Lyapunov characteristic exponents (LCEs) are convenient for categorizing asymptotic behaviors of dynamical systems. Following [6] , in table 1 we give a classification of (autonomous) continuous-time attractors on the basis of their Lyapunov spectrum, together with their Hausdorff dimension.

Table 1. LEC spectrum of continuous-time attractors

Topological dimension	Dynamic of the attractor	LEC spectrum	Hausdorff dimension
1	Fixed point	$-$	0
2	Periodic motion	$0\,-$	1
3	Torus T^2 Chaos C^1	$00\,-$ $+0\,-$	2 $2 < D < 3$
4	Hypertorus T^3 Chaos on T^3 Hyperchaos C^2	$000\,-$ $+00\,-$ $+\,+\,0\,-$	3 $3 < D < 4$ $3 < D < 4$
N	Fixed point Periodic motion (N-1)-Torus (N-1)- Chaos	$- \cdots -$ $0 - \cdots -$ $\underbrace{0\cdots0}_{l\geq2}\underbrace{-\cdots-}_{N-l}$ $\underbrace{+\cdots+}_{k\geq1}\ \underbrace{0\cdots0}_{l\geq1}\ \underbrace{-\cdots-}_{N-k-l}$	0 1 L $k+l < D < N$

3. Lyapunov Dimension

Kaplan and Yorke have suggested an interesting conjecture that relates the fractal dimension of the attractor to the Lyapunov spectrum:

$$D_L = j - \frac{\sum_{i=1}^{j} \lambda_i}{\lambda_{j+1}},$$

where the LCEs are ordered in the usual way as $\lambda_1 \geq \cdots \geq \lambda_n$ and where j is the largest integer such that $\lambda_1 + \cdots + \lambda_j > 0$. In particular, Kaplan and Yorke suggest that $D_L$is a lower bound of the capacity dimension, that is, $D_L \leq D_C$. For more details, see [3].

4. The 5D Hopfield-Type Neural Network

In this section, we will study a simplified Hopfield neural network as described by

$$x_i = -\, c_i x_i + \sum_{j=1}^{5} w_{ij} \tanh(x_i)\,, \quad i = 1, ..., 5 \tag{2}$$

where $W = (w_{ij})$ is an 5×5 matrix, called weight matrix or connection matrix describing the strength of connections between neurons.

In what follows, we will show by computer simulation that (2) dose exhibit periodic motion, 3-chaos and 4-torus dynamics for some carefully chosen weight matrices.

Now take $c_1 = c_2 = c_3 = 1$, $c_4 = 100$, $c_5 = 10$ and W to be

$$W = \begin{pmatrix} 1 & 0.7 & -2.6 & -0.9 & 0 \\ 0 & 2+p & 3.2 & 0 & 0 \\ 3 & -3 & 1 & 0 & 0 \\ 97 & 0 & 0 & 163 & 0 \\ 100 & 0 & 0 & 0 & 70 \end{pmatrix}$$

where p is an adjustable parameter. By numerical calculation, the system for $-1.2 < p < 0.3$ has only one unstable equilibrium point, which is just the origin. Because all the trajectories of system (2) are bounded, there must be some other dynamics such as periodic motion or even 3-chaos and 4-torus attractors. To illustrate this fact, we adjust p very slowly and calculate the the Lyapunov exponents λ_1,λ_2,..., λ_5 of this system whit initial condition $x_0 = [0.01,\ 0.01,\ 0,\ 0,\ 0]^T$by the algorithm proposed in [8] that coding in Matlab software. Since λ_4 and λ_5 are always negative, we merely give the first three Lyapunov exponent as function of p in Fig. 1 with $-1.2 < p < 0.4$. Fig.1 shows the evidence that there are periodic motion, 3-chaos and 4-torus attractors with respect to p.

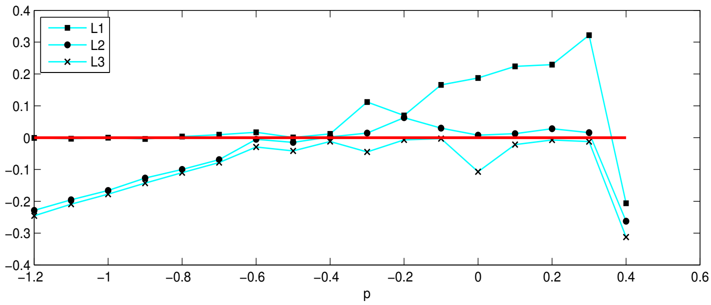

Figure 1: Lyapunov exponents as we adjust p with step 0.1.

For $p = -1$, we calculate the Lyapunov exponents as shown in Fig.2. Computer simulation shows that (2) has a periodic motion attractors as illustrated in Fig.2. In this case, Lyapunov dimension is 1.00095.

For $p = -0.09$, we calculate the Lyapunov exponents as shown in Fig.4. Computer simulation shows that (2) has a 3-chaos attractors as illustrated in Fig.5. In this case, Lyapunov dimension is 3.01988.

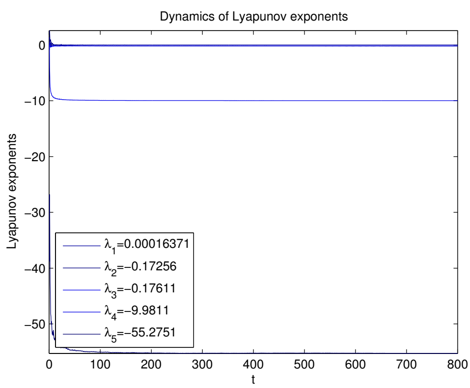

Figure 2: Lyapunov exponents for system (2) with p= -1

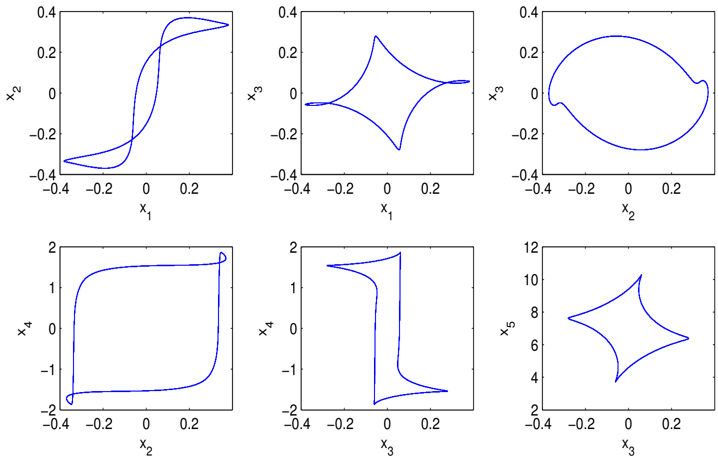

Figure 3: The limit circle of (2) observed with p= -1 and initial point (0.2479 , 0.3645, 0.0400 , 1.7842 , 9.9181).

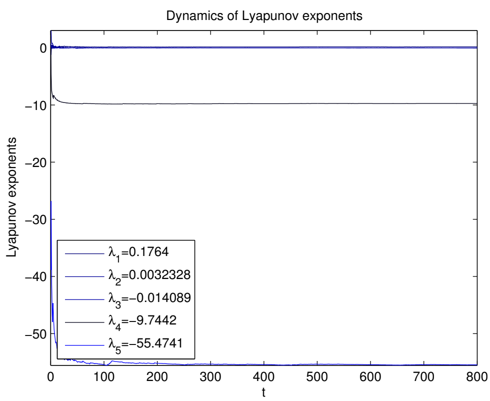

Figure 4: Lyapunov exponents for system (2) with p=- 0.09

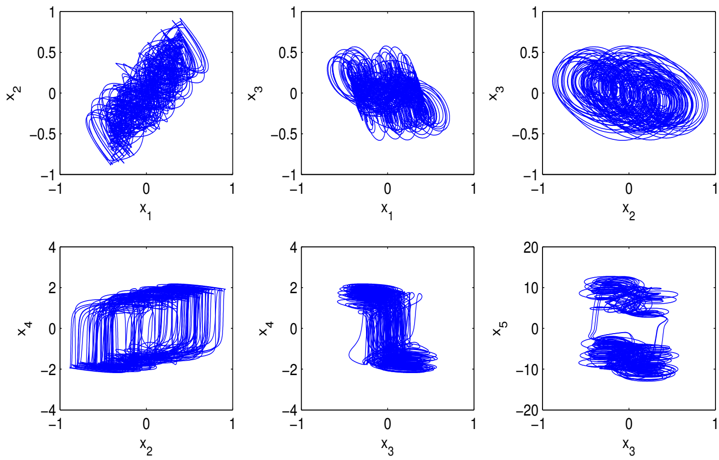

Figure 5: The 3-chaos of (2) observed with $p=$ = - 0.09 and initial point (-0.0668 -0.2141 -0.2783 1.3712 -7.5430).

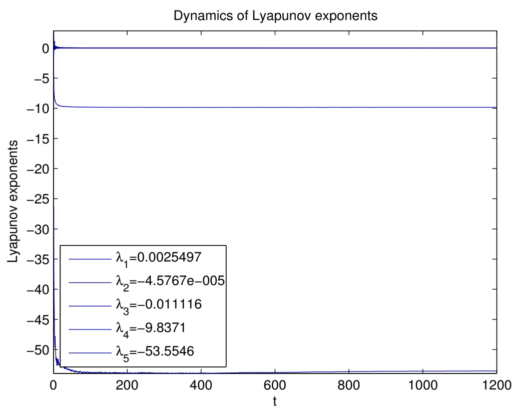

Figure 6: Lyapunov exponents for system (2) with p=-0.38

For $p = -0.38$, we calculate the Lyapunov exponents as shown in Fig.6. Computer simulation shows that (2) has a periodic 4-torus attractors as illustrated in Fig.7. In this case, Lyapunov dimension is 2.0146.

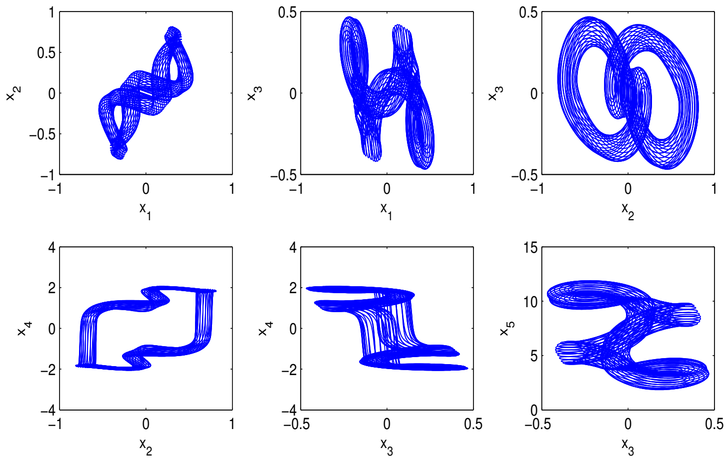

Figure 7 The 4-torus of (2) observed with $p =- 0.38$ and initial point (-0.3377 -0.8001 0.0003 -1.8739 3.6523).

Acknowledgments

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