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A COMBINATION OF THE LAPLACE TRANSFORM AND THE VARIATIONAL ITERATION METHOD FOR THE ANALYTICAL TREATMENT OF DELAY DIFFERENTIAL EQUATIONS

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Abstract: This paper is concerned with obtaining the exact solutions of linear and nonlinear delay differential equations (DDEs) via a combination of the Laplace transform and variational iteration method. In this approach, a correction functional is constructed by a general Lagrange multiplier, which is determined by using the Laplace transform with the variational theory. Numerical examples are given to elucidate the solution process, the simplicity, efficiency and reliability of the new approach.

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1. Introduction

Delay differential equations arises when the rate of change of a time dependent process in its mathematical modelling is not only determined by its present state but also at certain past state known as its history. Introduction of delays in models enriches the dynamics of such models and allow a precise description of real life phenomena. DDEs arise frequently in signal processing, digital images, control systems [24], lasers, traffic models [18], metal cutting, epidermology, neuroscience, population dynamics [25], chemical kinetics [20] and in many physical phenomena. In particular, they are fundamental when ordinary differential equation based models fail. Unlike ordinary differential equations (ODEs) where the initial conditions are specified at the initial point, DDEs require the history of the system over the delayed interval and are given as initial conditions. Due to this reason, delay systems are complex in nature. Because of this complexity, DDEs are difficult to analyze analytically and hence the need for numerical approach.

The variational iteration method (VIM) was developed by He [1-4] for solving a wide range of nonlinear problems. Subsequent works [1-8], [10-16] clearly reflect the flexibility, reliability and efficiency of the algorithm in VIM. This algorithm makes use of a general Lagarange multiplier which is determined by the variational theory. The VIM has been successfully applied on initial and boundary value problems, the schrodinger equations, integro-differential equations, fractional differential equations, partial differential equations, q-difference equations. quadratic Riccati differential equations, fractional diffussion wave equations etc [1-6], [8-15].

The application of VIM to differential equations usually involve the following steps

1. obtaining the correction functional
2. identifying the Lagrange multiplier
3. determining a good initial approximation

Various authors ([4], [11], [14], [25-26]) have identified this Lagrange multiplier via different approaches inorder to accelerate the convergence rate of solutions. Among such authors are Wu and Baleanu [14] who proposed the identification of the Lagrange multiplier via Laplace transform and variational theory. The basic motivation of this paper is the extension of this approach for solving linear and nonlinear delay differential equations which are otherwise difficult to analyze because of their complex nature and infinite dimensionality.

The paper is organised as follows: In Section 2, we give a brief description of how the VIM works and propose an algorithm with the Laplace VIM for DDEs. We applied the algorithm on some linear and nonlinear DDEs in Section 3. Finally, we give some concluding remarks in Section 4.

2. Analysis of the Laplace Variational Iteration Method (LVIM)

2.1. Basics of the VIM

Inorder to ilustrate the basic idea of the VIM for FIDEs, we consider the the following nonlinear differential equation.

$$\frac{d^m u}{dt^m} + R(u) + N(u) = f(t), \tag{1}$$

with the following the initial conditions

$$u^{(k)}(0) = u_0^k \qquad k = 0(1)(m-1), \tag{2}$$

where $u = u(t)$, R is a linear bounded operator, N is a nonlinear bounded operator and $f(t)$ is a given continuous function and $\frac{d^m u}{dt^m}$ is the term of the highest order derivative.

The basic idea of the VIM is to construct a correction functional for (1) of the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,\tau)\left(\frac{d^m u_n}{dt^m} + R(\widetilde{u}_n) + N(\widetilde{u}_n) - f(\tau)\right) d\tau. \tag{3}$$

The successive approximation u_n, $n \geq 1$, can be obtained by determining $\lambda(\tau)$, a general Lagrange multiplier which can be identified optimally via variational theory . The function $\widetilde{u}_n$ is regarded as a restricted variation which implies that $\delta\widetilde{u}_n = 0$. The Lagrange multiplier is first determined via integration by parts. Having identified the Lagrange multiplier $\lambda(\tau)$, the successive approximations $u_n(t)$ of the exact solution $u(t)$ will be readily obtained upon using a good initial approximation $u_0(t)$. The initial approximation is usually obtained from the initial conditions (2). As $n \to \infty$, $u_n(t)$ converges to the exact solution $u(t)$.

2.2. Laplace Variational Iteration Method

Inorder to accelerate the convergence rate of solutions, Wu and Baleanu [14] proposed the Laplace variational iteration method (LVIM). The basic steps involved are given as follows:

1. Take the Laplace transform of (1), then the correction functional is

$$U_{n+1}(s) = U_n(s) + \lambda(s)\left(s^m U_n(s) - s^{m-1}u(0) \right.$$
$$\left. - \cdots - u^{(m-1)}(0) + L\left[R(\widetilde{u}_n) + N(\widetilde{u}_n) - f(t)\right]\right). \quad (4)$$

2. Regarding the terms $L(R(\widetilde{u}_n) + N(\widetilde{u}_n))$ as restricted variations, we make (4) stationary with respect to U_n

$$\delta U_{n+1}(s) = \delta U_n(s) + \lambda(s)\left(s^m \delta U_n(s)\right). \quad (5)$$

From (5), we determine the Lagrange multiplier as

$$\lambda(s) = -\frac{1}{s^m}. \quad (6)$$

3. The successive approximations are obtained by taking the inverse Laplace transform to obtain

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^m}\left(s^m U_n(s) - s^{m-1}u(0) \right.\right.$$
$$\left.\left. - \cdots - u^{(m-1)}(0) - L\left[R(u_n) + N(u_n) - f(t)\right]\right)\right]$$
$$= L^{-1}\left(\frac{u(0)}{s} + \cdots + \frac{u^{(m-1)}(0)}{s^m}\right)$$
$$+ L^{-1}\left[\frac{1}{s^m}\left(L\left[R(u_n) + N(u_n) - f(t)\right]\right)\right],$$

with initial approximation

$$u_0(t) = L^{-1}\left(\frac{u(0)}{s} + \cdots + \frac{u^{(m-1)}(0)}{s^m}\right)$$
$$= u(0) + u'(0)t + \cdots + \frac{u^{(m-1)}(0)t^{m-1}}{(m-1)!}.$$

3. Numerical Applications

In this section, we apply the above procedure to some linear and nonlinear delay differential equations.

Example 3.1. We consider the first order DDE

$$u'(t) = \frac{1}{2}e^{t/2}u\left(\frac{t}{2}\right) + \frac{1}{2}u(t), \quad u(0) = 1.$$

The analytic solution is given by

$$\text{Exact}: \quad u(t) = e^t.$$

Taking the Laplace transform we obtain

$$sU(s) - u(0) = L\left[\frac{1}{2}e^{t/2}u\left(\frac{t}{2}\right) + \frac{1}{2}u(t)\right]$$

The iteration formula thus is

$$U_{n+1}(s) = U_n(s) + \lambda(s)\left[sU_n(s) - u(0) - L\left(\frac{1}{2}e^{t/2}u\left(\frac{t}{2}\right) + \frac{1}{2}u(t)\right)\right]$$

with the Lagrange multiplier

$$\lambda(s) = -\frac{1}{s}$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} u_{n+1}(t) &= u_n(t) - L^{-1}\left\{\frac{1}{s}\left[sU_n(s) - u(0) - L\left(\frac{1}{2}e^{t/2}u_n\left(\frac{t}{2}\right) + \frac{1}{2}u_n(t)\right)\right]\right\} \\ &= L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left\{\frac{1}{s}\left[L\left(\frac{1}{2}e^{t/2}u_n\left(\frac{t}{2}\right) + \frac{1}{2}u_n(t)\right)\right]\right\} \end{aligned}$$

Therefore,

$$u_{n+1}(t) = 1 + L^{-1}\left\{\frac{1}{s}L\left(\frac{1}{2}e^{t/2}u_n\left(\frac{t}{2}\right) + \frac{1}{2}u_n(t)\right)\right\} \tag{7}$$

with initial iteration $u_0(t) = 1$ and by using the above iteration formula (7) we obtain

$$u_1(t) = \frac{t}{2} + e^{\frac{t}{2}}$$

$$u_2(t) = -\frac{1}{6} + \frac{t^2}{8} + \frac{e^{\frac{t}{2}}}{2} + \frac{2e^{\frac{3t}{4}}}{3} + \frac{te^{\frac{t}{2}}}{4}$$
$$u_3(t) = -\frac{11}{84} - \frac{t^2}{12} + \frac{t^3}{48} + \frac{e^{\frac{t}{12}}}{2} + \frac{2e^{\frac{3t}{4}}}{3} + \frac{8e^{\frac{7t}{8}}}{21} + \frac{te^{\frac{t}{2}}}{8} + \frac{te^{\frac{3t}{4}}}{12} + \frac{t^2e^{\frac{t}{2}}}{32}$$
$$u_4(t) = -\frac{187}{2520} - \frac{11t}{168} - \frac{t^2}{48} + \frac{t^4}{384} - \frac{5e^{\frac{t}{56}}}{2} + \frac{7e^{\frac{3t}{4}}}{8} + \frac{4e^{\frac{7t}{8}}}{7} + \frac{64e^{\frac{15t}{16}}}{315} + \frac{te^{\frac{t}{2}}}{48} + \frac{te^{\frac{3t}{4}}}{12} + \frac{te^{\frac{7t}{8}}}{42} + \frac{t^2e^{\frac{t}{2}}}{64} + \frac{t^2e^{\frac{3t}{4}}}{192} + \frac{t^3e^{\frac{t}{2}}}{384}$$
$$\vdots$$

Example 3.2. As a second example, we consider the first order nonlinear DDE

$$u'(t) = 1 - 2u^2\left(\frac{t}{2}\right), \ u(0) = 0$$

The analytic solution is given by

$$\text{Exact} : u(t) = \sin(t)$$

Following the procedures of example 3.1, we obtain the iteration formula

$$u_{n+1}(t) = L^{-1}\left[\frac{1}{s}L\left(1 - 2u_n^2\left(\frac{t}{2}\right)\right)\right] \tag{8}$$

with initial approximation $u_0(t) = 0$. Thus, from the iteration formula (8), we obtain

$$u_1(t) = t$$
$$u_2(t) = t - \frac{t^3}{6}$$
$$u_3(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{8064}$$
$$u_4(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{61t^9}{23224320} - \frac{67t^{11}}{3406233600} + \frac{t^{13}}{12881756160}$$
$$\vdots$$

From these approximations, we can note that it is the series expansion of the exact solution $u(t) = \sin(t)$

Example 3.3. Consider the second order nonlinear DDE

$$u''(t) = \frac{3}{4}u(t) + u\left(\frac{t}{2}\right) - t^2 + 2, \quad u(0) = 0, \quad u'(0) = 0$$

The analytic solution is given by

$$\text{Exact} : u(t) = t^2$$

Taking the inverse Laplace transform, we obtain the iteration formula

$$\begin{aligned} U_{n+1}(s) &= U_n(s) \\ &+ \lambda(s)\left\{s^2U_n(s) - su(0) - u'(0) - L\left[\frac{3}{4}u(t) + u\left(\frac{t}{2}\right) - t^2 + 2\right]\right\} \end{aligned}$$

with the Lagrange multiplier

$$\lambda(s) = -\frac{1}{s^2}$$

Taking the inverse Laplace transform, we obtain

$$u_{n+1}(t) = L^{-1}\left\{\frac{1}{s^2}L\left[\frac{3}{4}u(t) + u\left(\frac{t}{2}\right) - t^2 + 2\right]\right\} \tag{9}$$

with initial approximation $u_0(t) = 0$. From (9), we obtain

$$\begin{aligned} u_1(t) &= t^2 - \frac{t^4}{12} \\ u_2(t) &= t^2 - \frac{13t^6}{5760} \\ u_3(t) &= t^2 - \frac{91t^8}{2949120} \\ u_4(t) &= t^2 - \frac{17563t^{10}}{67947724800} \\ &\vdots \end{aligned}$$

We can also observe the convergence of the approximations to the exact solution $u(t) = t^2$

Example 3.4. Consider the third order linear DDE

$$u'''(t) = -u(t) - u(t - 0.3) + e^{-t+0.3}, \quad u(0) = 1, \; u'(0) = -1, \quad u''(0) = 1$$

The analytic solution is given by

$$\text{Exact} : u(t) = e^{-t}$$

Applying the inverse Laplace transform, we have

$$s^3U(s) - s^2u(0) - su'(0) - u''(0) + L\left[u(t) + u(t-0.3) - e^{-t+0.3}\right] = 0$$

The iteration formula is thus

$$U_{n+1}(s) = U_n(s) + \lambda(s)\left[s^3U(s) - s^2u(0) - su'(0) - u''(0) + L\left(u(t) + u(t-0.3) - e^{-t+0.3}\right)\right]$$

with the Lagrange multiplier

$$\lambda(s) = -\frac{1}{s^3}$$

Taking the inverse Laplace transform, we have the iteration formula

$$u_{n+1}(t) = 1 - t + \frac{t^2}{2} - L^{-1}\left\{\frac{1}{s^3}\left[L\left(u(t) + u(t-0.3) - e^{-t+0.3}\right)\right]\right\}$$

Taking $u_0(t) = 1 - t + \frac{t^2}{2}$, we obtain the approximations

$u_1(t) = 2.34986 - 2.34986t + 1.17493t^2 - 0.390833t^3 + 0.0958333t^4 - 0.0166667t^5 - 1.34986e^{-t}$

$u_2(t) = 5.52184 - 5.52184t + 2.76092t^2 - 0.920298t^3 + 0.230051t^4 - 0.0459643t^5 + 0.00759722t^6 - 0.00103175t^7 + 0.000992063t^8 - 4.52184e^{-t}$

$u_3(t) = 12.9755 - 12.9755t + 6.48777t^2 - 2.16259t^3 + 0.540647t^4 - 0.108129t^5 + 0.0180203t^6 - 0.00257303t^7 + 0.000320547t^8 - 0.0000349427t^9 + 3.19665\times 10^{-6}t^{10} - 11.9755e^{-t}$

$u_4(t) = 30.4907 - 30.4907t + 15.2453t^2 - 5.08178t^3 + 1.27044t^4 - 0.254089t^5 + 0.0423481t^6 - 0.00604971t^7 + 0.000756188t^8 - 0.0000840012t^9 + 8.38816\times 10^{-6}t^{10} - 7.56846\times 10^{-7}t^{11} + 6.09601\times 10^{-8}t^{12} - 4.11112\times 10^{-9}t^{13} + 1.83532\times 10^{-10}t^{14} - 29.4907e^{-t}$

$\vdots$

Example 3.5. Lastly, we consider the third order nonlinear DDE

$$u'''(t) = -1 + 2u^2\left(\frac{t}{2}\right), \quad u(0) = 0, \ u'(0) = 1, \quad u''(0) = 0$$

The analytic solution is given by

$$\text{Exact} : u(t) = \sin(t)$$

Following the procedures of the previous examples, we obtain the successive approximations

$$
\begin{aligned}
u_1(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} \\
u_2(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} + \cdots \\
u_3(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}}{39916800} + \frac{t^{13}}{622702800} + \cdots \\
u_4(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}}{39916800} + \frac{t^{13}}{622702800} \\
&\quad - \frac{t^{15}}{1307674368000} + \frac{t^{17}}{355687428096000} + \cdots \\
&\vdots
\end{aligned}
$$

which converges to the exact solution $u(t) = \sin(t)$ as $n \to \infty$.

4. Conclusion

The Laplace-Lagrange multiplier is proposed from the Laplace transform and incorporated with the techniques of VIM to produce exact/approximate solutions to linear and nonlinear delay differential equations. With this approach, the Lagrange multipliers can be identified easily and new variational iteration formulae can be derived. The proposed algorithm is employed without using linearization, discretization or unrealistic assumptions. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result. The approach is implemented on some DDEs to show its simplicity, accuracy and efficiency.

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